THE PROPERTIES OF ENGINEERING MATERIALS

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WITH 290 ILLUSTRATIONS IN THE TEXT AND 34 PLATES

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PREFACE

For the engineer who is designing structural members and machine parts, a knowledge of the stresses induced in these members by the application of known loads is of paramount importance. In addition, he should have some knowledge of the strength properties of the materials which he intends to use in forming the members.

The present volume deals with these matters. Part I considers the relation between load and stress in the more usual shapes used for structural members, on the assumption that the material is homogeneous and elastic. Part II is devoted to a discussion of the properties peculiar to the materials themselves, with some reference to modes of manufacture; and in this connexion reference is made to the various ways in which the materials are tested. In the second part such matters as the behaviour of material under repeated stress, effect of wear and deterioration, microscopic analysis, and Coker's optical method of exhibiting stress variation, also receive attention.

It was felt that although excellent books exist on the Strength of Materials, there was room for one more dealing with the subject from the descriptive as well as the mathematical standpoint.

This volume is intended mainly for the use of third and fourth year students in the Engineering Schools of the Universities and of the first-class Technical Schools. It is hoped that a good deal of the earlier work can be used by second-year students. It may also be used in the professional examinations, such as those of the Institution of Civil Engineers. There is a great deal in both parts of the book to appeal to engineers in practice, and it is hoped that it will be of value to them.

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THE PROPERTIES OF ENGINEERING MATERIALS

PART I

CHAPTER I

STRESS AND STRAIN IN ELASTIC MEMBERS

1. Introductory—When an engineer is called upon to design a structural member or machine part he should have knowledge of the following :

1. The loads which are likely to be applied to the member, both as to kind and magnitude.
2. The stresses and strains which the above loads may be expected to develop in the various parts of the member.
3. The qualities of the materials available, in order to make it possible to select the most suitable material for the purpose in view.
4. The probable effect of time and local conditions on the materials selected.

Of these four directions in which knowledge is desirable (1) is dealt with in the "Theory of Structures," and it will be assumed in what follows that this part of the designer's information has already been supplied. (2) is largely mathematical in nature, and is dealt with in that part of "Applied Mechanics" called "Strength of Materials." This forms an important part of what follows. (3) and (4) are chiefly dependent upon experimental results and come under the head of "Testing of Materials." It is intended to follow this scheme in the following pages.

2. The Terms used—The external forces applied to a structural member or piece of material are spoken of as loads. These are of various kinds. In "tension" loading the applied force exerts a pull on the piece; "compression" loading is the reverse of this and subjects the material to a pressure; when the loading is "transversal" the tendency is towards breaking

across and the effect is bending; in "torsion" loading the piece is twisted.

The application of load causes internal stress in the material. Such stress is the force transmitted from a part of the material to the portion with which it is in contact, or it may be called the force which the material opposes to the tendency to alter its shape or size.

Loads are generally expressed in pounds or tons in Anglo-Saxon countries. The term "stress" generally means "intensity of stress" or force upon unit area; it is expressed as so many pounds or tons per square inch or square foot. Application of force always results in "deformation" or change in shape or size of the piece. Thus, tension causes "stretch," compression "shortening," transverse loading causes "bending," and the effect of torsion is "shear" or "twist."

If the piece returns to its original form and dimensions when the force has been removed its material is said to be "elastic." All engineering materials are sensibly elastic under working conditions. In such materials as mild steel, wrought iron, and copper, relatively high loads cause permanent changes in shape of such a kind that the material can be described as "plastic." In some other materials, such as stone and concrete, loading beyond the elastic limit results in crushing or crumbling. In timber, when stressed either along or inclined to the direction of the grain, there is very little plastic stage in tension; the plastic stage in compression is more pronounced, tearing asunder or crushing of the wood structure accompanying the highest loads. In materials such as glass and hard steel an elastic condition persists almost up to fracture.

It is safe to say that there are very few materials which do not show the quality of elasticity at some stress; even such unlikely substances as clay and lead have been found elastic under very small loads. Some materials, such as cast iron and concrete, are never quite elastic, but within the total range of working stresses the elasticity is sufficiently evident to warrant its assumption in calculations of dimensions of structural parts. In materials where the deformation is only partly elastic, that portion which remains after the removal of the load is referred to as "permanent set."

The stress on any plane section of a loaded piece is either normal, tangential, or a combination of both. Thus in Fig. 1, ABCD represents two cubical parts of the material in contact. If a load tends to lift BCD vertically from ABD the stress developed normal to the plane BD is a direct stress; if the tendency of the force is to cause the upper cube to slide on the one beneath, the stress on the dividing plane is tangential, and is referred to as a

shear stress. The above direct stress tending to separate the cubes is tensile, and one acting in the reverse direction and pressing the cubes together would be compressive.

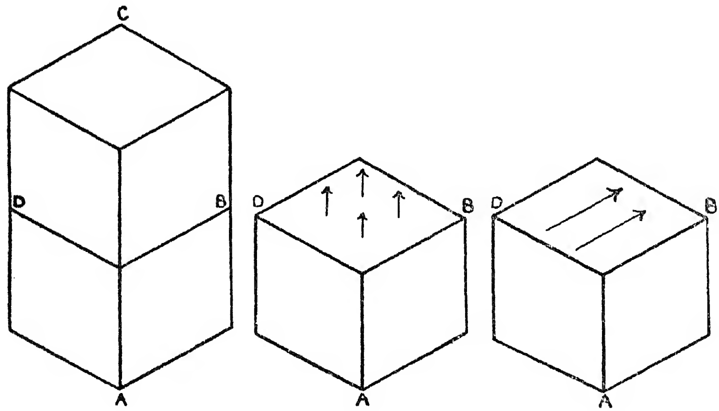


FIG. 1.

3. Strain—If the stress is defined as the resistance offered by the material to deformation, this deformation, which occurs when the resisting stress is overcome, is properly called the "strain," which as explained may be either elastic, plastic, or part elastic and part plastic.

The term "strain" may be used either to mean simple deformation or in a more precise sense. With this latter meaning it is more correctly described as the "proportional strain"; which represents the fraction which the magnitude of the strain forms of an unloaded dimension. Thus if a bar 100 in. long is stretched until it becomes 100·1 in., the magnitude of the tensile strain is 0·1 in., the proportional strain being $\frac{0 \cdot 1}{100}$ or 0·001. This is the correct way to denote strain.

4. Relations between Stress and Strain in an Elastic Rod—In all bodies under an increasing load there is some stress up to which no set or deformation is observed on the removal of the load; this is the *elastic limit*, the point below which the material may be regarded as quite elastic. Within this limit it is found that the strain is proportional to the stress; this is known as Hooke's Law.

Stated more precisely the law is, "Within the elastic limit the stress is directly proportional to the resulting strain." This is true of all materials in the elastic condition under any kind of stress, whether direct, resulting in longitudinal strain, shear

with angular strain, or normal stress over the whole surface of a body as hydrostatic pressure, resulting in volumetric strain.

This may be stated by saying that for a given elastic material the stress is equal to the strain multiplied by a constant or

$$\text{Stress} = \text{strain} \times \text{constant}.$$

The relation between stress and strain for an elastic material is accordingly linear.

In the case of an elastic rod subjected to longitudinal stress the constant is denoted by E and is known as the Modulus of Elasticity for the particular material, and also as Young's Modulus. The above relation may be written

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\text{stress}}{\text{proportional extension}} = \frac{\frac{P}{a}}{\frac{x}{l}} = \frac{Pl}{ax}$$

where P is the load, a the area upon which it acts, x the change in length of the rod, and l the original length. For example, a copper trolley wire 0·100 sq. in. in cross-section and 100 in. long is found to be extended 0·030 in. by a load of 500 lb. What is its modulus of elasticity?

$$E = \frac{\text{stress}}{\text{strain}} = \frac{Pl}{ax} = \frac{500 \times 100}{0{\cdot}100 \times 0{\cdot}030} = 16{,}650{,}000 \text{ lb. per sq. in.}$$

In this case the load is known or can be measured, the elongation is also measured, the original length and cross-section are known, so that it becomes easy to calculate the modulus of elasticity. The value thus found is a constant for the material in question. In strictness the area of the rod corresponding with the load and not the original area should be used to calculate the stress, but for most materials the change in area is so small within the elastic stage that no appreciable error is incurred by using the unstressed area.

Deformation and Stress of Various Kinds—The different kinds of deformation under load important to engineers are:—

1. Elongation under a pull, the stress being tensile.

2. Shortening or contraction under compressive stress, caused by a push.

3. Distortion under a pair of parallel forces acting normally to the axis of the piece but in opposite directions and tending to cause slide. In this case the stress is shear.

4. Twist caused by torsional moment. Here the stress is also shear, but is sometimes spoken of as torsional.

5. Deflection or bending from the straight by three or more forces acting at right-angles to the length of the piece. There

are in this case, at the same time, two kinds of stress of varying intensity, i.e., tensile and compressive, and in most cases shear also.

The intensities of the stresses are generally uniform across the section, or nearly so, in (1), (2), and (3). In the case of (4) the stress is one varying from the centre to the surface. In (5) there are at the same time tensile and compressive stresses, both varying uniformly from zero to a maximum; if shear is present this also varies from zero to a maximum.

5. The Shear Modulus—In Fig. 2 ABCD is a side view of a cube of elastic material and ABC′D′ is the view of the cube when it has been subjected to shear. The shear stress across a surface is given by the total tangential force divided by the area of the surface; this is denoted by s in the figure. In order to produce shear two pairs of parallel forces are necessary which act as shown; if only one pair acted the result would be simply a bodily rotation of the cube. The shear strain is given by

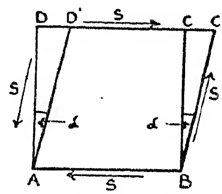


FIG. 2.

$$\frac{DD'}{AD} = \frac{CC'}{CB} = a.$$

Calling the shear modulus or modulus of rigidity "G," the conditions may be expressed by

$$G = \frac{\text{stress}}{\text{strain}} = \frac{s}{a}.$$

This may otherwise be interpreted as meaning: If a shear stress s will distort the cube through an angle a it will need (if elasticity can be maintained) a stress G to distort it through unit angle, i.e., one radian. The above definition of a is only true when the shear strain is small, as is nearly always the case in practice. If the shear strain is great the sides of the cube shown in the figure increase appreciably in length and a is represented by a more involved expression. The expression for G is of course correct, whatever the amount of strain, provided elasticity persists.

In the above cases the material is under either direct stress tensile or compressive, or the stress is shearing and the strain distortion. As the modulus of elasticity for direct stress can be found experimentally by measuring loads and changes in length, so it can also be obtained from experiments in cross-bending when the relations between the loads and the direct stresses are known, by observing loads and measuring the deflections.

This applies also to direct shear and to torsion, which latter

is also a case of shear. When the value of the shear modulus is desired it is generally more convenient to find it by means of torsion tests than by experiments in direct shear.

6. Induced Stresses and Strains—When a body is acted upon by a certain kind of load, such as a tension, the immediate effect is a direct stress on surfaces normal to the direction of the load. At the same time other stresses are caused on planes making various angles with the direction of load, and these induced stresses are accompanied by their corresponding strains. The most common case is where a tensile or compressive stress in one direction gives rise to shear stresses on planes which make angles of 45 deg. with the direction of the direct stress. This is indicated by the photograph in Plate I. Here a round bar of ductile material, such as mild steel, is loaded to just beyond the elastic limit, when lines begin to show on the surface, making angles of 45 deg. with the axis of the bar. These shear lines suggest a commencement of shear as the first indication of failure. The lines show clearly the existence of shear stresses along directions making 45 deg. with the direct stresses.

The same thing can be seen in cast iron under compression where the commencement of failure manifests itself by the appearance of minute cracks on the surface, all lying at about 45 deg. with the axis. These are both cases of shear stresses induced by direct stress. On the other hand, the original stress may be shear and may be the cause of tensile or compressive stress on surfaces inclined at 45 deg. with its own direction. Such a case is that of a circular torsion specimen of brittle material. In the case of cast-iron torsion specimens, failure takes place by the material breaking under a tensile stress induced by the shear; the result is shown by the helical fracture. Just as the applied loads induce stresses, so there are induced strains, and in some cases these can be measured.

7. Longitudinal Strain and Lateral Contraction or Dilatation When an elastic body is stretched or shortened under direct stress it either contracts or dilates laterally and there is a corresponding change in volume. In Fig. 3 is shown the case of a prismatic bar of length l' and diameter d'; this is stretched (being always elastic) until l' has become l'' and the diameter d' has shrunk to d''. The longitudinal proportional strain y will be

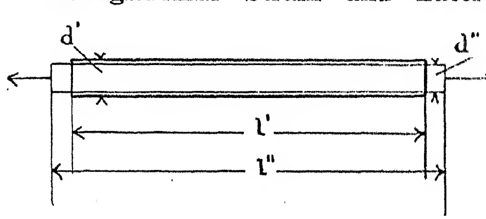


Fig. 3.

$$\frac{l''-l'}{l'}=y.$$

At the same time the lateral proportional contraction z is

$$\frac{d'-d''}{d'}=z.$$

The ratio which the lateral strain bears to the longitudinal strain, that is $\frac{z}{y}$, is spoken of as Poisson's Ratio, and has different values for different materials. For india-rubber, when the strains are small, Poisson's Ratio, $\frac{1}{m}$, is found to have a value of $\frac{1}{2}$. For most engineering materials its approximate value is generally taken as $\frac{1}{4}$. The authors found $\frac{1}{m}$ for concrete to be 0·200.

For some of the other materials, the following figures, quoted by Johnson from Wertheim, are interesting:

Values of Poisson's Ratio for several Common Materials:

Glass: 0·2451.	Brass: 0·3275.	Steel: 0·2686
Delta Metal: 0·3399	Copper: 0·3270	Lead: 0·4282

8. Change of Volume accompanying Longitudinal and Lateral Strain—Referring to Fig. 3, it is seen that the original volume of the body, previous to the stretch, was $V'=(d')^2l'$ and the volume after stretch $V''=(d'')^2l''$, in both cases leaving out $\frac{\pi}{4}$.

From this it follows that the fractional change in volume will be the following increase—

$$\frac{V''-V'}{V'}=\frac{(d'')^2l''-(d')^2l'}{(d')^2l'}.$$

Taking $\frac{1}{m}=\frac{1}{4}$ and writing this expression in terms of y, it finally reduces to $\frac{y}{2}-\frac{y^2}{16}+\frac{y^3}{16}$.

As y (the proportional change in length) is rarely greater than 0·001, its powers in the above expression greater than the first may be neglected. The proportional change in volume is thus sensibly equal to one-half the proportional change in length.

9. Direct and Shearing Stress—On Fig. 4 is shown a rectilinear body, of which EFGH is the vertical square forming one end. The solid rests on a horizontal plane and upon its top face acts a force P in such a manner as to apply a uniform pressure on the face and a uniform vertical stress upon every other hori-

zontal section. The area of the top face upon which P acts $=$EF (the length may be taken as unity). Call this top area a. The vertical stress upon it will be $\frac{\mathrm{P}}{a} = p_d$, which is the vertical stress upon any horizontal section.

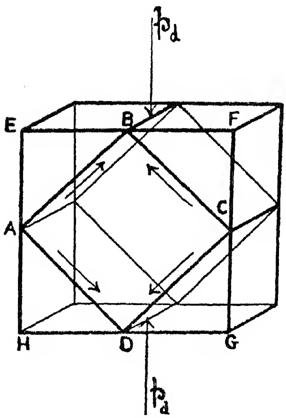


FIG. 4.

Consider the piece of material whose end is AEB, the vertical force on the half EB will be $p_d \cdot$ EB, being the product of the vertical stress and the magnitude of the area itself.

Resolve this normally and parallel to AB, which is inclined at an angle of 45 deg. with the horizontal.

The normal component is p_dEB cos 45.

The tangential component is p_dEB sin 45.

Dividing these component forces by the areas upon which they act, the stress on AB$= \frac{p_d\mathrm{EB}\cos 45}{\mathrm{EB}\sec 45} = p_d \cos^2 45 = \frac{p_d}{2}$ and the shear stress on AB$= \frac{p_d\mathrm{EB}\sin 45}{\mathrm{EB}\sec 45} = \frac{p_d}{2}\sin 90 = \frac{p_d}{2}$. Similar results can be obtained for the surfaces BC, CD, and DA. The result is that where there is a longitudinal stress of a certain intensity on planes at right angles to the axis of the prism, there will be a shear stress of one-half the intensity on planes making 45 deg. with the axis.

10. Shear Stress—When dealing with shear stress it is important to recognize that if there exists shear stress of a certain intensity on a given series of parallel plane surfaces, there must exist a shear stress of equal intensity on a series of parallel plane surfaces perpendicular to the first.

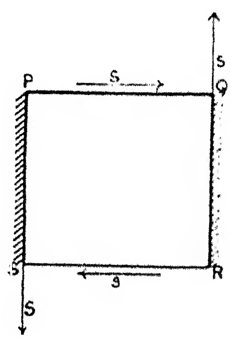


FIG. 5.

Consider the end view of a square prism of material of unit length shown in Fig. 5 as PQRS. First suppose that a shear stress s acts along the faces PS and RQ as shown by the arrows. The total forces on the two faces will be s(PS) and s(RQ) respec-

tively. These form a couple $s(PS)(PQ)$, and in order to prevent rotation there must be an equal and opposite couple: that is, for equilibrium and shear, there must be shearing forces acting along the other two faces. As the areas of these

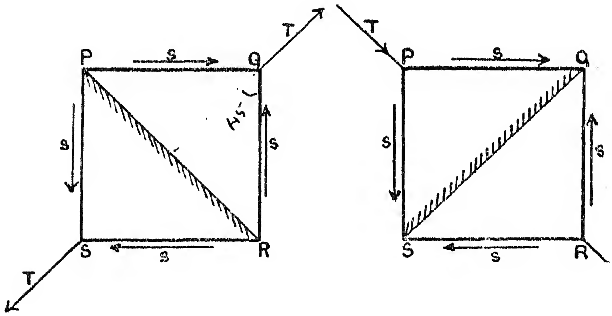


FIG. 6.

faces are equal to those of the first pair, and the arms of the couples are equal, it follows that the stresses are also equal.

It may also be useful to consider the connexion between the shear stress and the induced direct stresses from another point of view than the one already given.

In Fig. 6 the square PQRS is the end view of a prism of unit length and is subjected to the shear stresses shown. The forces along the faces RQ and PQ, i.e., $s(RQ)$ and $s(PQ)$ respectively will have components acting normal to the diagonal PR. This total normal force is

$$T = \{s(RQ) + s(PQ)\} \sin 45°$$

and the stress normal to PR is given by

$$\frac{\{s(RQ) + s(PQ)\} \sin 45°}{PR} = s.$$

The compressive stress on QS is found in the same way and has the same value; in other words, all three stresses, namely the shear stress and the direct stresses on the diagonal planes, have the same magnitude.

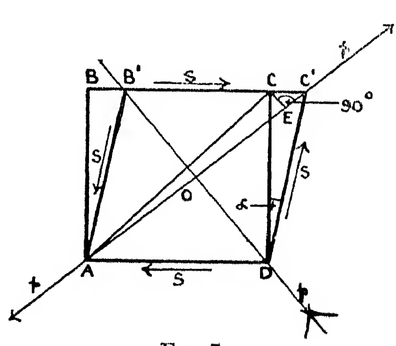


FIG. 7.

11. Shear Strain—Relation between Shear and Direct Modulus—In Fig. 7 AB′C′D is the view of the face ABCD of a

cube of elastic material which has been subjected to shear stresses. It was shown in the last article that the effect of these shear stresses was to induce direct stresses of equal intensity to the shear stresses across diagonal planes of the cube (these are denoted by p in the Figure), and it follows that if the shear stresses s acted alone the result would be the same as if the direct stresses p acted alone.

In the Figure the triangles CEC′ and ADO are similar, hence

$$\frac{CC'}{AD}=\frac{EC'}{AO}$$

but

$$\frac{CC'}{AD}=\frac{CC'}{CD}=\alpha=\text{shear strain}$$

and $\dfrac{EC'}{AO}=\dfrac{2EC'}{AC}=2e$ where e is the direct strain. $\therefore \alpha=2e$.

In the above analysis the distortion of the cube is assumed small.

Let $G=$ shear modulus, $E=$ direct modulus, and $\dfrac{1}{m}=$ Poisson's Ratio.

The shear strain $=\dfrac{s}{G}$.

The direct strain e consists of two parts, one caused by the tension in the direction AC′; this is equal to $\dfrac{p}{E}$. The other is caused by the compression in the direction DB′, which results in a shortening along the direction DB′ and an extension along the direction AC′. This extension is $\dfrac{p}{mE}$. The total direct strain is therefore given by $e=\dfrac{p}{E}+\dfrac{p}{mE}$, hence

$$\frac{s}{G}=\frac{2p}{E}\left(1+\frac{1}{m}\right),\text{ and since } s=p.$$

$$G=\frac{mE}{2(m+1)}\quad\text{or}\quad\frac{1}{m}=\frac{E}{2G}-1$$

$$\text{If }\frac{1}{m}=\frac{1}{4}\qquad G=\frac{2}{5}E.$$

The above equation gives a relation between E, G, and m, and it will appear that if any two of these three constants are known the other can be calculated. In particular Poisson's Ratio $\dfrac{1}{m}$ can be determined if the values of E and G are known.

12. Elasticity of Volume, Bulk Modulus—When a body

of elastic material is pressed upon all faces by a uniform pressure its volume is reduced and the amount of the reduction divided by the original volume is the volumetric strain. In this case, as in the cases of direct and shear strain, the strain is proportional to the stress or

$$K = \frac{\text{stress}}{\text{volumetric strain}}.$$

K is termed the bulk modulus.

In Fig. 8 is shown a cube of elastic material which is subjected to a compressive stress of intensity p on each of its six faces. Assume that the lengths of the side of the cube are unity and consider the decrease in height caused by the stresses p. This will consist of two parts, a shortening caused by the vertical stresses p which is given

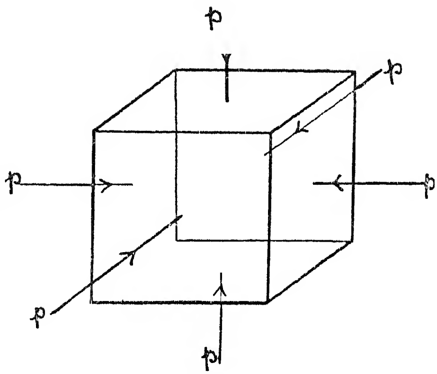


FIG. 8.

by $\frac{p}{E}$ and a lengthening caused by the horizontal stresses which is given by $\frac{2p}{mE}$.

The total decrease in height which is given by the algebraic sum of the above is

$$\frac{p}{E}\left(1 - \frac{2}{m}\right).$$

A similar shortening will occur in each of the two horizontal directions, so that the reduction in volume will be sensibly

$$\delta v = \frac{3p}{E}\left(1 - \frac{2}{m}\right).$$

A consideration of the expression will show that if m is less than two, δv will be negative or there will be an increase in volume, which is evidently impossible for compressive stresses. The value of m therefore must lie between 2 and infinity, or Poisson's Ratio $= \frac{1}{m}$ between 0 and $\frac{1}{2}$.

Now consider the expression

$$K = \frac{\text{stress}}{\text{volumetric strain}}.$$

The original volume is unity, hence

$$K = \frac{p}{\delta v} = \frac{p}{3\frac{p}{E}\left(1 - \frac{2}{m}\right)}$$

$$= \frac{mE}{3(m-2)} \quad \text{or} \quad \frac{1}{m} = \frac{1}{2} - \frac{E}{6K}$$

$$\text{If} \quad \frac{1}{m} = \frac{1}{4} \qquad K = \frac{2}{3}E.$$

13. Relations between the Constants E, G, $\frac{1}{m}$, and K—For a value of $\frac{1}{m} = \frac{1}{4}$ the following two ratios have been found:

$$G = \frac{2}{5}E \text{ and } K = \frac{2}{3}E.$$

These are deduced from

$$G = \frac{mE}{2(m+1)} \text{ and } K = \frac{mE}{3(m-2)}$$

which are also written

$$\frac{1}{m} = \frac{E}{2G} - 1 \text{ and } \frac{1}{m} = \frac{1}{2} - \frac{E}{6K}$$

These result in the general equation

$$E = \frac{9KG}{3K + G}$$

As an example of these relations it is well known that E for steel is never far removed from 30,000,000 lb. per sq. in., and also that G for steel is found to be somewhere near 12,000,000 lb. per sq. in., that is, $\frac{2}{5}$ of 30,000,000 lb. per sq. in.

14. Total Strain—In an elastic material, Fig. 9, let there be three stresses, p', p'', and p''', acting in directions mutually at right-angles, and call the fractional strains corresponding with these directions e', e'', e''' respectively. Let the modulus of elasticity be denoted by E and Poisson's Ratio by $\frac{1}{m}$.

When p' acts alone $e' = \frac{p'}{E}$, and in two directions at right-angles to the direction of stress there will be a contraction equal to $\frac{p'}{mE}$. Similarly when p'' acts alone there will be a strain $e'' = \frac{p''}{E}$, and at right-angles to this a lateral contraction $= \frac{p''}{mE}$.

Also p''' acting alone will cause the strain $e'''=\frac{p'''}{E}$ and a contraction in all directions normal to its own of $\frac{p'''}{mE}$. If there are three principal stresses, p', p'', and p''', acting at a point in an elastic material in directions mutually at right-angles, the total strain in each direction will be the algebraic sum of the strains produced by the three stresses acting independently. In the case of tensile stress the strain in any direction will be an extension minus two contractions. Thus the total strains in the three directions are :—

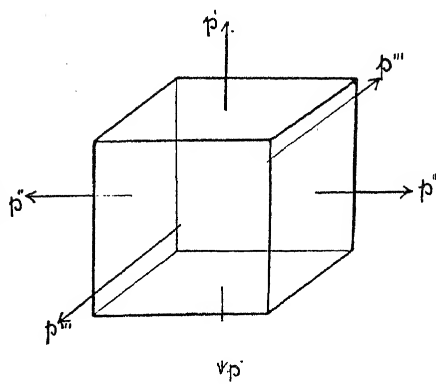


FIG. 9.

In the direction 1 total strain x'

$$=\frac{p'}{E}-\frac{p''}{mE}-\frac{p'''}{mE}=\frac{p'}{E}-\frac{p''+p'''}{mE}$$

In the direction 2 total strain y'

$$=\frac{p''}{E}-\frac{p'''}{mE}-\frac{p'}{mE}=\frac{p''}{E}-\frac{p'''+p'}{mE}$$

In the direction 3 total strain z'

$$=\frac{p'''}{E}-\frac{p'}{mE}-\frac{p''}{mE}=\frac{p'''}{E}-\frac{p'+p''}{mE}$$

15. Modulus when Lateral Strain is prevented—The direct modulus E is usually given by the direct stress divided by the strain produced by it, and no account is taken of the lateral strains which are assumed free to take place. If by the application of other principal stresses, in addition to p' the lateral contraction could be modified the modulus would have a different value. If, for example, such stresses p'' and p''' were applied as would be necessary wholly to prevent lateral contraction, the total strains y' and z' given in the last paragraph would be zero, and the total strains would become

$$x'=\frac{p'}{E}-\frac{p''+p'''}{mE}$$

$$0=\frac{p''}{E}-\frac{p'''+p'}{mE}$$

$$0=\frac{p'''}{\mathrm{E}}-\frac{p'+p'}{m\mathrm{E}}$$

which may be written

$$\mathrm{E}x'=p'-\frac{p''}{m}-\frac{p''}{m}$$

$$0=p''-\frac{p'''}{m}-\frac{p'}{m}$$

$$0=p'''-\frac{p'}{m}-\frac{p''}{m}$$

If p'' and p''' are so applied to control the lateral strain they will for reasons of symmetry be equal to one another. From the last two equations we have, taking $p''=p'''=p$,

$$0=p\left(1-\frac{1}{m}\right)-\frac{p'}{m}$$

$$p=\frac{p'}{(m-1)}$$

Again

$$\mathrm{E}x'=p'-\frac{p''}{m}-\frac{p'''}{m}$$

$$=p'-\frac{2p}{m}$$

$$=p'\left(1-\frac{2}{m(m-1)}\right)$$

$$=p'\left\{\frac{m^2-m-2}{m(m-1)}\right\}$$

or

$$x'=\frac{p'}{\mathrm{E}}\left\{\frac{(m-2)\ (m+1)}{m(m-1)}\right\}$$

Giving m its approximate value for metals of 4, the above becomes $x'=\frac{10}{12}\frac{p'}{\mathrm{E}}$. This means that by preventing lateral contraction the elongation under the given stress p' is reduced in the ratio 10 : 12. The modulus calculated from the extensions so limited by the shrinkage will be correspondingly increased in the ratio 12 : 10 above what it would have been with free extension and full contraction.

Stresses in Prisms—Some idea of the behaviour of prismatic bodies when under the action of tensile or compressive loads in simple cases has been given, and it is intended to extend this. It has been pointed out that the effect of longitudinal loading of a prismatic body is to develop stresses, both direct and tangen-

tial, on planes making angles of 45 deg. with the axis. This is a special case, and in what follows more general cases will be discussed.

16. Stresses normal and tangential to any Plane intersecting a loaded Prism—The diagram in Fig. 10 represents

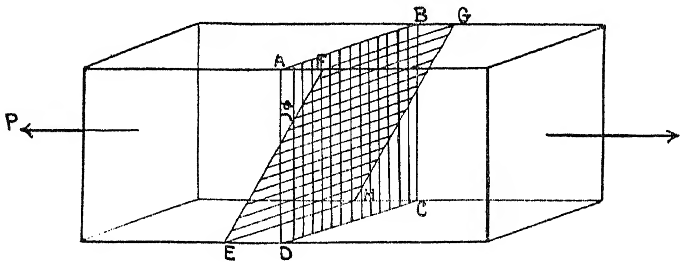


Fig. 10.

a bar or prism subjected to tension by a load P, which acts at right-angles to normal plane sections of the prism as ABCD. The normal stress on the surface is $\frac{P}{a}$, where a is the area of ABCD. Call the stress p. A second plane surface EFGH intersects the prism. The angle between ABCD and EFGH being θ, it is required to find the normal and tangential stresses on the second plane.

The force normal to the inclined surface is $P \cos \theta = pa \cos \theta$, and the tangential force is $pa \sin \theta$. Dividing these by the areas of the surfaces on which they act, the normal and tangential stresses on the inclined surface are obtained: call these p_d and p_s, then

$$p_d = \frac{pa \cos \theta}{a \sec \theta} = p \cos^2 \theta$$ and the tangential ~~strain~~

$$p_s \quad \frac{pa \sin \theta}{a \sec \theta} = p \cos \theta \sin \theta = \frac{p \sin 2\theta}{2.}$$

Of these p_s attains its greatest value when $2\theta = 1$, that is when $\theta = 45$ deg., so that the maximum value of p_s is $\frac{p}{2}$. The value of the normal stress corresponding with $\theta = 45$ deg. is $p_d = \frac{p}{2}$.

17. The Effect of two Direct Stresses at Right-angles to one another—This case is shown in Fig. 11, where there are direct stresses on two opposite faces of the solid ABCDEFGH :

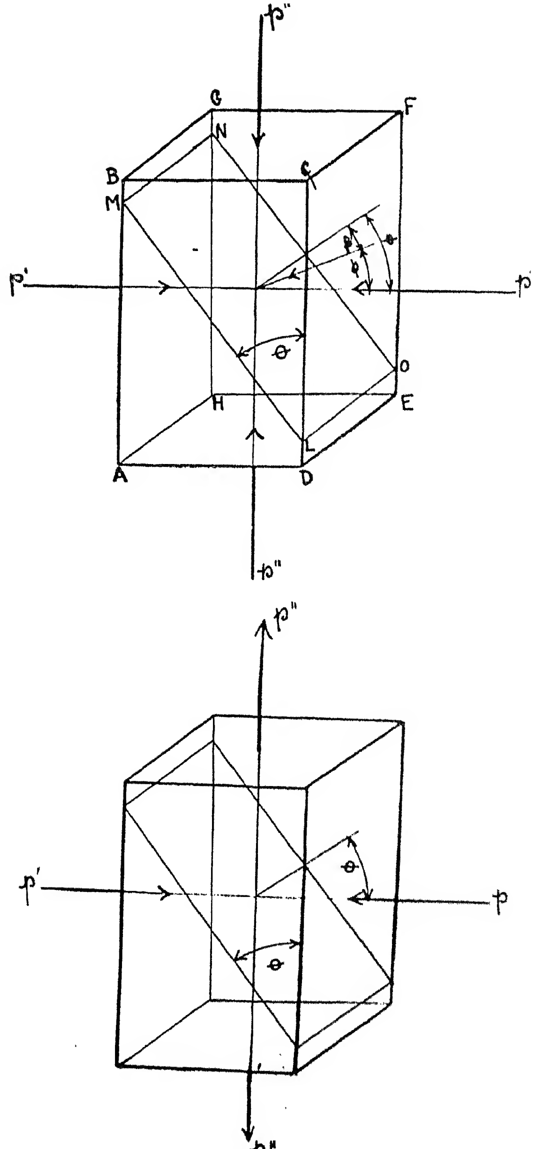


FIG. 11.

it is required to find the resulting normal and shear stresses on an inclined plane surface LMNO. The angle between the direction of the horizontal stress and the normal to this plane is θ. In this case the two given stresses p' and p'' are compressive. Resolving p' normal to LMNO the direct stress is p'_d.

$$p'_d = \frac{p' \cos \theta}{\sec \theta} = p' \cos^2 \theta$$

and resolving p'' normal

$$p''_d = \frac{p'' \sin \theta}{\operatorname{cosec} \theta} = p'' \sin^2 \theta.$$

The total normal stress on the planes will be the sum of these:

$$p_d = p_d' + p_d'' = p' \cos^2 \theta + p'' \sin^2 \theta.$$

Resolving along LMNO for the shear stress,

$$p'_s = \frac{p' \sin \theta}{\sec \theta} = p' \sin \theta \cos \theta$$

and similarly

$$p''_s = \frac{p'' \cos \theta}{\operatorname{cosec} \theta} = p'' \cos \theta \sin \theta.$$

From a consideration of the Figure it is seen that the shear stresses oppose one another, and hence the total shear stress is

$$p_s = p'_s - p''_s = (p' - p'') \sin \theta \cos \theta$$

$$= \frac{p' - p''}{2} \sin 2 \theta$$

This is a maximum as before when $\theta = 45$ deg. and the maximum shear stress becomes

$$\frac{p' - p''}{2}.$$

The direct stress when $\theta = 45$ deg. is $\dfrac{p' + p'}{2}$ In the case where one of the stresses, say p'', is tensile, the total resultant stresses are given by changing the sign of p'' in the above expressions. Thus

$$p_d = p'_d - p''_d = p' \cos^2 \theta - p'' \sin^2 \theta.$$

$$p_s = p'_s + p''_s = \frac{p' + p''}{2} \sin 2 \theta.$$

18. Resultant Stress on Plane LMNO—The total normal stress on the plane surface LMNO and the total shear stress have been found. Let the area of this surface be A; then the projected areas on the faces DEFC and BCFG are A cos θ and A sin θ respectively and the resultant of the total forces on

these areas must be equal to the total force on the area A. Let the resultant stress on the area A be q.

$$\text{Then} \quad qA = \sqrt{(p'A\ \cos\ \theta)^2 + (p''A\ \sin\ \theta)^2}$$
$$q = \sqrt{(p'\cos\theta)^2 + (p''\sin\theta)^2}$$
$$q = \sqrt{p_d^2 + p_s^2}$$

The component stresses of the stress q on the plane LMNO are $p'\cos\theta$ and $p''\sin\theta$ in the directions of p' and p'' respectively, and if ϕ is the angle which q makes with the direction of p',

$$\text{then}\ \tan\phi = \frac{p''\ \sin\ \theta}{p'\ \cos\ \theta} = \frac{p''}{p'}\tan\theta$$

Also if β is the angle which q makes with the normal to LMNO

$$\tan\beta = \frac{p_s}{p_d}.$$

19. Ellipse of Stress—To construct the ellipse of stress for the last case begin by describing about a point O the two circles BAPCD and EQF with radii respectively equal to the two stresses p' and p'' (Fig. 12). Draw LM through O at the proper

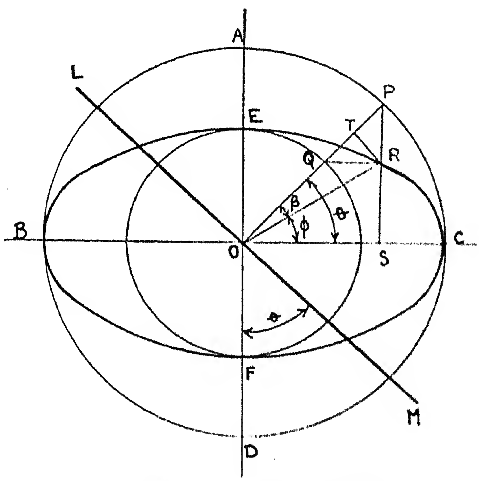


FIG. 12.—Ellipse of Stress.

angle θ and draw OP perpendicular to LM to meet the larger circle in P and the smaller one in Q. Now from Q and P draw horizontal and vertical lines to meet in R. Then OR represents the resultant stress on LM both in direction and magnitude. The point R evidently lies on an ellipse whose principal diameters

are equal to the diameters of the given circles. This is the "Ellipse of Stress" for the principal stresses p' and p''.

If the construction be made for various values of θ, then corresponding with each value there will be various values of the angle β between the resultant stress and the normal to the plane. This angle is evidently a maximum when the angle ORT is a right-angle where T is the mid-point of QP.

Since $\quad \mathrm{TR} = \mathrm{TP}$

$$= \frac{p' - p''}{2} \text{ and } \mathrm{OT} = \frac{p' + p''}{2}$$

$$\sin \beta = \frac{p' - p''}{p' + p''}$$

Also since QT = TR the angle TQR = the angle TRQ = θ and

$$\text{when TRO} = \frac{\pi}{2} \text{ the angle QRO} = \frac{\pi}{2} - \theta$$

and angle PQR = angle POR + angle ORQ

or $$\theta = \beta + \frac{\pi}{2} - \theta \quad \text{or} \quad \theta = \frac{\beta}{2} + \frac{\pi}{4}$$

and this value of θ corresponds with the expression for the maximum value of β.

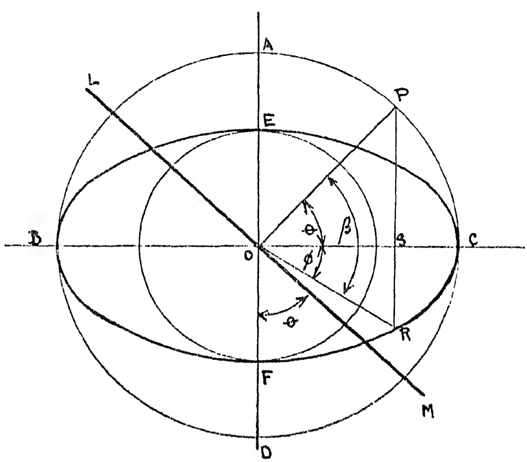


FIG. 13.—Ellipse of Stress.

In the second of the two cases above p' is positive and p'' negative. To represent this on the ellipse of stress (Fig. 13) draw as before, but draw the vertical from P until it cuts the

ellipse below the horizontal diameter; then the resultant stress on LM will be OR.

20. Principal Stresses and Ellipsoid of Stress—If in a stressed body a plane be found across which the stress is wholly normal, the plane is called a principal plane and the stress a principal stress. In the cases of combinations of stress which have been referred to above, the stresses given were principal stresses acting in given directions; and the problem was to find the stress across a given plane as an effect of the simultaneous action of the given stresses. Conversely if the plane and the magnitude and direction of the resultant stress across it are given, the ellipse of stress can be constructed and hence the principal planes and stresses determined. From this it follows that corresponding with every position of the plane and the resultant stress there will be two series of planes perpendicular to one another across which the stress is wholly normal. Also in the case of three-dimensional stress there are at any point in the body three mutually perpendicular planes across which the stress is wholly normal. These are the principal planes, and the stresses normal to them the principal stresses.

The equation to the ellipse in Fig. 13 is

$$\left(\frac{OR\cos\phi}{OC}\right)^2+\left(\frac{OR\sin\phi}{OE}\right)^2=1$$

or

$$\frac{l^2}{p'^2}+\frac{m^2}{p''^2}=\frac{1}{q^2}$$

where l and m are the direction cosines of q with respect to the direction of p' and p'' respectively.

Similarly the equation to the stress ellipsoid is

$$\frac{l^2}{p'^2}+\frac{m^2}{p''^2}+\frac{n^2}{p'''^2}=\frac{1}{q^2}$$

where p''' is the third principal stress perpendicular to p' and p'' and n is the direction cosine of q with respect to the direction of p'''. In this equation it is seen that p', p'', and p''' are the lengths of the semiaxes of the ellipsoid and q, the resultant stress, is equal to the length of the central radius vector in the direction l, m, n.

21. Ellipsoid of Strain—Imagine a small sphere centred in an elastic body. When the body is strained this sphere will become an ellipsoid whose principal planes will be the principal planes of strain, and the length of any radius vector of the strained sphere or ellipsoid will be the changed length of the corresponding radius vector in the unstrained sphere.

22. Principal Stresses—It is useful and important to be able to find from the principal stresses the magnitude and direc-

tion of the resultant stress across any plane whose inclination is known. What is often of greater importance is to find the magnitudes and directions of the principal stresses resulting from the simultaneous action of two or more known forces.

Well-known instances of such cases are: (*a*) Principal stress in a loaded beam; here there are known tensile and compressive stresses as well as shear stresses at a given point in the material of the beam, and knowing these it is possible to find the principal stresses resulting and the positions of the surfaces upon which they act. (*b*) Another well-known case is that of combined torsion and bending, where there is a shear stress from the torsion combined with the direct stresses due to bending. (*c*) A third case is that of a thin tube subjected at the same time to circumferential stress due to the pressure and longitudinal stress from tension, these two causing tensile stress on a surface making an angle with the axis which can be calculated.

23. Two Direct Stresses at Right-angles combined with a Shear Stress—Fig. 14 represents a rectangular slab of material,

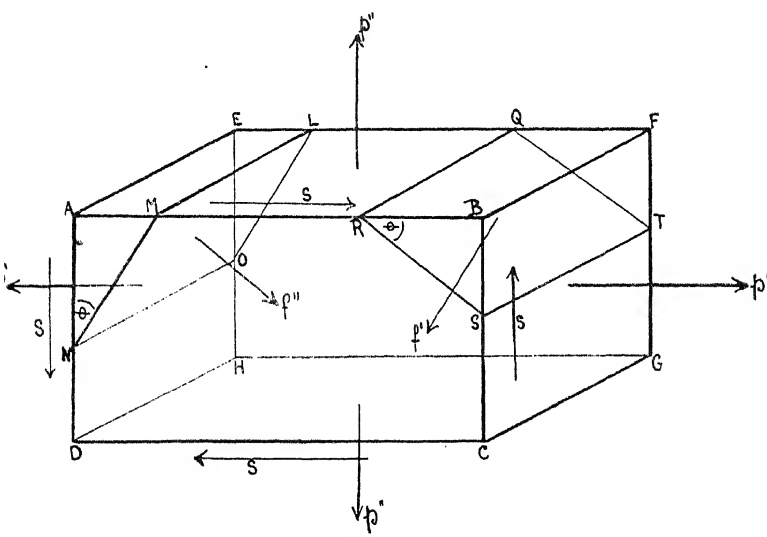


FIG. 14.

with a direct stress acting normally to its ends, a second direct stress normal to the top and bottom faces, and at the same time, shear stresses on the top, bottom, and end faces. The problem involved is to find the principal stresses in magnitude and direction and also the position of the principal planes. A case such as

this might occur when a thin tube is exposed to internal pressure and longitudinal tension and is at the same time resisting a torsion. In the figure the slab in question is ABCDEFGH. The given stresses are—a horizontal stress p', acting on BCGF and ADHE; a vertical stress p'' on AEFB and CDHG: and a pair of shear stresses on the end surfaces with complementary shear stresses on top and bottom of intensity s. Forming portions of these two solids are shown two wedge-shaped pieces AELMNO and BFQRST. In order to obtain the desired principal stresses the equilibrium of these wedges must be investigated. Each piece is supposed to be so small that there can be no variation in stress intensity at different points on its faces: also there is no stress on the front or back faces of the solid. For simplicity the slab thickness is taken as unity.

Consider the balance of forces on the wedge BFQRST, which—as BF is unity—may be considered as the triangle BRS.

The stress f' acting normally to RS gives a force $f' \times RS$. This may be resolved horizontally and vertically with the following results :—

$$f'RS \sin \theta = p'.BS + s.RB$$
$$f'RS \cos \theta = p''.RB + s.BS.$$

Divide these equations respectively by RS, which gives

$$f' = p' + s \cot \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$
$$f' = p'' + s \tan \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (ii)$$

These may be written

$$f' - p' = s \cot \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iii)$$
$$f' - p'' = s \tan \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iv)$$

Subtract (iii) from (iv), which gives

$$p' - p'' = s(\tan \theta - \cot \theta) \text{ or}$$

$$\tan 2\theta = \frac{p' - p''}{2s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (v)$$

Now multiply (iii) and (iv)

$$(f' - p')(f' - p'') = s^2$$

from which is obtained

$$f' = \frac{p' + p''}{2} \pm \sqrt{\left(\frac{p' + p''}{2}\right)^2 - p'p'' + s^2} \quad . \quad . \quad (vi)$$

Which can also be written

$$f' = \frac{p' + p''}{2} \pm \sqrt{\frac{1}{4}(p' - p'')^2 + s^2} \quad . \quad . \quad . \quad (vii)$$

The above equation (v) gives the angle θ made by the planes of principal stresses with the horizontal under the condition given.

There are evidently two of these planes whose angles of inclination differ by 90 deg.

The intensity of the principal stresses is given by (vi) or (vii), of which there are evidently two values, represented by f' and f'' in the Figure.

In any given case the values found for θ and f' depend on the conditions of the case as represented by the signs and values of the given stresses. Thus in the problem given one direct stress might be changed from tension to compression, or the shear stress might be non-existent or changed in direction. As there are two mutually prependicular planes on which the stress is wholly normal, so there will be two planes along which the stress is all shear, and a maximum value is best found from the proof already given (Art. 17), in which it was shown that two principal stresses at right-angles give rise to a pair of shear stresses also at right-angles and on planes inclined at 45 deg. to the principal planes. This shear stress q has a maximum value

$$
\begin{aligned}
q &= \frac{1}{2}(f' - f'') \\
&= \sqrt{\frac{(p' + p'')^2}{4} - p'p'' + s^2} \\
&= \sqrt{\frac{(p' - p'')^2}{4} + s^2}
\end{aligned}
$$

24. Elastic Properties of Timber—Most engineering materials can be considered as "isotropic," i.e., they have the same elastic properties in every direction, and the elastic properties of such bodies are completely defined if the values of two elastic constants are known. The two constants generally quoted to specify the elastic properties of such a body are Poisson's Ratio $\left(\frac{1}{m}\right)$ and Young's Modulus (E). Most of what has been given applies only to isotropic bodies, and it is now intended to give a brief account of the elastic properties of timber, which is an aelotropic material.

In Fig. 15 is shown a piece of timber with the direction of the grain in the direction oz. The radii of the annual layers are supposed great compared with the width of the piece, and the layers are considered as plane surfaces. In the Figure these surfaces are parallel to the planes oyz, and ox is normal to the plane oyz and sensibly coincides with the direction of the radii of the annual layers. From a consideration of the Figure it will appear that the structure of the piece in each of these three directions ox, oy, and oz is different, but that the structure in every direction parallel to any one of these directions is the same.

In other words, the piece can be considered as having three planes of symmetry of structure, *oyz*, *ozx*, and *oxy*, which are also planes of elastic symmetry. The number of elastic constants involved in the complete definition of the elasticity of the piece can be found by assuming it to be subjected to different types of stresses. First suppose that a direct stress is applied in the direction *oz*. This will cause a corresponding strain in the direction *oz*, and the relation between the stress and the strain will be the direct modulus of elasticity for the direction *oz*; call this Ez.

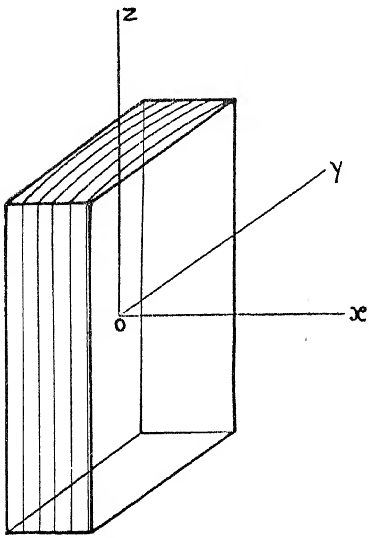


FIG. 15.

The stress will also cause lateral strains in the piece, and will involve two values of Poisson's Ratio, one corresponding with the direction *oy* and *oz* and the other with the directions *ox* and *oz*.

Poisson's Ratio corresponding with the direction *oy* and *oz*, is given by the relation

$$\frac{\text{lateral strain in direction } oy}{\text{longitudinal ,, \quad ,, \quad ,, } oz} = \sigma_{zy}.$$

Similarly corresponding with the other two directions it will be σ_{zx}.

Now suppose a stress to be applied in the direction *oy*, then by a similar argument it will involve three constants, i.e. E_y, σ_{yz} and σ_{yx}. Also corresponding with the direction *ox* the three constants involved will be E_x, σ_{xy}, σ_{xz}.

Next assume that shear stresses act in planes parallel to *ozx*. These will tend to cause slide in the direction *ox* in planes parallel to *oxy* and also in the direction *oz* in planes at right-angles to these, i.e., parallel to *oyz*. Call this shear modulus involved μ_{zx}. Corresponding with shear in the other two pairs of directions the shear modulus is in a similar manner denoted by μ_{xy} and μ_{yz}.

The total number of constants now specified are :—

Three values of the direct modulus : E_x, E_y, and E_z.

Six values of Poisson's Ratio : σ_{yz}, σ_{zy}, σ_{zx}, σ_{xz}, σ_{xy}, σ_{yx}.

and three values of the shear modulus : μ_{yz}, μ_{zx}, and μ_{xy}, or a total of twelve.

It is shown in books on elasticity that there are three symmetrical relations connecting the direct modulus with Poisson's Ratio; these are $\frac{\sigma_{zy}}{E_z} = \frac{\sigma_{yz}}{E_y}$, $\frac{\quad}{E_x} = \frac{\quad}{E_z}$ and $\frac{\sigma_{yx}}{E_y} = \frac{\sigma_{xy}}{E_x}$, so that if the three values of the shear modulus be known and six values of the direct modulus and Poisson's Ratio, the elastic properties of the piece are completely defined. The following table of values of the elastic constants for spruce obtained by one of the authors (H. Carrington)[1] is interesting:—

Density when tested, lb./cu. ft.	No. of Annual Rings per in.	Moisture per cent.	E_x 10^6lb./□″	E_y 10^4lb./□″	E_z 10^6lb./□″	μ_{yz} 10^4lb./□″	μ_{zx} 10^4lb./□″	μ_{xy} 10^3lb./□″
32·8	22·2	12·4	11·4	9·06	2·44	12·4	9·05	5·09

σ_{zy}	σ_{yz}	σ_{xz}	σ_{zx}	σ_{yx}	σ_{xy}
·520	·0233	·0182	·364	·335	·427

Each value is the mean of about twelve results, and they were all obtained from pieces cut from one balk whose density and humidity was sensibly constant throughout. It may be noted that variations in the value of any particular constant were found to be mainly dependent upon the density and humidity, the width of the annual rings having no appreciable effect.

EXAMPLES

(1) Distinguish between stress and strain. An iron bar 20 ft. long and 2 in. diameter is stretched $\frac{1}{20}$ of an inch by a load of 7 tons applied along the axis. Find the intensity of stress on a normal cross-section and the coefficient of elasticity of the material. (I.C.E.)

(2) Show that for isotropic elastic material Poisson's Ratio lies between 0 and $\frac{1}{2}$ and that the modulus of rigidity lies between one-third and two-thirds of Young's Modulus.

(3) A bar of iron is at the same time subjected to a direct tensile stress of 5,000 lb. per sq. in., and to a shear stress of 3,500 lb. per sq. in. What is the resultant equivalent tensile stress in the material? (S. and A.)

(4) A sufficiently rigid bar weighing 100 lb. is suspended in a horizontal position by three wires of the same length and all having the same diameter of $\frac{1}{10}$ in. The centre wire, fixed at the mid-point of the bar, is of copper with an elastic modulus of 17,000,000 lb. per sq.

[1] *Phil. Mag.*, June, 1921, May and July, 1922.

in., and the other two, placed symmetrically with respect to the centre wire, are of steel with an elastic modulus of 29,000,000 lb. per sq. in. Find the load borne by each wire.

(5) A bar of steel 4 in. × 2 in. in cross-section is subjected to a longitudinal tension of 40 tons. Find the normal and shear stresses on a section inclined at 30 deg. to the axis of the bar. (I.C.E.)

(6) A piece of steel shaft 4 in. diameter and 5 ft. long is subjected to a hydrostatic pressure of 2 tons per sq. in. Taking Young's Modulus and Poisson's Ratio as 30×10^6 lb. per sq. in. and $\frac{1}{4}$ respectively, find the decrease in volume of the shaft.

(7) A steel tyre $\frac{1}{2}$ in. thick is to be shrunk on to a solid steel disc 2 ft. diameter. Find the internal diameter to which the tyre should be machined if a tension of about 7 tons per sq. in. is to be obtained. Take Young's Modulus as 13,000 tons per sq. in. (Victoria.)

(8) A bar of steel 1 in. diameter and 10 ft. long is heated to 100° F. above atmospheric temperature, and is then firmly fastened at the ends. After it has cooled to atmospheric temperature again the end fastenings are found to be $\frac{1}{40}$ in. nearer together than they were when the bar was hot. Find the total pull exerted by the cold bar (coefficient of linear expansion for steel per 1° F. = 0·0000062. Modulus of Elasticity = 13,000 tons per sq. in.) (S. and A.)

(9) The longitudinal and circumferential stresses in a boiler plate are 2 and 4 tons per sq. in. respectively. Find the magnitude and direction of the maximum shear stress.

(10) Steel rails are welded together and are unstressed at a temperature of 60° F. They are prevented from buckling and cannot expand or contract. Find the stresses when the temperature is (1) 20° F., (2) 120° F., taking steel as expanding 0·0012 of its length for a temperature change of 180° F. E = 30×10^6 lb. per sq. in. If the elastic limit is 40,000 lb. per sq. in., at what temperature would it be reached? (I.C.E.)

CHAPTER II

EFFECTS OF DIRECT STRESS—STRESS-STRAIN DIAGRAMS

25. Brittle Materials in Tension—The brittle materials used by engineers include Cast Iron, Hardened Steel, and vitreous materials, such as Stone, Brick, and Portland Cement.

If a piece of any one of these be placed in a testing machine and subjected to a gradually increasing pull there will be a corresponding stretch, and both the amounts of the pull and the stretch can be measured. These are shown graphically in Fig. 16. Here the increasing loads are plotted vertically and the resulting strains horizontally. The line rises from O and slopes off very slightly towards B. This line corresponds with very brittle materials such as chilled cast iron, hardened steel, or hard glassy stone and brick, and it is seen that OB is almost perfectly straight. B represents the load and strain at fracture. It is difficult to find B except by using autographic means or by interpolation from the earlier points.

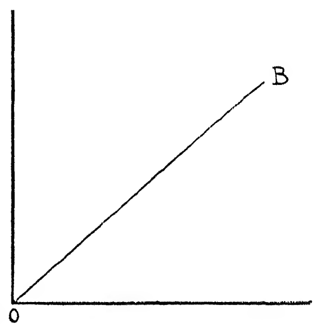


FIG. 16.

The more usual form of curve is shown in Fig 17, which is the kind of graph given by a material whose brittleness is nearly but not quite perfect. Thus, in the case of good Portland cement concrete, or unhardened high-carbon steel, the brittleness is only imperfect. The line remains straight as far as the point E, and afterwards curves somewhat towards the right, ending at B. If the load were to be removed when E had been reached the line would return to O along its former path. In such a case the material would be described as elastic within the range of load from zero to E, and no set would be visible after removal of the load. If, however, the load were to be released after E had been passed the line would not return to E but would follow a course roughly parallel to OE. With materials like cement this return line is nearly straight, but with iron and steel the

line curves backwards towards the left, thus showing partial recovery of the set.

The position of the elastic limit load is for engineering purposes fixed by placing a straight edge along the line from O and noting the position of the tangent point. The elastic limit may also be found by gradually increasing the load and removing it after every increment; in such a case the limit would be fixed as the maximum load up to which no set is observed after removal of the load. This process is tedious but effective.

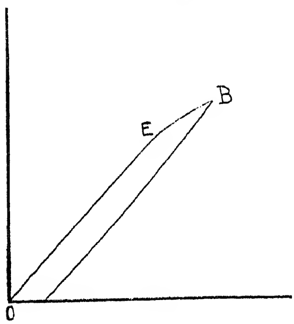


FIG. 17.

In Fig. 18 is shown the kind of graph obtained from a specimen of ordinary grey cast iron. The line is never quite straight, but is approximately so for some distance from O towards G. If at any point D the load is removed and the strain measurements taken, it will be found that the line returns to a point C, being slightly concave towards O, with CO as set. A second loading will show a line CG straighter than OD which will form a loop with DC. On a second removal of load the graph returns to F and the line formed by a third increase of load would make a loop with GF and the width of this loop would be less than the width of the first. If a series of such loops were formed the width of each loop would be less than the one immediately before it. The failure to return to the starting-point in any case indicates set, the curved return line corresponding with the removal of the load shows a partial recovery of the material after straining, and the fact that the loop gets narrower with each repetition confirms the suggestion that the recovery is only partial and that there is a gradually accumulating permanent set. Cast iron may be called semi-elastic, and takes small permanent set under low tensions. The straining of vitreous materials shows similar results. Thus the first engineering quality which may be determined is the elastic strain accompanying tensile stress; from this the modulus of Elasticity can be calculated in the manner already given.

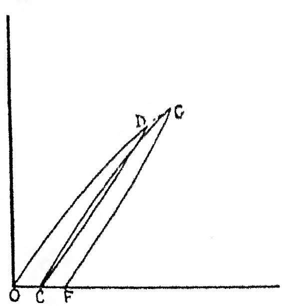


FIG. 18.

The second and the most generally recognized engineering quality is the Tensile Strength or the average tensile stress on the normal section of the piece, where fracture takes place. In brittle materials tensile fracture occurs on surfaces which are normal to the direction of the stress (Fig. 19). In the parallel

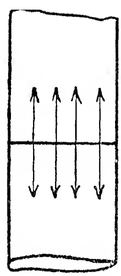


Fig. 19.

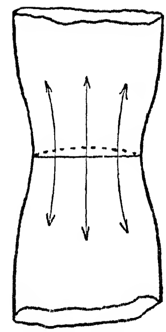


Fig. 20.

part of long round bars the fracture is straight across; where the parallel portion emerges from an enlarged end fracture often follows the surface suggested on Fig. 20.

The ultimate tensile strength

$$= \frac{\text{Breaking load}}{\text{Initial area of normal cross-section.}}$$

The area of the unstrained section is nearly always used in this expression. In the case of brittle materials the change in area under stress is as a rule inappreciable.

Elongation after fracture is also small in brittle materials, and, for this reason, besides being very difficult to measure, is not quoted in test reports.

When a brittle material is broken by tensile stress the meaning is that the molecular attraction between the particles on one side of the fracture and those on the other side has been broken down. It has been estimated that if the surfaces can be forcibly separated by as much as one-hundred-millionth of an inch the attraction fails and the piece is broken.

The following are average values of the chief engineering constants for a number of representative materials :—

ENGINEERING QUALITIES OF SOME OF THE BRITTLE MATERIALS

Material.	Modulus of Elasticity. Per sq. in.		Tensile Strength. Per sq. in.	
	Pounds.	Tons.	Pounds.	Tons.
Hard cast iron .	15,000,000	6,700	31,300	14
Medium cast iron.	14,000,000	6,250	22,400	10
Hard steel. . .	30,000,000	13,400	224,000	100
Medium steel . .	30,000,000	13,400	112,000	

26. Ductile Materials in Tension—The complete load-strain diagram drawn autographically for a bar of typical ductile material, when subjected to a pull, from zero load up to the point of fracture, is given on Fig. 21. The material here referred to is mild steel, with carbon content of about 0·2 per cent.

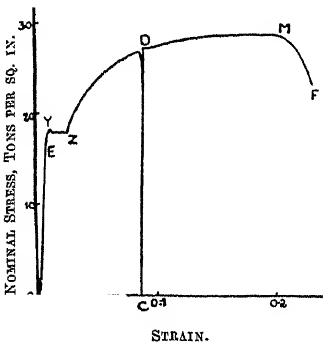


FIG. 21.

Loading commences at O, and from this point to E the diagram is a straight line, and during this period the material is elastic. Beyond the elastic limit, E, the line curves slightly towards the right, thus indicating a small amount of set, with a failure of truly elastic conditions.

The curvature persists to the point Y, when there is a rapid increase in the stretch. This goes on, with the diagram following a roughly horizontal step as far as Z. The stress at Y is the Yield Stress, sometimes spoken of as "the commercial elastic limit." The form of the yield can be seen more in detail by inspection of Fig. 22. In this diagram the portion of the curve from O to Z is drawn to an exaggerated strain scale. In some cases the waviness of the yield line of which YZ is a portion is more pronounced than that shown.

Up to the point E there is no plastic failure; from E to Y there is some local failure, limited to minute patches of the entire bulk of the material. This failure rapidly spreads to all parts of the piece until sudden and complete release of the tension takes place; this is shown by the rapid downward plunge taken by the curve when Y has been reached. After a short interval of time this general release of stress comes to an end, and the material takes hold of the load again and the diagram slopes

upwards, but does not again get up as far as the original Y. The diagram proceeds with diminishing fluctuations until it finally settles down to the horizontal line which points towards Z.

The curve in Fig. 22 is a small part of the complete graph in

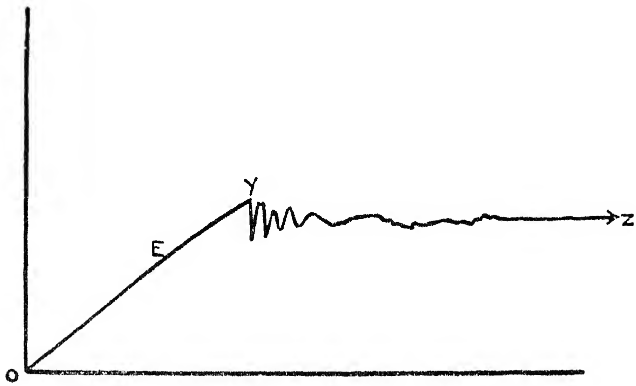


FIG. 22.

Fig. 21; and the wavy portion appears as an approximately horizontal step in Fig. 21.

Beyond Z the elongation has become almost wholly plastic, and is relatively very much greater than in the earlier elastic stretch and continually increases.

After the yield point has been passed plastic strain sets in, the piece stretches, and at the same time contracts laterally. During the elastic stage the stretch is due to a simple tension: in the plastic stage shear stress becomes predominant and the stretch is mainly the result of local shear taking place throughout all parts of the bar. Stretch proceeds steadily until fracture is approached, when a period of instability sets in and the piece begins to become thinner at one part, often the middle of its length. Immediately the thinning begins the load decreases. In Fig. 21 the part of the curve from M to F corresponds with the beginning of thinning to fracture which takes place by a shearing tear. With ductile material separation of the two parts of the piece takes place by shearing along surfaces inclined at 45 deg. with the axis. Two typical fractures are shown in Fig. 23. In (*a*) fracture is on a 45 deg. surface clean across the bar; this often occurs in certain kinds of flat steel bars. In round bars the form (*b*) is the most usual, consisting of a conical shear terminating in a rough, flat extreme end. In some flat bars there are four surfaces of shear forming a trough-like end to one portion of the bar and a short pointed end to the other portion. The

harder steels break square across. Often such steels as nickel-chromium steels have a square fracture across with a rim of 45 deg. surface round the edge.

27. Persistence of Elasticity—It must not be thought that when a ductile bar has been extended that the elastic condition has been departed from or that elasticity has failed. Up to fracture ductile steel remains perfectly elastic; plastic strain has changed the form and dimensions of the piece, but it continues as an elastic material. Fig. 21 shows this point very well. When the point D was reached—well on in the elongation the load was released to zero at C, and then reapplied. The effect is shown by the straight line DC.

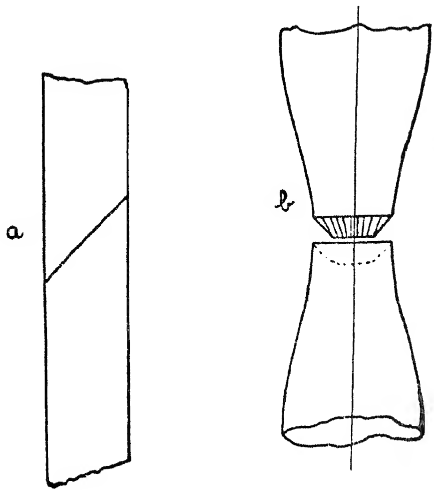


FIG. 23.

A second example of this persistence of elasticity is shown in Fig. 24. This diagram was taken autographically during the crushing test of a short length of nickel-chromium steel tubing. E is the elastic limit, Y the yield stress, M the maximum stress, and D the stress where the load was released, and the line DC is sensibly straight and parallel to EO. The meaning is that all steels remain elastic within elastic stresses, no matter to what mechanical treatment they may have been subjected. Thus suppose a test bar is cut from a slab of normal steel. When this is loaded in a testing machine within its elastic limit stress it will show an elastic stress under the load and will return to its original length di-

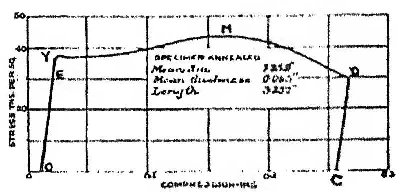


FIG. 24.

mension when the load is again removed. The original slab of steel may now be subjected to a very large amount of plastic strain and a second piece cut out. This second specimen will show perfect elasticity up to its elastic limit. Mechanical

elastic straining does not destroy the primitive elasticity of a steel. Thus in Fig. 21 the material is elastic up to E; on release of tension at any point D the elastic conditions prevail from D to zero load at C. At D the steel is in a high state of plasticity and yet springs back when the load is removed, to an extent represented by the line OC, and the same in Fig. 24.

28. Effect of Time—In neither the elastic stage of a test nor the plastic stage does the stretch take place instantly, but requires time. For engineering purposes the time effect on elastic strain may be neglected; the plastic strain is more markedly affected by the lapse of time and must be considered. Elastic time effect is exhibited in Fig. 25. Here a bar is loaded quickly to A and allowed to rest under this same load; further extension AB will take place during the period of rest. If the load had been applied very slowly in the first instance the load-strain condition would have been expressed by the line OB. Rapid release of load gives the straight line BC parallel to AO, and if the bar is again allowed to rest a small amount of the set DC will disappear. If the load had been released slowly the return line would have been represented by BD.

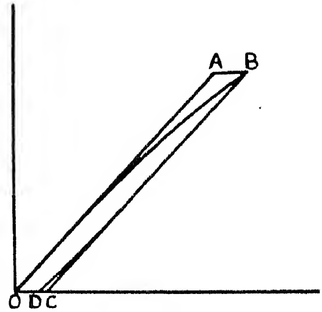


FIG. 25.

During plastic strain the time effect is more clearly evident. In Fig. 26 part of the load-strain curve during the plastic stage is represented by AD (see Fig. 21). AD shows the form the line would take with quickly applied loads and quick measurements, or when drawn with an autographic appliance with rapid loading. If the loading were slow instead of fast the line would become as in AE. It is seen that every point on AD moves to the right owing to such increases as BC and DE in the extensions. This effect is the more marked the nearer the ultimate load is approached. Experiments have been made in which length measurements were taken after each increment of load has remained on

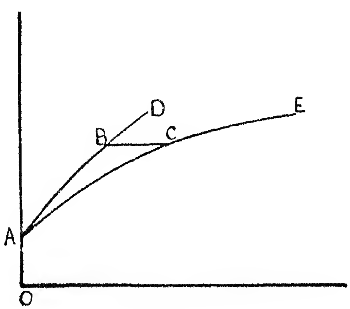
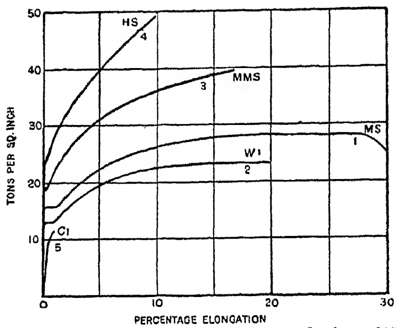


FIG. 26.

the specimen for so long as twenty-four hours. The curve from such a test may lie still further to the right of AE. For engineering purposes there is a practical limit to the time effect, and the experimenter soon finds out where these limits are.



Here CI refers to cast iron; WI, wrought iron; MS, mild steel; MMS, medium steel; HS, high-carbon steel.

FIG. 27.

29. Typical Effects.—In Fig. 27 are shown a number of typical load-strains in diagrams.

(1) is for soft structural steel with well-marked yield and great total elongation.

(2) is that of good wrought iron, with fairly evident yield, but less total extension than the last.

(3) is a medium carbon steel with faintly marked yield and smaller elongation.

(4) is from a very hard steel with no yield and little elongation.

(5) shows the kind of curve obtained from a bar of grey cast iron; here there is no yield. The permanent stretch is small, as is also the breaking load. The yield beyond the elastic limit is characteristic of the softer ductile irons and steels.

Fig. 28 shows the successive shapes taken by a bar of ductile steel during a tensile test to destruction.

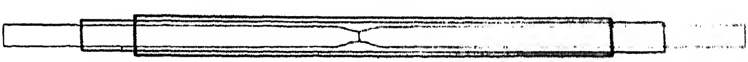


FIG. 28.

All plastic failure begins with shear taking place on planes inclined to normal planes. Thus in Fig. 29 the two main vertical lines represent a portion of the bar under tensile stress. XO is a normal section on which acts the principal tensile stress caused by the pull P. Call this principal tensile stress p. It was shown in Art. 16 that the effect of the pull was to induce stresses along and normal to planes as XY inclined at any angle θ to normal planes. Let the stresses along and normal to XY be p_s and p_d respectively. Then from Art. 16 $p_s = \frac{p \sin 2\theta}{2}$, $p_d = p \cos^2\theta$, and

p_s is a maximum when $\theta = 45$ deg. and is equal to $\frac{p}{2}$, p_d is a maximum when $\theta = 0$, that is when XY is normal to the axis of the bar and is then equal to p. Hence the direct tensile stress p induces shear stresses p_s and normal stresses p_d whose maximum values are equal respectively to $\frac{p}{2}$ and p. In ductile materials the power of the material to resist the induced shear stresses is less than one-half its power to resist the induced direct stresses, and hence elongation and fracture occur mainly by shear.

30. Eccentric Loading—In Fig. 30 is shown part of a tension

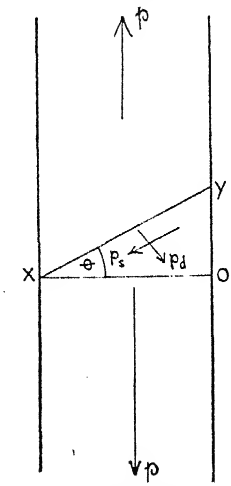


FIG. 29.

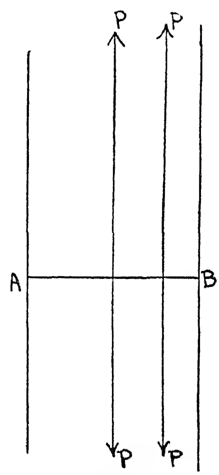


FIG. 30.

bar under a load P acting along a load line which is nearer B than A, where AB represents a normal section. If P acts along the geometric axis of the bar the stress intensity will have equal values at A and B. With the load line in some such position as the one shown the tensile stress will be variable, with a maximum value at B and a minimum at A. That is, the effect of placing the load line nearer one side of the bar is to increase the stress intensity on that side, and the result is local stress greater than the average stress got by dividing the load by the cross-sectional area. This eccentricity of loading is to be avoided as likely to give unreliable results, especially with brittle materials. With ductile materials the effect is the same when finding the elastic limit yield load, but the plastic distortion preceding the maximum

stress serves to level up the inequalities of stress and bring the load line into the geometric axis.

31. Rate of Loading—The rate at which the load is applied must have some effect on the ultimate strength, in both brittle and ductile materials. As a rule in brittle material quick loading is likely to give higher results than slower loading, but this is not invariably the case. In ductile materials both the tensile stress and the elongation are affected. Plastic strain requires time for completion, and the longer the time spent over a test, the greater the elongation shown. For a given material, the ultimate strength will be greater the greater the final area of section, and as the reduction of area is dependent on the lapse of time, it follows that quick tests should give higher results as regards the ultimate strength, with less elongation.

32. Load and Stress—In engineering test results elastic-limit stresses, yield-point stresses, and maximum or ultimate stresses are quoted as so many pounds or tons on the original cross-sectional area, and not on the area at the moment when the particular stress occurs. This is a matter of pure convention, and is found to be the most convenient way from all points of view.

Determination of Elastic Limit—In any material the elastic limit may be found by applying the load by equal increments and measuring the corresponding increments of strain by means of an extensometer. An examination of the tabulated strains will show that the increments are sensibly constant up to a certain point, when they begin to increase. The stress at which this increase begins is the elastic limit of the material. A better way is to plot the loads against the strains, and the elastic limit is given by the tangent point of the elastic line and the curve. As the elastic limit is the stress at which the proportionality of stress and strain ceases to hold, it is sometimes also called the "Limit of Proportionality."

Fixing the Yield-point Stress—In most of the softer steels and also generally in the case of wrought iron, the yield point shows itself in an unmistakable manner. The testing machine beam which is supporting the increasing load on the specimen suddenly falls when the yield point is reached and tension in the bar is released for a moment. Or, a length may be scribed out along the bar, and one point of a pair of dividers held in position at one end of the length and the other point placed lightly on the other end and watched while the load is being applied. When yield begins the scribed mark will leave the divider point rapidly and to a very definite extent—often as much as $\frac{1}{4}$ in. in 10 in.—before the beam is again raised and the addition of more load becomes necessary to cause further elongation and fracture. This is

perhaps the better way. In some mild steels, high-carbon steels, and most non-ferrous metals the yield point is absent. In these cases the autographic records would be smooth curves and the machine beam would remain steady until fracture occurred. When the yield point is absent or very small the best plan appears to be to use the equivalent of a pair of dividers, generally a bent piece of steel with hardened points at a specified distance apart. A favourite distance is 2 in., but it depends on the specification. Previous to loading, a centre dot is placed on the specimen and an arc set out by the compass (Fig. 31). The load is increased

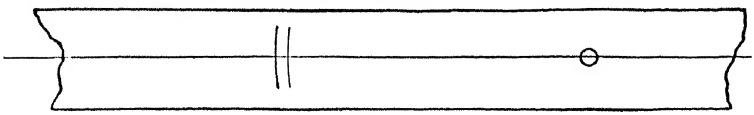


FIG. 31.

until the specified yield point is reached and then released and a second arc struck from the same point. If the two arcs coincide there is no set and has been no yield, and the test can proceed to the fracture. If any yield is shown the material may be rejected without going further. This mode of procedure is very definite and leaves no room for doubt. Some specifications define the yield point as so much elongation on a stated length. An instance of this is the Institution of Civil Engineers' definition, which states that the yield point is to be considered as having been reached when the proportional elongation is $\frac{1}{200}$.

The Maximum Stress—When the elastic limit and yield have been passed the load goes on increasing until the maximum load is reached. This does not necessarily mean the maximum real stress, unless the local contraction is negligible and the fracture square across. The real stress at any load is the load divided by the area corresponding with it, but the usual convention is that of obtaining stresses by dividing loads by the original cross-sectional area. In order to draw the true stress-elongation curve it would be possible to measure the cross-dimensions of the bar after each increment of load and divide the loads by the corresponding areas. This is tedious and not done unless for some special reason. If the real curve is desired the best plan is first to get the load-elongation curve (Fig. 32) and from it construct the stress-elongation curve by using a graphical method which assumes constancy of volume throughout.

In Fig. 32 OPBMG is a stress-elongation curve of the usual type for ductile steel, the stress ordinate at any point being the load divided by the initial area of the bar. OA is set off equal to L—

the original length of the bar involved. From any point B on the curve draw the straight line BD normal to OY. Now draw a straight line from A through D to cut the vertical through B at

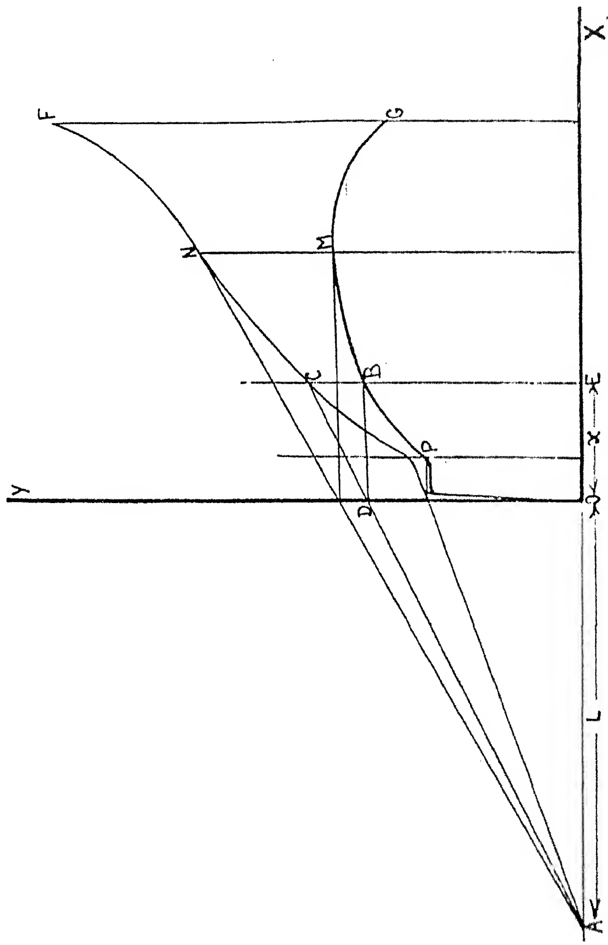


Fig. 32.

C. Then if EB represents the load divided by the initial area or nominal stress, EC represents the real stress. Call the initial area A, the nominal stress EB = p, and the area and extension corresponding with this stress a and x respectively. Assuming constancy of area

$$(L + x)a = LA \text{ or } \frac{L + x}{L} = \frac{A}{a}$$

In the triangles AEC and AOD $\frac{EC}{OD} = \frac{AE}{AO}$ or

$EC = EB\left(\frac{AE}{AO}\right) = p\left(\frac{L + x}{L}\right) = \frac{pA}{a} = \frac{\text{load}}{\text{corresponding area}} = \text{real}$ stress. This construction will apply to any point on the nominal stress-strain curve up to M, the maximum load, because the bar remains parallel over this range. From M to fracture at G the construction fails because local contraction is taking place and a waist forming under new conditions. The point F can be found by measuring the area of the fracture when the bar is broken and completing the curve by drawing a smooth curve from N to F.

33. Criteria of Ductility—Just as strength properties are given in terms of Yield Stress, Ultimate Stress, and possibly Elastic Limit Stress, so the ductility is expressed in terms of elongation or contraction after fracture.

To determine the elongation two centre dots or scratches are marked on the bar a known distance L apart. When the bar is broken the fractured portions are placed together and the distance between the points measured. Denoting the increase in the distance by l, the proportional elongation is $\frac{l}{L}$. This is generally given as a percentage. The value of L is largely a matter of convenience, a favourite dimension for constructional steel being 8 in. A better value where the length of the bar permits is 10 in. Many other lengths are used, one largely used for high-carbon steels and copper alloys being 2 in. It is a good plan to mark out the bar in separate inches from end to end, so that if fracture occurs outside the middle length it is still possible to get a measurement. When there are heads on the bar the marked length should be well clear of these.

The elongation l can be considered as made up of two distinct parts. One part, say l', is present when the maximum load is reached just before the formation of the waist. This is evidently proportional to L, the original unstressed length, or $l' = AL$, where A is a constant. When the bar is broken there is a further extension l'' due to the formation of the waist which is constant and independent of L. The total elongation is therefore given by $l = AL + l''$. On test reports the elongation is sometimes given for two lengths in order to make it possible to determine the constants in the above equation, and hence to calculate the elongation for some other length.

When it is desired to compare the elongations for bars with different cross-sectional dimensions there should be geometrical

similarity between the cross-sectional dimensions and the length L over which the elongation is measured. In the case of round bars the ratio $\frac{\quad}{\text{diameter}}$ should be the same for each bar.

Elongation is the most commonly used method of measuring ductility both general and local. It is a simple and easily made measurement.

When using contraction as a criterion of the ductility the original cross-dimensions are found, and also the cross-dimensions at fracture. If the original area be A and the area at fracture a, the proportional contraction is $\frac{A-a}{A}$, and is generally given as a percentage. In this connexion it is useful to remember that for a round bar the above may be written

$$\frac{\frac{\pi}{4}D^2 - \frac{\pi}{4}d^2}{\frac{\pi}{4}D^2} = 1 - \left(\frac{d}{D}\right)^2$$

Where D and d are the initial and final diameters respectively.

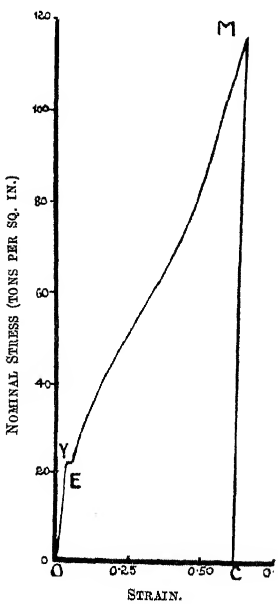


FIG. 33.

34. The Effects of Crushing Stress—The effects are not unlike those for tension. Elastic shortening takes place until an elastic limit is reached, and materials with a yield point in tension have also a yield in compression. After the yield the load continues to increase; there is no fracture in the tensile sense. In brittle materials crushing takes the form of splitting into parallel prisms, or shearing along inclined planes, or mere crumbling. With ductile materials the yield is followed by shortening, which in some cases ends in a shearing fracture. In Fig. 33 is shown a stress-proportional-contraction curve drawn autographically for a piece of mild steel in compression. E is the elastic limit, Y the yield stress, and M the point where the load was removed. The return line MC is sensibly parallel to OE, and hence as in tension the material is almost perfectly elastic when considerably deformed. What

happens to a piece of ductile material in compression is shown in Fig. 34. As the piece shortens under the load it becomes barrel-shaped, and if the load is sufficiently great, longitudinal or sometimes inclined cracks appear on the surface. The effective area increases and the nominal stress at any load is greater than the actual stress. This is the reverse to what happens in tension, where the area decreases under the load and the nominal stress is less than the real stress.

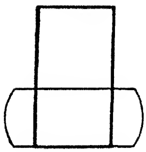
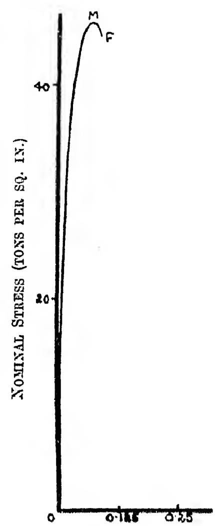


FIG. 34.

In Fig. 35 is shown an autographic record for a piece of cast iron in compression. The curve bends over gradually towards the right until the point M, corresponding with the maximum load, is reached. A sudden general yielding then takes place and the curve falls slightly to F, where fracture occurs. Fracture generally takes the form of a shear across a plane the normal to which is inclined at about 55 deg. with the axis of the piece: this is shown in Fig. 36. If the length of the piece is less than the vertical distance between A and B, a clear shear across such planes as AB is prevented and the piece breaks up into a number of small pyramids. The load required to break such a short piece is generally greater than if the piece were longer.



STRAIN.

FIG. 35.

Denoting the angle between the normal to the plane and the axis of the piece by θ, it was shown in Art. 16 that the shear stress is given by $p_s = p \sin\theta \cos\theta$, where p is the direct stress. This is a maximum when $\theta = 45$ deg. and p_s is then equal to $\frac{p}{2}$. It was also shown that the stress normal to the plane is $p_d = p \cos^2\theta$. In the case of cast iron it would appear that the normal pressure on planes whose normals are inclined at 45 deg. to the axis of the piece caused friction forces along them such that the resistance to shear was greater than across planes whose normals are inclined at 55 deg. Let q be the maximum resistance of the material to shear stress. Then the resistance to shear across a plane whose normal is inclined at an angle θ with the axis of the piece is $q + \mu p \cos^2\theta = p \sin\theta \cos\theta$ where μ is the coefficient of friction,

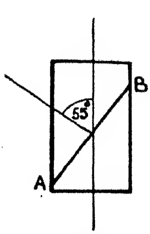


FIG. 36.

$$\therefore p = \frac{q}{\sin\theta\cos\theta - \mu\cos^2\theta}$$

If shear stress takes place across a plane whose normal is inclined at an angle θ which involves a minimum value of the direct stress p, the denominator in the above expresion will be a maximum, hence differentiating

$$\cos^2\theta - \sin^2\theta + 2\mu\sin\theta\cos\theta = 0$$

$$\cos 2\theta + \mu\sin 2\theta = 0$$

$$\cot 2\theta = -\mu = -\tan\phi = \cot\left(\frac{\pi}{2} + \phi\right)$$

where ϕ is the angle of friction.

Hence $\theta = \frac{\pi}{4} + \frac{\phi}{2}$, and since $\theta =$ about 55 deg. the value of ϕ is approximately 20 deg.

35. Timber—In the case of timber under direct stress along the grain the general form of the stress-strain curve is as shown in Fig. 17. Elastic strain persists up to an elastic limit, after which the curve bends over to the right; there is no yield point. For tensile stress the elastic limit approaches to the breaking stress, and failure is generally accompanied by longitudinal splitting. If the piece is properly shaped and gripped so that a pure tensile break is obtained the tensile stress is found to be very high, over 11 tons per sq. in. having been obtained by the authors for spruce.

The maximum strength in compression is less than the tensile strength, and the elastic limit is further removed from the maximum stress than is the case for tension. Failure generally occurs in compression by shear along a plane inclined to normal planes, and the position of the plane is usually such that the lines resulting from its intersection with normal planes are in the direction of the radii to annual rings. The stress-strain curves corresponding with direct stress in directions inclined to the grain are also similar to that shown in Fig. 17. For both tensile and compressive stress the elastic limit and maximum stress decrease as the direction of the stress approaches to directions normal to the grain, where the elastic limit is very low and incipient creeping is evidenced at comparatively small strains. If a specimen is left under a stress a little higher than the elastic limit the creeping will decrease asymtotically to zero. If the stress is increased the period during which creeping occurs also increases until a stress is reached under which creeping will continue for hours and even days until finally fracture occurs. For tensile stress normal to the direction of the grain fracture generally occurs by a clean break across a normal section. In the case of

compressive stress, shortening and lateral bulging ensue and failure is caused by the piece splitting into a number of pieces. The values of the elastic limit and maximum stress corresponding with every direction vary significantly with the humidity and density of the timber. The values decrease as the humidity increases until the fibre-saturation point is reached, after which the addition of more moisture only serves to fill up the cavities in the wood and the values remain practically constant. There appears to be no simple connexion between the values and densities of different species of timber, but for a particular species the values increase with the density.

EXAMPLES. II

(1) From the autographic diagram in Fig. 21 find (1) the Yield Stress, (2) the Ultimate Stress or nominal Max. Stress, (3) the Percentage Elongation excluding fracture, (4) the work done per cubic in. of material up to the maximum load. Why is it impossible from the diagram and the data given to calculate the elongation on a length including fracture ?

(2) A bar of steel 1·380 in. wide and 0·380 in. thick yields at a load of 10·3 tons and breaks after a maximum load of 15·1 tons has been reached. The increase in an 8 in. length after fracture is 2·10 in. Find the Yield Stress, Ultimate Stress, and Percentage Elongation on 8 in.

(3) A bar of steel $\frac{7}{8}$ in. diameter has an elongation on 3 in. of 36 per cent. Another bar $\frac{1}{2}$ in. diameter has an elongation on 8 in. of 28 per cent. and on 2 in. of 34 per cent. Compare the elongations of the two bars.

(4) A bar of ductile material is loaded at the rate of 1 ton per minute, and withstands a maximum load of 17 tons. A similar bar of the same material is loaded at the rate of 50 tons per minute. How would the maximum load and elongation for the latter test compare with those of the former ? Would you expect any difference between the values of the maximum stresses obtained by dividing the actual breaking loads by the areas of the fractures ?

(5) A number of compression pieces $\frac{7}{8}$ in. diameter are cut from a piece of good cast iron. Some of the pieces are $1\frac{1}{4}$ in. long, and an equal number $\frac{3}{4}$ in. long. Why would you expect the mean of the results on the longer pieces to be lower than that on the shorter ? Which of the two series of results would you expect to be the most consistent ?

(6) Two similar tensile test pieces of the same steel are $\frac{7}{8}$ in. diameter and 9 in. between the shoulders. One breaks in the middle and has an elongation between the shoulders of 25 per cent. The other breaks 2 in. from one of the shoulders and has an elongation of 23·5 per cent. If the rates of loading are the same in the two tests, give the reason for the difference in the elongations.

(7) Explain why at ordinary rates of loading a ductile tensile test

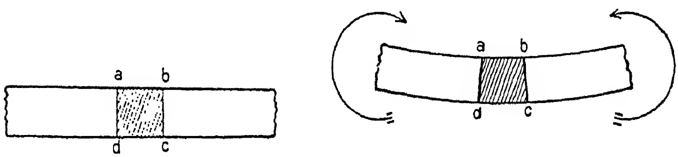
piece can break in only one place at once. Why is it much more likely that a brittle test piece may break in two or more places at the same time?

(8) A bar of high tensile steel 1·784 in. diameter is placed in a testing machine and the extension is measured with Marten's Extensometer for increments of load of 2 tons. From a graph of the points it is found that the extension is 0·00665 in. for a load of 36 tons. If the extension is measured on 15·0 cm., find the value of Young's Modulus.

CHAPTER III

TRANSVERSE LOADS—BEAMS

36. In all structural members coming in the above categories, the external loads act obliquely or at right-angles to the length of the pieces. The principal effect is to produce moments which cause bending and thus develop direct stresses in the material. In addition to the stresses induced by the moments, there are generally simultaneous shear stresses induced, and a complete analysis of the resultant stresses in the members involves the study of both systems of stresses. The general effect of a moment on a member is seen in Fig. 37. Here (*a*) is



(*a*) (*b*)

Fig. 37.

part of the long piece of material which forms the beam and upon which there is no force acting; the member therefore retains its originally straight form. In the square *abcd*, *ad* and *bc* are perpendicular to the length of the piece. On the application of a moment the piece is forced to take the curved form shown at *b*. The effect of this is to shorten *ab* and lengthen *cd*. This means that the concave face of the beam is put into a state of compression and the convex face in tension. Somewhere between *ab* and *cd* is a sensibly plane surface along which the stress is zero. This is the neutral surface whose intersection with a normal plane section of the piece results in a slightly curved line which is usually considered as straight, called the neutral axis of the section. The above represents a case of pure bending where there is no shear to be considered. In the more general cases of transverse loading shear has to be reckoned with; this is seen in Fig. 38, *a*, *b*, *c*, *d*. At (*a*) is shown a length of beam built into a wall so as to project horizontally outwards and supporting a load P near to its free end. A short length of the beam between the load and the fixed end is cut completely

away and replaced by the links AB, CD, and AC. If the link AB is severed there will be no tension exerted between A and B and the right-hand half of the beam will rotate about the point C as shown at *b*. Similarly if the link DC is removed and AC and AB are intact, there can be no compressive force between C and D and the beam will rotate about the point A as indicated at (*c*). At (*d*) is shown what happens when the link AC is severed. The right-hand portion of the beam will fall with respect to the left and the frame ABCD will depart from its original rectangular shape and become a parallelogram.

This indicates what tends to take place in most beams and girders, whether in the form of open frames or solid. The material near the top and bottom faces resists the bending action as shown at (*b*) and (*c*), and that in the body of the beam resists

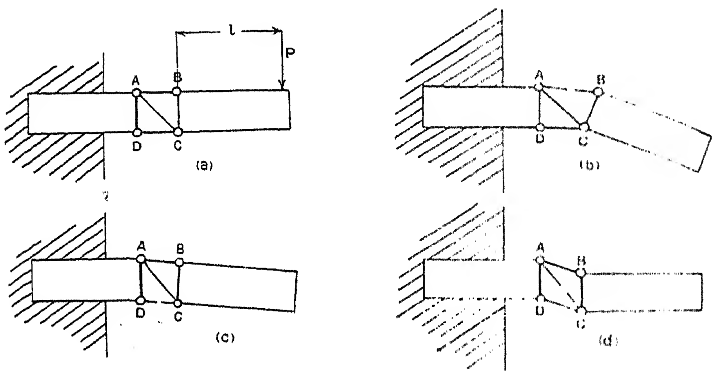


FIG. 38.

the sliding action caused by a force or forces acting parallel to the section in question. In the case described (Fig. 38) actual failure might occur by the tearing apart of AB, buckling of CD, and the tearing of AC or buckling of DB. This should give an elementary notion of what happens in actual beams and should also indicate the kind of stresses which have to be provided for in the economic distribution of the material. In Fig. 37 it is seen that the curvature which takes place shortens *ab* and lengthens *cd*. As the material is assumed elastic, this shortening and lengthening means stress in the material of numerical value proportional to the strain. The same is true in the case of Fig. 38. The tension in AB and compression in CD indicate corresponding lengthening and shortening of the respective bars, which allow the beam to become bent. Strain of the shear bar AC or BD allows AD and BC to remain parallel and causes the

portion of the beam to the right of BC to move bodily with respect to the remaining portion. In Fig. 38 (*a*) the force tending to cause that part of the beam to the right of the normal section BC to move bodily with respect to the left is P. This is the shearing force across the section and is evidently the same for any normal section between P and the fixed end. Also the force P has a bending effect at the section BC given by P*l*; this is evidently proportional to *l*, the distance of the section from P. More generally the effect of the forces acting at a normal section of a loaded beam can be separated into two distinct

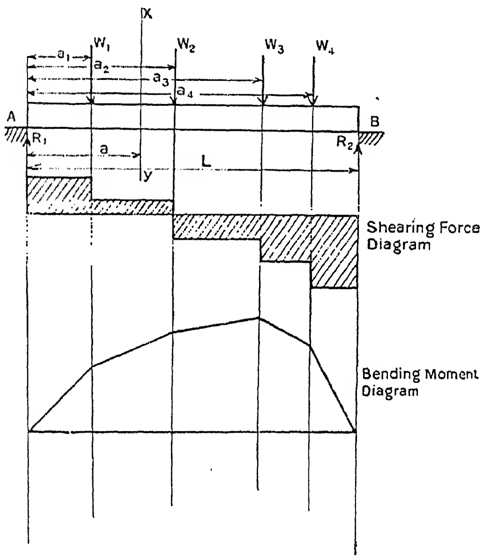


FIG. 39.

types, one having a shearing effect at the section and the other a bending effect.

37. Shearing Forces and Bending Moments—The following definitions are important :—

(A) The shearing force acting at any section of a loaded beam is the algebraic sum of the components normal to the beam of all the forces on one side of the section.

(B) The bending moment acting on any section of a beam is the algebraic sum of the moments of all the forces acting on one side the section. Bearing these statements in mind, it becomes easy to obtain the above values for any given case.

The instance in Fig. 39 is a simple one. Here is a beam

resting in a horizontal position across a span L and carrying a number of vertically applied concentrated loads, W_1, W_2, W_3, and W_4, acting at distances from the left-hand support of a_1, a_2, a_3, and a_4 respectively. The effect of these loads will be to cause pressures at A and B upon the supports which may be considered as reactions R_1 and R_2 of the supports upon the beam. To find these take moments about A. The moment of all the forces about A is $W_1a_1 + W_2a_2 + W_3a_3 + W_4a_4$. This is a clockwise moment and is balanced by the anti-clockwise moment R_2L.

Hence $R_2 = \dfrac{W_1a_1 + W_2a_2 + W_3a_3 + W_4a_4}{L}$. R_1 may be found from the fact that $R_1 + R_2$ is equal to the total load on the beam. Where the forces to be dealt with are many, it is a good plan to find R_1 and R_2 independently and so get an absolute check on the work.

Having fixed the reactions, it becomes easy to calculate the shearing force and the bending moment at any section such as XY. Thus:

Shearing Force at XY $= R_1 - W_1$.

Bending Moment at XY $= R_1 \times a - W_1(a - a_1)$.

38. Diagrams of Shearing Forces and Bending Moments—If the values of the shearing forces and bending

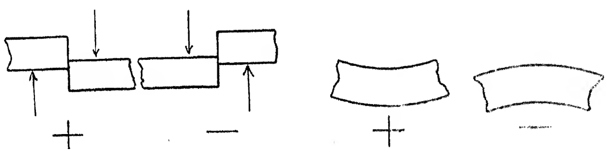


FIG. 40.

moments as found in the above case be plotted, the result will be the figures shown on the lower part of Fig. 39, called the diagrams of shearing forces and bending moments. In doing this it is well to plot plus values upwards and minus values downwards. The choice of the plus and minus sign is immaterial, but it is best to adopt some definite convention. Such a scheme is shown in Fig. 40, and it is seen that when the shearing force at a section tends to make the portion of the beam to the left of the section move upwards with respect to the right, the shearing force is positive. When the tendency to motion of the portions with respect to one another is the reverse of this, the shearing force at the section is negative. Also the bending moment is called positive when it tends to bend the beam concave upwards and negative when it tends to bend it concave downwards.

Uniformly Distributed Loading.—The loads dealt with so far

have been concentrated at points on the beam and have acted perpendicular to its length. If the direction of the forces is inclined to the length of the beam as shown in Fig. 41, the normal

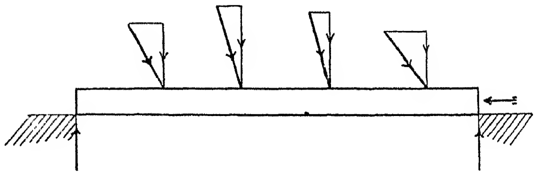


FIG. 41.

components should be found and used as shown above in the determination of the shearing forces and bending moments. When the forces are so inclined the stresses at any section of the beam will be caused by three systems of forces: (1) the shearing force at the section, (2) the bending moment at the section, and (3) the resultant horizontal component of the inclined forces at the section.

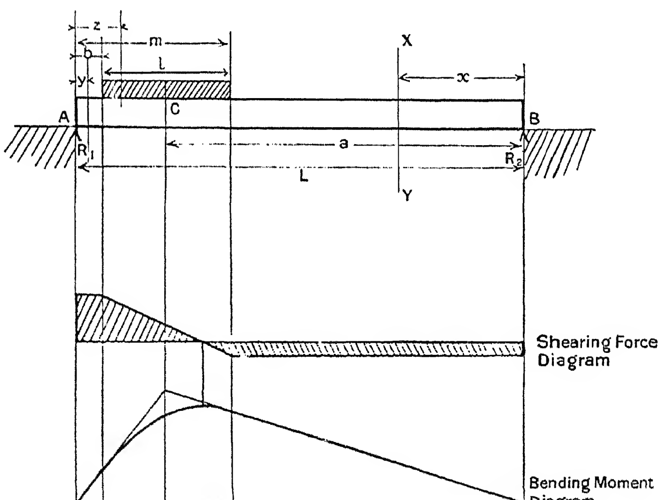


FIG. 42.

A case where the load is not concentrated but distributed along a portion of the beam is shown in Fig. 42. Here is a horizontal beam AB supported at its ends. It carries a uniform load of w pounds (or some other unit) on each unit of length

covering a length l, with one end of the load at a distance b from A. To obtain the reaction R_1, take moments about B. The weight of the uniform load will act through its middle point C. Its moment about B is wla, which is balanced by the moment of R_1 about $B = R_1L$. From this is obtained $R_1 = \frac{wla}{L}$. Also the other reaction R_2 is equal to the difference between wl and R_1, or, taking moments about A, it is given as

$$R_2 = \frac{wl(L-a)}{L}$$

The bending moment at any section as XY between B and the right-hand end of the load and distant x from B is given by

$$R_2x = M_x$$

This is the equation of the bending-moment diagram for values of x between $x=0$ and $x=a-\frac{l}{2}$ and is evidently a straight line.

Consider next the bending moment at any section of the beam between A and the left-hand end of the load. Let the distance of such a section from A be y, the bending moment at this section is $M_y = R_1y$, which is also linear and is true for all values of y between $y=0$ and $y=b$. These lines form part of the bending-moment diagram and are shown in Fig. 42. To obtain the expression for the bending moment for the portion of the beam covered by the load, consider a section distant z from A. The bending moment at the section is given by

$$M_z = R_1z - \frac{w(z-b)(z-b)}{2}$$

$$= R_1z - \frac{w}{2}(z-b)^2$$

This is the equation of a parabola and forms the remainder of the bending-moment diagram, which is shown drawn complete in Fig. 42. The shearing-force diagram is obtained in a similar manner. The shearing force at any section distant y from A for values of y between $y=0$ and $y=b$ is R_1; at any section of the beam which is covered by the load and distant z from A it is $R_1 - w(z-b)$, and at any section distant x from B for values of x between $x=0$ and $x=a-\frac{l}{2}$ it is R_2. These are all straight lines, and when drawn form the shearing-force diagram shown in Fig. 42.

In Fig. 43 is shown a beam of length L carrying a uniform load of w units per unit of length over its entire length. The

bending moment at any section XY distant x from A is $M_x = R_1x - \frac{wx^2}{2}$, or in terms of R_2 it is $M_x = R_2(L-x) - \frac{w(L-x)^2}{2}$. Since $R_1 = R_2 = \frac{WL}{2}$, which is independent of x, the curve is a parabola and the lengths of the ordinates corresponding with five values of x have the following values: When $x=0$ and $x=L$, $M_x = 0$, at one quarter and three-quarters of the span, where x is $\frac{1}{4}L$ and $\frac{3}{4}L$ respectively M_x becomes $\frac{Wl}{8} \times \frac{3}{4}L = \frac{3}{32}wL^2$. Its value in the middle of the span comes out at $M_x = \frac{wL^2}{8}$. The

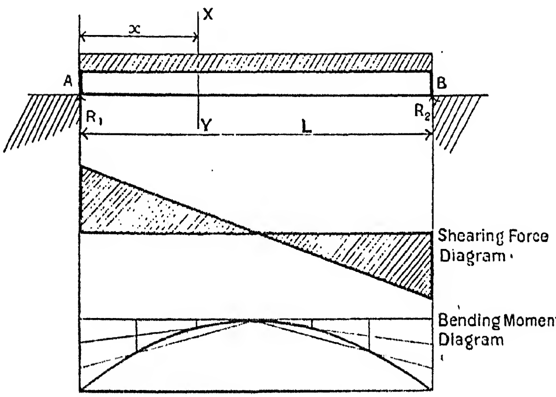


FIG. 43.

parabola is shown drawn in Fig. 43, and the fine lines indicate a method of constructing it by which only the value of the maximum ordinate is necessary.

The shearing force at the section XY is $F_x = R_1 - wx$ or $F_x = w(L-x) - R_2$. This is evidently a straight line; when $x=0$, $F_x = R_1$, and when $x=L$, $F_x = -R_2$. The shearing-force diagram is also shown drawn in Fig. 43.

The case of a cantilever carrying a uniform load of w units per unit length is shown in Fig. 44. The bending moment at any section XY distant x from the fixed end is $M_x = \frac{w(L-x)^2}{2}$. This is a parabola whose maximum ordinate occurs when $x = 0$ and is given by $\frac{wL^2}{2}$ and is shown drawn in Fig. 44, the fine lines

indicating a geometrical method of construction. The shearing force at the section XY is $F_x = w(L - x)$, which is evidently a straight line and is a maximum when $x = 0$. F_x is then equal to wL.

In some cases where it is desirable to draw the shearing-force and bending-moment diagrams for a beam carrying a considerable number of loads, it is often easier to draw the diagrams for each load separately or to split the loading into sections and draw the diagrams for each section. The shearing-force and bending-moment diagrams for the complete loading are then given by the algebraic sum of the diagrams corresponding with

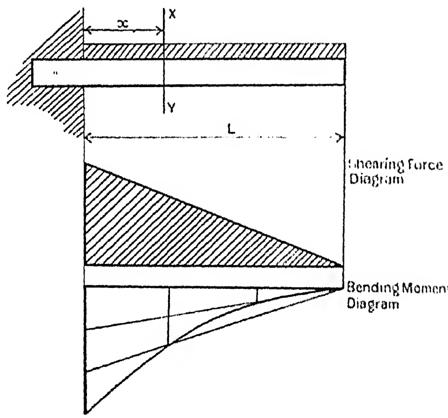


FIG. 44.

each load or section of loads. This method is demonstrated in the following example: A beam ABCD (Fig. 45) 40 ft. long is supported horizontally at the end A and at a point C distant 30 ft. from A and carries a uniformly distributed load of 1 ton per ft. length. An upward vertical force of 10 tons is applied to the beam at B distant 20 ft. from A. It is required to draw the shearing-force and bending-moment diagrams. Consider first the bending-moment diagram, assuming the upward force at B is removed. Taking moments about A to find R_2'—the reaction at C corresponding with the distributed load:

$$R_2' \times 30 = 1 \times 40 \times 20.$$

$$R_2' = \frac{80}{3} = 26\tfrac{2}{3} \text{ tons}$$

$$\text{and } R_1' = 40 - 26\tfrac{2}{3} = 13\tfrac{1}{3} \text{ tons.}$$

Hence the bending moment at any section XY between A and B and distant x from A is

$$M_x = R_1'x - \frac{1 \times x^2}{2} = x\left(13\tfrac{1}{8} - \frac{x}{2}\right)$$

This is the equation of a parabola whose ordinate vanishes when $x = 0$ and when $x = 26\frac{2}{3}$. Its value when $x = 30$ is -50

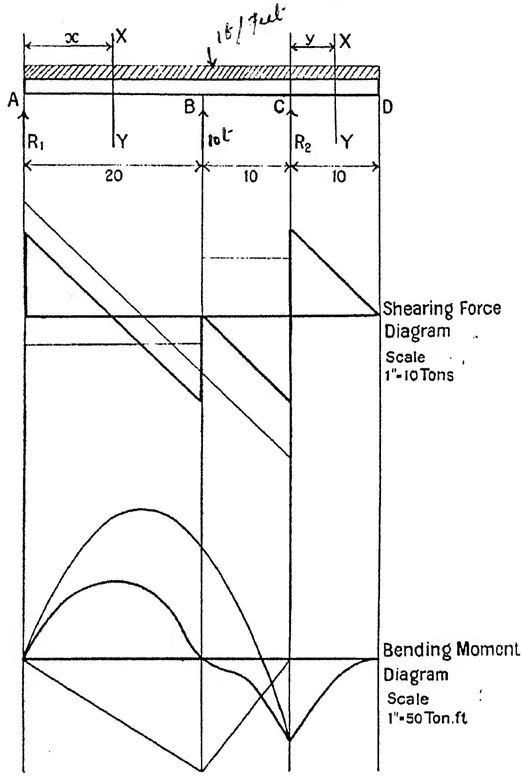


FIG. 45.

ton-ft. The bending moment at any section XY distant y to the right of C is $M_x = -\frac{(10 - y)^2 \times 1}{2}$. This is also a parabola whose ordinate vanishes when $y = 10$, i.e., at the end of the beam. Its value when $y = 0$ is -50 ton-ft. These two parabolas form the bending-moment diagram for the continuous loading, and are shown drawn in Fig. 45.

Now suppose that the continuous load is removed and the upward force at B applied. The new reactions R_1'' and R_2'' will now act downwards. To find the reaction R_2'' take moments about A

$$10 \times 20 = R_2'' \times 30 \qquad R_2'' = 6\tfrac{2}{3} \text{ tons}$$

$$\therefore \ R_1'' = 10 - 6\tfrac{2}{3} = 3\tfrac{1}{3} \text{ tons.}$$

The bending moment at any section XY between A and B and distant x from A is $M_x = -R_1''x = -3\frac{1}{3}x$. This is the equation of a straight line, and the value of M_x is 0 when $x = 0$; when $x = 20$ $M_x = -66\frac{2}{3}$. By a similar method the equation of the part of the bending-moment diagram between B and C is found to be a straight line. The diagram is shown drawn in Fig. 45, and also the algebraic sum of the diagrams, which is the bending-moment diagram for the combined loading.

To determine the shearing-force diagrams first, assume that the force at B is removed. The shearing force at any section XY between A and C and distant x from A is $R_1' - x \times 1$, which vanishes when $R_1' = x = 13\frac{1}{3}$ ft. Its value when $x = 0$ is $R_1 = 13\frac{1}{3}$ tons and when $x = 30$ it is $13\frac{1}{3} - 30 = -16\frac{2}{3}$ tons. The shearing force at any section XY distant y from C is $(10 - y) \times 1$. This vanishes at the end of the beam, where $y = 10$, and its value is 10 tons when $y = 0$. These straight lines form the shearing-force diagram for the continuous load, as shown in Fig. 45. Now suppose that the continuous load is removed and the upward force applied. The shearing force at any section between A and B is constant and equal to $-R_1'' = -3\frac{1}{3}$ and at any section between B and C it is also constant and equal to $R_2'' = 6\frac{2}{3}$. These lines are shown drawn in the figure together with the sum of the two diagrams, which is the complete shearing-force diagram of the two systems of loads.

39. Vector and Link Polygon—There is a graphical method for drawing the bending-moment diagram which is especially convenient when a number of loads have to be dealt with. This will now be considered. In Fig. 46 is shown a beam carrying a number of concentrated loads W_1, W_2, W_3, W_4, and W_5. The spaces between the forces on the beam, which include both loads and reactions, are lettered so that any force can be identified by the letters in the spaces on either side of it. This method of identifying the forces is known as Bow's notation. In the diagram to the right of the beam the vertical line *af* represents to scale the total load on the beam, and this is divided into segments such that the length of each segment represents to scale the corresponding load on the beam. Thus the length *ab* represents the load AB. The pole O is chosen, a convenient known

distance from af, and approximately opposite its mid-length, and straight lines are drawn from the end of each segment to O. The resulting diagram is called the vector polygon. Next the lines of action of the loads and reactions on the beam are produced, and starting from any point q on the line of action of R_1, the line qs is drawn across the space A and parallel to Oa. Similarly, the line sk is drawn across the space B parallel to Ob, and so on for the other lines. Finally, the closing line qz is drawn and the resulting figure is known as the link polygon. If now the line Og in the vector polygon is drawn parallel to qz, then the lengths of fg and ga represent to scale the amounts of the reactions R_2 and R_1 respectively. This can be proved by assuming that the

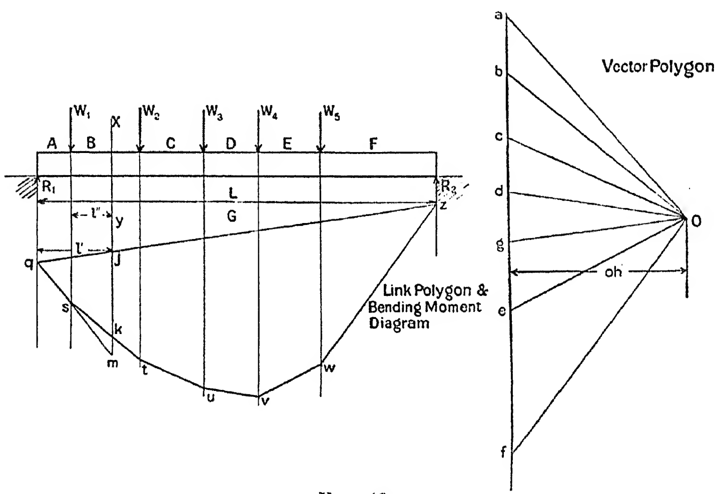


FIG. 46.

loads are suspended on a rope whose ends are fixed at the points q and z. Then if the length of the rope is suitably adjusted its shape will be similar to that of the link polygon. This will be evident when it is noted that each set of three forces acting at the points s, t, u, v, w are in equilibrium, for each set is represented by a triangle of forces which forms part of the vector polygon.

The three forces acting at s are represented by the triangle Oab, and thus the tension in the portion sq of the rope is represented by Oa, and for equilibrium at q there must be another force acting in addition to R_1. Similarly, the tension in the portion of the rope wz is represented by Of in the vector polygon and for equilibrium at z there must be another force acting in

addition to R_2. Also for equilibrium these forces at q and z must be equal in magnitude and act in the same straight line, that is, along the line qz.

Since Og is parallel to zq the triangle Oga is the triangle of forces for the forces acting at q, and hence ga represents the reaction R_1. Similarly, Ofg is the triangle of forces for the point z, and hence fg represents R_2.

The link polygon is also the bending-moment diagram corresponding with the loads on the beam, for its vertical depth at any point such as jk represents to a certain scale the bending moment at the corresponding section XY of the beam. If in the vector polygon the load scale is 1 in. = w tons and the scale of spacing out the load is 1 in. = s ins. and the polar distance $oh = q$ in., then the scale of the bending-moment diagram is 1 in. = wsq ton-in.

The proof of this can be effected by considering the similar triangles in the link and vector polygons.

The triangles qjm and Oga are similar.

Hence $\dfrac{jm}{ga} = \dfrac{l'}{oh}$ and since $ga = R_1$, $jm = \dfrac{l'R_1}{oh}$

also the triangles skm and Oba are similar and therefore

$$\frac{km}{ab} = \frac{l''}{oh} \text{ and since } ab = W_1,\ km = \frac{l''W_1}{oh}$$

$$\text{Hence } jm - km = jk = \frac{l'R_1 - l''W_1}{oh}$$

or $l'R_1 - l''W_1 = jk \times oh =$ bending moment at the section XY.

A frequent case that occurs in practice is where the beam carries a continuous load due to its own weight and the permanent material upon it, as well as a number of isolated loads. The above method of drawing the bending-moment diagram can be applied to continuous loading by dividing the load into sections and taking the weight of each section as an isolated load acting through the centre of gravity of the section. The bending-moment diagram thus obtained can be converted into that for continuous loading by drawing a smooth curve tangent to the straight lines which formed the bending-moment diagram for the isolated loads.

40. Moments of Resistance.—The bending moment and shearing force at any section of a beam whose dimensions and manner of loading are known can be found in the manner described, and from this point the problem may be carried further.

The bending moment and shearing force on a beam section arising from the external forces induce forces in the material of

the beam which must balance the external forces. The two sets of forces so induced are best obtained separately and the resultant of the forces acting at a section can then be found, if necessary, by superposing the two sets. The forces induced by the bending moment are generally of greater importance than those induced by shear and in some cases that occur in practice the shearing effect can be neglected in comparison. In what follows the stresses induced by bending will be considered; the stresses induced by shear will be treated later on.

The simplest case is that of a girder or a framed structure having two flanges joined by a web. In this case it is generally assumed that the material in the flanges resists the bending moment and the material in the web the shearing effect. Referring to Fig. 38, the bars AB and CD are equivalent to the flanges. Let F be the pull or thrust in the flanges, H the perpendicular distance between them, and M the bending moment on a normal section of the beam which intersects the flanges. Then for equilibrium $M = F \times H$. Hence the force in the flanges of a girder or similar framed structure can be found corresponding with any section, if the depth and bending moment at the section are given.

It was shown in Art. 36 and Fig. 37 that the stresses along the beam varied from a max. compression on the upper concave surface to a max. tensile on the lower convex surface. In the case of the girder just considered it was assumed that the stress was constant across the flanges and that no stress was induced in the web by the bending moment. Such assumption cannot be made for solid beams and the treatment becomes more involved.

In the case of the general theory of bending the assumptions made are as follows:—

(1) The material of the beam remains elastic throughout the loading.

This is a matter of choice of material; most of the materials used in engineering are to all intents and purposes perfectly elastic under the customary working conditions.

(2) The elastic properties of the material of the beam are the same in tension as in compression.

Experiment has shown that there is little or no difference in the elastic properties of materials whether under tension or compression.

(3) Longitudinal filaments of the beam are free to expand or contract laterally. This is sensibly true except when the section of the beam is very broad compared with its depth.

(4) A normal section of the beam which is plane previous to bending remains plane during and after the application of the

bending moment. This assumption, which is due to Bernoulli, will be considered later on in Art. 42 and in the section on shear in beams.

In Fig. 47(a) is shown a short length of an unloaded beam. A′B′C′D′ is a rectangle, A′B′ and C′D′ being the planes by which the beam is cut; E′F′ is the neutral surface which is plane when the beam is unloaded. Loading changes the shape to that at (b). Here A′B′ is lengthened to AB and C′D′ is shortened to CD; the neutral surface retains its original length but becomes slightly

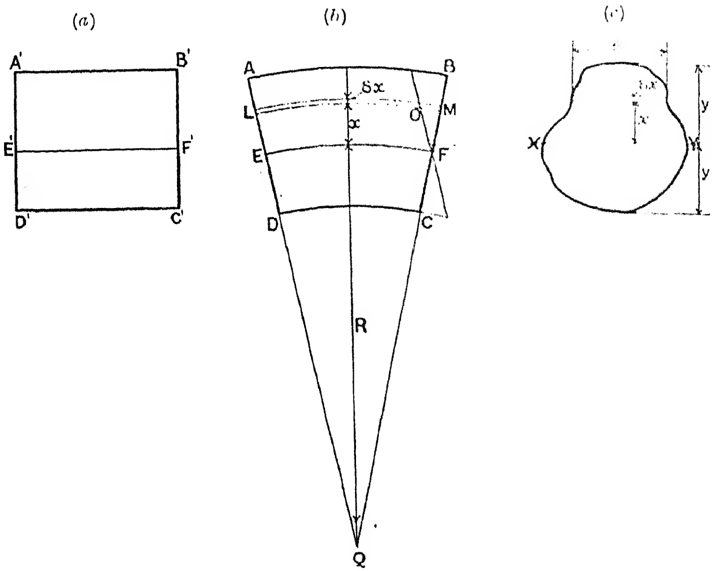


Fig. 47.

curved. Under the loaded conditions the planes AD and BC, originally parallel, become inclined and meet in a line whose end view is the point Q. Consider any thin layer of material LM with a thickness δx and mean distance x from the neutral surface. A line drawn through F parallel to AD will exhibit the strain of AB (stretch), of LM (stretch), and of DC (shortening). As the surfaces AD and BC are plane before and after bending, it is evident that the strain, whether stretch or shortening, is proportional to the distance of the layer from the neutral surface. The strain is evidently a maximum on the surface AB, which is furthest from the neutral surface, so that (Strain of material at LM) : (Strain of material at AB) :: $x : y$, where y is the distance

of the extreme surface AB from the neutral surface. Since the material is elastic the stress is proportional to the strain, from which it follows that if p is the stress at LM and f the stress at AB, then $p : f :: x : y$ or $p = \frac{xf}{y}$. This gives the stress intensity at any distance x from the neutral surface.

At (c) is shown a normal section of the beam viewed normal to itself, that is, in the direction along the beam. The width of the strip LM is denoted by z, and hence its area is $z\delta x$, so that the total force acting upon it is $pz\delta x$. By dividing the area of the beam into a number of such strips and adding together the force in each strip the total force acting in the direction of the length of the beam will be obtained. Let F denote this total force; then $F = \Sigma\, pz\delta x$ where Σ is the sign of summation. Since $p = \frac{xf}{y}$ then, using the sign of integration, $F = \frac{f}{y}\int_{y^1}^{y} zx dx$. Since there is no external force acting along the length of the beam, F must be equal to zero, and hence $\int_{y^1}^{y} zx dx = 0$, which means that the neutral axis must pass through the centre of gravity of the section. The neutral axis is denoted by the line XY in (c).

The total force on the strip is $pz\delta x$, and the moment of this force about the neutral axis is $(pz\delta x)x$, which constitutes the resisting moment of the strip. By dividing the area of the beam into a number of such strips and adding together their resisting moments, the total resisting moment of the section will be obtained. This must equal the bending moment M at the section, so that it is now possible to write $M = \Sigma\, pzxdx$. Since $p = \frac{xf}{y}$ and, using the integration sign, $M = \frac{f}{y}\int_{y^1}^{y} zx^2 dx$.

In this expression for the moment the integral portion is known as the moment of inertia of the section and is generally denoted by the symbol I. The expression thus becomes $M = \frac{fI}{y}$. The term $\frac{I}{y}$ is often denoted by the symbol Z and called the modulus of the section, and the equation becomes $M = fZ$. It will be shown later that, when using a certain graphical method for finding the moment of inertia of unsymmetrical sections, it is easier to find Z direct instead of I and get I by multiplying Z by y. The expressions Moment of Inertia and Modulus of Section are often puzzling to students, chiefly because they cannot be readily visualized. It must be remembered that they are simply names

for certain mathematical expressions which often occur in this and other subjects (see Art. 44).

In the expression $M = \frac{fI}{y}$, f is the stress in all fibres of the beam distant y from the neutral surface. In beams of most frequent shape of cross-section the neutral surface is equidistant from the top and bottom surface. This is so with circles, squares, rectangles, and all figures symmetrical about a horizontal axis. When this is not the case and the maximum fibre stress is required, care should be taken to insert the greatest value of y in the above expression.

41. Radius of Curvature—Referring again to Fig. 47(b) it will be noted that the figures FOM and QEF are similar, hence $\frac{OM}{EF} = \frac{MF}{FQ}$. The length of the portion of the beam considered is assumed small so that EF can be considered as bent into a circular arc of radius R = FQ. The above expression can therefore be written

$\frac{OM}{LO} = \frac{x}{R}$. Here $\frac{OM}{LO}$ is the strain in the strip LM and the corresponding stress is p, hence

$\frac{OM}{LO} = \frac{p}{E} = \frac{x}{R}$ where E is the modulus of elasticity.

Since $p = \frac{x}{y}f$ or $\frac{p}{x} = \frac{f}{y}$ the expression becomes

$\frac{f}{y} = \frac{E}{R}$, from which the radius of curvature at any section of a bent beam can be calculated.

Example.—A tube of high-tensile steel 1 in. outside diam. and 0·05 in. thick rests in a horizontal position on supports 24 in. apart. Find the total uniformly distributed load that the tube will carry if the max. stress in the material is 20 tons per sq. in. Also find the minimum radius of curvature of the neutral surface of the bent tube, taking the modulus of elasticity (E) as 30×10^6 lb. per sq. in.

Let w be the load in tons per in. run. The bending-moment diagram for this case is the same as that shown in Fig. 44 and the max. bending moment occurs in the middle of the span and is equal to $\frac{w \times 24^2}{8} = 72w$. The expression first to be used is $\frac{M}{I} = \frac{f}{y}$. I is the moment of inertia of the section about its

neutral axis, which is $\frac{\pi}{4}(R^4 - r^4)$ where R and r are the external and internal radii of the tube respectively. (The method of deducing this expression is given in Art. 44.) Hence $I = \frac{\pi}{4}(1^4 - 0{\cdot}90^4) = 0{\cdot}270$ in ^{4} units, $f = 20$ tons per sq. in., and $y = 0{\cdot}5$ in., therefore

$$72w = \frac{20 \times {\cdot}270}{0{\cdot}5}$$

$$w = 0{\cdot}15 \text{ tons.}$$

Hence the total load is equal to $0{\cdot}15 \times 24 = 3{\cdot}6$ tons. The radius of the curvature can be obtained from either the expression $\frac{M}{I} = \frac{E}{R}$ or $\frac{f}{y} = \frac{E}{R}$. In this case the latter is the more convenient. The radius of curvature R is evidently a minimum when $\frac{f}{y}$ is a maximum, that is, in the middle of the span, hence

$$R = \frac{30 \times 10^6 \times 0{\cdot}5}{20 \times 2{,}240 \times 12} = 279 \text{ ft.}$$

42. Brief Recapitulation of Beam Theory—From the explanation given above it is possible to obtain a general view of the conditions surrounding the theory relating to beams as recognized by engineers. The basic fact is the planeness of normal cross-sections of the beam under all loads, which, with the assumption of perfect elasticity at all parts of the material and the truth of Hook's Law, make it possible to deduce that the stress in longitudinal fibres is proportional to their distances from the neutral surface where the stress is zero. From this deduction the determination of expressions for the resisting moment and radius of curvature of the beam at any section presents little difficulty. As a matter of physical fact, a normal plane section of an elastic beam remains plane under a bending moment if shear is absent or has a constant value, but if shear is variable the section becomes slightly distorted and takes a form of double curvature.*

The error involved by assuming no distortion of sections when considering the bending effect under all types of loading is, however, so small as to be altogether negligible in the great majority of engineering problems relating to beams. The planeness of the beam section which is shown by the straightness of the line forming its end view has been several times demonstrated by experiment. In a paper dealing with experiments on reinforced concrete beams one of the authors (W. C. Popple-

* Todhunter and Pearson's "Theory of Elasticity," vol. II, pt. i.

well, "Min. Proc. Inst. C.E.," Vol. CLXXVII) shows how he estimated the shape of the section as shown on the side of a loaded beam. The portion of the beam considered was subjected to pure bending only and the shape of the section was determined by measuring the longitudinal strain in the loaded beam at different points along its depth by placing a Martens' extensometer on the side. By plotting the strains against the corresponding depths the shape of the surface was obtained. One of the graphs is reproduced in Fig. 48, and it will be noted that on the compression side of the beam the line is straight, which was the case for the other graphs obtained by the same method. For concrete material Hooke's Law can be assumed to hold for all practical purposes up to the maximum compressive stress used (about 600 lb. per sq. in.), and hence the straightness of the line on the compression side demonstrates the fact that the section remained plane after bending. On the tension side the line is distorted by the presence of the steel bars.

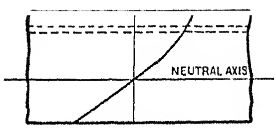


FIG. 48.

Professor Bovey of Montreal did the same thing for a beam of steel and found that the longitudinal strains, both tensile and compressive, were represented by one straight line.

In both the above cases the point where the strain was zero gave the position of the neutral axis.

43. Anticlastic Curvature—When an elastic beam is bent the longitudinal strains in the material induce lateral strains of amount equal to the longitudinal strains multiplied by Poisson's Ratio, which cause a distortion of the boundaries of normal cross-sections. In Fig. 49 is shown the shape of a normal section of a bent beam which was rectangular before the beam was bent. The neutral surface is denoted by the line XY, and it will be noted that this surface and all surfaces parallel to it which were originally plane are distorted into surfaces of anticlastic curvature. Thus the surfaces have a shape similar to the inside of an anchor ring.

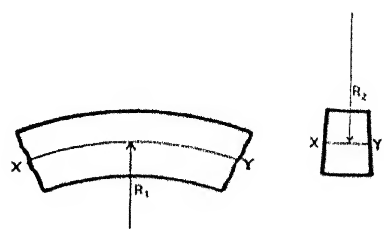


FIG. 49.

The ratio of the lateral to the longitudinal strains is Poisson's Ratio $\left(\frac{1}{m}\right)$. Hence for the radius of curvature R_1, and refer-

ring to Art. 41 and Fig. 49, $\frac{e}{y}=\frac{1}{R_1}$, where e is the strain in fibres distant y from the neutral surface, and E is Young's Modulus for the material. Similarly for the radius of curvature R_2, $-\frac{e}{my}=\frac{1}{R_2}$ From the above expressions is obtained $\frac{1}{m}=\frac{R_1}{R_2}$ (Since R_1 and R_2 are of opposite sign (Art. 49) Poisson's Ratio is negative, which will be otherwise noted when it is remembered that it is the ratio of a decrease in length to an increase. The sign is generally dropped by engineers unless specially required.) This relation suggests a method of determining Poisson's Ratio by measuring the principal curvatures of the neutral or parallel surfaces of a bent beam.*

The truth of the above relation depends on the assumption that longitudinal fibres of the material of the beam are free to expand laterally, which is not the case if the beam is very broad compared with its depth. If lateral strain in the direction of the breadth is wholly prevented, the value of the modulus of elasticity can be shown to be given by $\frac{Em^2}{m^2-1}$ by the method given in Art. 15. This will mean that the radius of curvature R_2 is infinite and that the longitudinal curvature is increased from $\frac{EI}{M}$ to $\frac{EIm^2}{M(m^2-1)}$.

The ordinary theory of bending applies only when the section is comparatively narrow and the anticlastic bending due to lateral strain is substantially free to take place, which holds good in most actual beams. The lateral curvature is not generally of practical importance.

In such materials as steel, wrought iron, aluminium, and some of the bronzes, constancy of E may be assumed at the usual working stresses; in cast iron, concrete, mortar, timber, and brickwork there are minute permanent strains at all but the very lowest stresses, and often at the working stresses. These, however, are not as a rule great enough to invalidate the safety of structures.

44. Moments of Inertia.—The moment of inertia of a plane section about any axis may be defined as the sum of all the elementary areas into which the section may be divided multiplied by the square of their respective distances from the axis.

* "Phil. Mag.," Feb. 1921.

Thus in Fig. 50 the moment of inertia of the area about the line XY is $\Sigma y^2 \delta A$.

Rules and examples relating to the moments of inertia of circles, ellipses, rectangles, and triangles, as well as figures made up of combinations of these, will now be considered. The following rules are important:

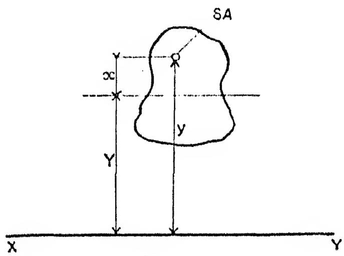


FIG. 50.

(1) The moment of inertia of a section about an axis in its plane not passing through its centre of gravity is equal to the moment of inertia of the section about the line passing through its centre of gravity and parallel to the axis plus the area of the section multiplied by the square of the distance of its centre of gravity from the axis.

(2) The moment of inertia of a section about an axis normal to its plane is equal to the sum of the moments of inertia about perpendicular axes in the plane of the section which intersect on the normal axis.

To prove (1) consider Fig. 50. Let the area of the section be denoted by A; the distance of the centre of gravity of the area from the axis XY is Y. The moment of inertia of the section about the axis is I_{xy} =

$$\begin{aligned}\Sigma y^2\delta A &= \Sigma(y+x)^2\delta A\\ &= \Sigma y^2\delta A + \Sigma x^2\delta A + \Sigma 2yx\delta A\\ &= y^2\Sigma\delta A + \Sigma x^2\delta A + 2y\Sigma x\delta A.\end{aligned}$$

The last term of this expression is zero, and hence $I_{xy} = Ay^2 + I_G$, where I_G is the moment of inertia of the section about the line passing through its centre of gravity and parallel to the axis.

In the case of (2) an expression for the moment of inertia of the section (Fig. 51) is required about the normal axis whose end view is the point O. Calling the area of the section A and I_O the moment of inertia about the axis, then

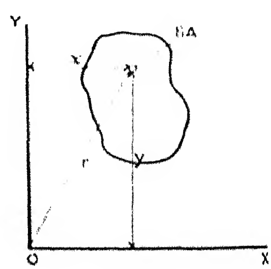


FIG. 51.

$$I_O = \Sigma r^2\delta A = \Sigma(x^2+y^2)\delta A = \Sigma x^2\delta A + \Sigma y^2\delta A = I_x + I_y$$

where I_x and I_y are the moments of inertia of the section about the axis OX and OY respectively. A few cases will now be considered:—

Case 1.—Rectangle (Fig. 52), breadth B, depth D, about the line XY

$$\mathrm{I} = \int_{-\frac{\mathrm{D}}{2}}^{+\frac{\mathrm{D}}{2}} x^2 \mathrm{B}dx = 2\mathrm{B}\int_{0}^{\frac{\mathrm{D}}{2}} x^2 dx = \frac{2\mathrm{BD}^3}{24}$$

$= \dfrac{\mathrm{BD}^3}{12} = \dfrac{1}{12}\mathrm{D}^2\mathrm{A}$ where A is the area of the rectangle.

Case 2.—Circle of radius R and area A about the normal axis through its centre (Fig. 53)

$$\mathrm{I}_0 = \int_{0}^{\mathrm{R}} 2\pi r^3 dr = \frac{\pi \mathrm{R}^4}{2} = \frac{1}{2}\mathrm{R}^2\mathrm{A}$$

Case 3.—Circle about a diameter (Fig. 53). If the moments of inertia of the circle about the perpendicular lines OX and OY be denoted by I_x and I_y respectively, then by rule (2) above $\mathrm{I}_x + \mathrm{I}_y = \mathrm{I}_0$. But for a circle $\mathrm{I}_x = \mathrm{I}_y = \mathrm{I}$, hence $\mathrm{I} = \dfrac{\mathrm{I}_0}{2} = \dfrac{1}{4}\mathrm{R}^2\mathrm{A}$.

Case 4.—Ellipse about a principal diameter (Fig. 54). Let the

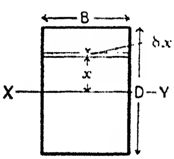


FIG. 52.

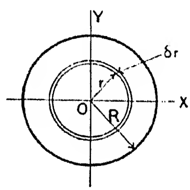


FIG. 53.

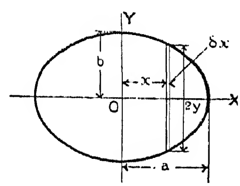


FIG. 54.

area of the ellipse be A and its equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then

$$\mathrm{I}_x = \int_{-a}^{+a} \frac{2}{3} y^3 dx.$$

This can be integrated by putting $x = a\cos\phi$ and $y = b\sin\phi$, which is permissible because these equations satisfy the equation to the ellipse, hence—

$$\mathrm{I}_x = 2\int_{0}^{\frac{\pi}{2}} \frac{2}{3} ab^3 \sin^4 \phi d\phi.$$

$$= 2\left(\frac{2}{3} \times \frac{3}{16} ab^3\right) = \frac{1}{4} ab^3 = \frac{1}{4} b^2\mathrm{A}.$$

The above results may be summed up by the following rule. The moment of inertia of a circle or ellipse about a principal

axis or a square or rectangle about an axis through its centre of gravity and parallel to a side is given by

$$\frac{(\text{area of figure}) \times (\text{length of semi-axis of fig. normal to axis})}{4 \text{ (for circles and ellipses) or } 3 \text{ (for rectangular figures)}}$$

Case 5.—Triangle about the axis through apex and parallel to base (Fig. 55)

$$I_{xy} = \int_0^H xy^2 dy \text{ and } \frac{y}{H} = \frac{x}{B} \text{ hence } x = \frac{B}{H} y$$

$$I_{xy} = \int_0^H \frac{B}{H} y^3 dy = \frac{1}{4} BH^3 = \frac{1}{2} H^2 A,$$

where A is the area of the triangle.

Case 6.—Triangle about axis through its centre of gravity. Call the required moment of inertia I_G

$$I_{xy} = I_G + \left(\frac{2}{3} H\right)^2 \times \frac{BH}{2}$$

$$I_G = \frac{1}{4} BH^3 - \frac{}{18} BH^3$$

$$\frac{BH^3}{36} = \frac{1}{18} H^2 A$$

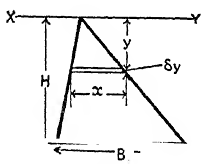


Fig. 55.

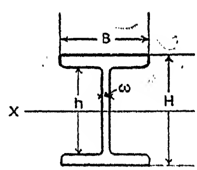


Fig. 56.

Case 7.—Triangle about its base

$$I_{base} = I_G + \left(\frac{1}{3} H\right)^2 BH$$

$$= \frac{1}{36} BH^3 + \frac{1}{18} BH^3$$

$$= \frac{1}{12} BH^3 = \frac{1}{6} H^2 A.$$

With the aid of the above results the moment of inertia of an area about an axis in its plane which can be divided up into rectangles, circles, ellipses, or triangles can be found, for it is the sum of the moments of inertia of the several figures.

I section (Fig. 56) about the line XY

$$I_{xy}=\frac{BH^3}{12}-\frac{(B-\omega)h^3}{12}=\frac{1}{12}(BH^3-(B-\omega)h^3).$$

H section about the line XY (Fig. 57)

$$I_{xy}=\frac{2Bt^3}{12}+\frac{h\omega^3}{12}=\frac{1}{12}(2Bt^3+h\omega^3).$$

Circular ring about a diameter (Fig. 58)

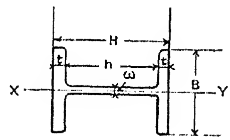


FIG. 57.

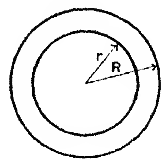


FIG. 58.

$$I=\left(\frac{1}{4}\pi R^4-\frac{1}{4}\pi r^4\right)$$

$$=\frac{1}{4}\pi(R^4-r^4)$$

Tee section, angle section, and channel (Fig. 59).

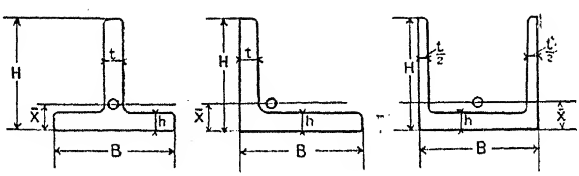


FIG. 59.

First find the distance $\bar{x}$ of the line through the centre of gravity from the base.

$$\bar{X}=\frac{\frac{Bh^2}{2}+\frac{t(H-h)(H+h)}{2}}{Bh+t(H-h)}$$

$$I_{xy}=\left[\frac{t(H-h)^3}{12}+t(H-h)\left(H-\bar{X}-\frac{H-h}{2}\right)^2\right]$$

$$+\left[\frac{Bt^3}{12}+Bh\left(\bar{X}-\frac{h}{2}\right)^2\right]$$

$$=t\left[\frac{(H-h)^3}{12}+(H-h)\left(\frac{H+h}{2}-\bar{X}\right)^2\right]+B\left[\frac{t^3}{12}+h\left(\bar{X}-\frac{h}{2}\right)^2\right]$$

Fig. 60 represents two common sections of considerable importance used for cast-iron beams (much more usual formerly than at present) and the frames of machines. As cast iron and materials of similar nature are stronger in compression than in tension, an economic design is effected by making the section unsymmetrical about the neutral axis in order to bring the neutral axis nearer to the tensile face than the compressive. As a rule it is the tensile stress which governs the strength of the beam, and hence the least value of y—that is, the distance of the tensile face from the neutral surface—should be used to determine the stress. The distances of the centres of gravity of these and similar sections from their bases and also their moments of inertia can easily be found by the methods given.

EXAMPLES. III

(1) A round steel rod $\frac{1}{2}$ in. in diameter resting upon supports A and B 4 ft. apart projects 1 ft. beyond A and 9 in. beyond B. The

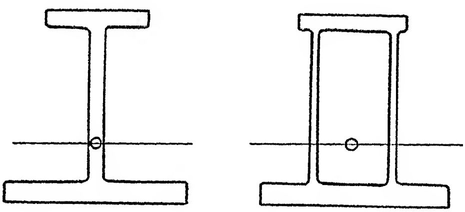


Fig. 60.

extremity beyond A is loaded with a weight of 12 lb. and that beyond B with a weight of 16 lb. Neglecting the weight of the rod, investigate the curvature of the rod between the supports and calculate the greatest deflection between A and B. Find the greatest intensity of stress in the rod due to the two applied forces ($E = 30,000,000$ lb. per sq. in.) (Victoria).

(2) A balk of timber 9 in. square rests on supports 25 ft. apart and carries a load of 1 ton in the middle. Find the main stress in the material, allowing for its own weight, if the timber weighs 45 lb. per cu. ft.

(3) A rolled-steel joist is 18 in. deep with flanges 7 in. wide and 1 in. thick. It is freely supported at its ends over a span of 12 ft. If it carries a load of 2 tons per ft. run, find the stress in the flanges.

(4) It is desired to cut from a balk of timber 12 in. diameter the strongest beam of rectangular cross-section. Find the ratio of the sides.

(5) A cast-iron beam is 18 in. deep with a web 1 in. thick. The tension flange is 15 in. wide and $1\frac{1}{2}$ in. thick and the compression flange is $3\frac{1}{2}$ in. wide and 1 in. thick. If the skin stress on the tension side is 1 ton per sq. in., find the safe uniformly distributed load it will

or energy and is called the **molal surface energy.** It represents the work involved in producing the surface, S_0, against the surface tension, γ. It is analogous to the molal volume energy, pv_0, in the case of a gas (II, 4).

Between the molal surface energy of pure (*i.e.*, non-associated) liquids and the temperature there exists a relation, discovered by Eötvös[a] in 1886, which is perfectly analogous to equation (13, II) for gases. It is expressed mathematically by the equatior,

$$\frac{-\mathrm{d}(\gamma S_0)}{dt} = -a\,\frac{\mathrm{d}\left[\gamma\left(\frac{M}{D}\right)^{2/3}\right]}{dt} = const., \text{ or } \frac{-\mathrm{d}\left[\gamma\left(\frac{M}{D}\right)^{2/3}\right]}{dt} = K_s \quad (9)$$

where K_s is a constant which has the same value for all pure liquids. Expressed in words this equation states that: **The temperature rate of change of the molal surface energy of a pure liquid is independent of the temperature and of the nature of the liquid.** The independence with respect to the temperature is illustrated by the data given in Table V.

TABLE V

Illustrating the equation of Eötvös, $\frac{-\mathrm{d}\left[\gamma\left(\frac{M}{D}\right)^{2/3}\right]}{dt} = K_s$. Values of K_s for benzene at various temperatures between 11° and 120°. t_c = 288°.

	Measurements by Renard and Guye					Measurements by Ramsay						
$t°$ =	11.4	31.2	55.1	68.5	78.3	80	90	100	110	120	200	250
K_s =	2.10	2.13	2.12	2.10	2.10	2.09	2.10	2.10	2.10	2.10	2.10	2.08

The integral of equation (9) is

$$\gamma\left(\frac{M}{D}\right)^{2/3} = -K_s t + K_s K' = K_s(K' - t) \quad (10)$$

where K_sK' is the integration constant. (Cf. equation 14, II.) Experiments made by Ramsay[b] and Shields using a number of different liquids showed that the constant, K', has approximately the value,

$$K' = t_c - 6 \quad (11)$$

[a] Baron Roland von Eötvös (*pr.* Āutvush), Professor of Physics in the University of Budapest.

[b] William Ramsay, K. C. B., F. R. S. (1852–1916). Professor of Chemistry at University College, London.

CHAPTER IV

SECTION MODULUS AND MOMENT OF INERTIA DETERMINED GRAPHICALLY

45. Reasons for adopting Graphical Methods.—In the cases so far given the moments of inertia are found without much trouble by straightforward calculation from the dimensions of the section. This method is as a rule the best when it can be applied, but there are many sections bounded by curves or irregular lines which will not allow themselves to be treated in this simple way, and graphical methods have then to be adopted.

In Fig. 61 the outer line a is the boundary of an area whose moment of inertia is required about the line XY. The modulus of the area is also required, as well as the distance of its centre of gravity from XY. Draw the line AB parallel to XY and then draw a line such as PQ across the area and also parallel to XY. Now draw Pp and Qq normal to AB and join p and q to any point O on XY. Then where pO and qO cut the line PQ—that is at P′ and Q′—are points on another figure called the first modulus figure. Other points are found by a similar procedure, and the new figure is drawn by joining the points by a smooth curve. This figure is denoted by b in Fig. 61. The new figure is then treated in a similar way and another figure c is obtained called the second modulus figure. Thus P″ and Q″ are points on the second modulus figure. With a knowledge of the areas of the three figures a, b, and c, and the perpendicular distance between the lines XY and AB, the moment of inertia of the original figure and also the distance of its centre of gravity from the line XY can be calculated.

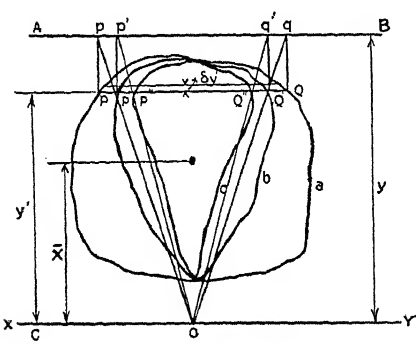


FIG. 61.

Let the width of a strip of average length PQ be denoted by $\delta y'$. Then the area of the figure a is $\Sigma PQ\ \delta y'$ between the proper limits and similar expressions represent the areas of figures b and c. Now consider the triangles pOq and $P'OQ'$

$$\frac{pq}{P'Q'} = \frac{PQ}{P'Q'} = \frac{y}{y'} \therefore P'Q' = PQ\frac{y'}{y}$$

$$\text{and } \Sigma P'Q'\delta y' = \frac{1}{y} \cdot \Sigma PQ\ y'\delta y'.$$

If A denotes the area of the original figure and $\bar{X}$ the distance of its centre of gravity from XY, then

$$A\bar{X} = \Sigma PQ\ y'\delta y' = y\Sigma P'Q'\delta y'$$

$$\text{or } \bar{X} = \frac{y\Sigma P'Q'\delta y'}{A}$$

Thus the distance of the centre of gravity of the original figure from the line XY is given by the area of the first modulus figure divided by the area of the original figure and multiplied by the perpendicular distance between the lines XY and AB.

Next consider the triangles $p'Oq'$ and $P''OQ''$:

$$\frac{p'q'}{P''Q''} = \frac{P'Q'}{P''Q''} = \frac{y}{y'} \therefore P'Q' = \frac{y}{y'}P''Q''$$

and it has just been shown that $P'Q' = PQ\frac{y'}{y}$;

hence $PQy'^2 = y^2P''Q''$, and

$$\Sigma PQ\ y'^2\delta y' = y^2\Sigma P''Q''\delta y'.$$

That is, the moment of inertia of the original figure—which is the left-hand side of this equation—is given by the area of the second modulus figure multiplied by the square of the distance between the two lines XY and AB.

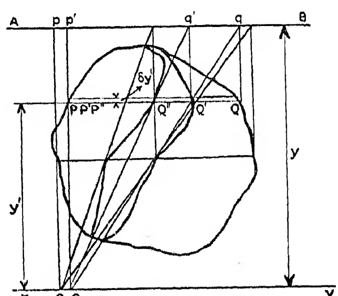


FIG. 62.

In Fig. 62 is shown another method of drawing the modulus figures by which part of the boundary of the original figure is caused to coincide with part of those of the modulus figures. This method is the more convenient when the original figure is unsymmetrical about a line perpendicular to XY because by the previous method the complete modulus figures have to be drawn, whereas when it is symmetrical only half of the figure is necessary. By this method a different pole O_1, O_2 . . . is in general necessary to correspond with different

positions of the lines PQ. The letters in Fig. 62 correspond with those in Fig. 61, so that the proofs given will apply to both.

It has now been shown how to find the moment of inertia of an area about a line such as XY and also the distance of the centre of gravity from the line, and it is easy to find the moment of inertia about any line in the plane of the area and parallel to XY by the rule given in Art. 44. For instance, let I_G denote the moment of inertia of the area about the line passing through its centre of gravity and parallel to XY. Then, as was shown in Art. 44, if I_{xy} denotes the moment of inertia about XY, $\overline{X}$ the

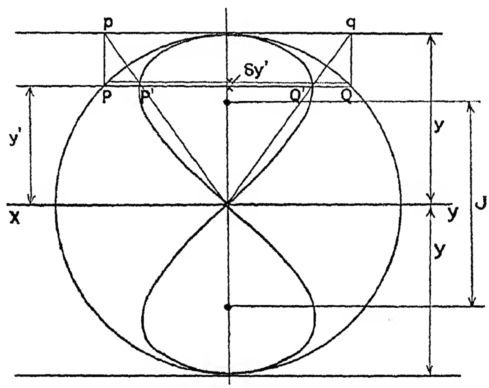


FIG. 63.

distance of the centre of gravity of the area from XY, and A the area, $I_{xy} = I_G + A\overline{X}^2$, from which I_G can be obtained.

46. Significance of First Modulus Figure—In Fig. 63 the first modulus figure is drawn for a circle with respect to the line XY which passes through its centre; this line is the neutral axis of the section. Let f' and f denote the stresses along the surfaces distance y' and y respectively from the neutral surface, then it was shown in Art. 40 that $\dfrac{f'}{y'} = \dfrac{f}{y}$.

The total force in the strip of width $\delta y'$ and average length PQ is $f'\text{PQ}\ \delta y' = f\dfrac{y'}{y}\text{PQ}\,\delta y$, but $\dfrac{y'}{y}\text{PQ} = \text{P}'\text{Q}'$.

Hence $f'\text{PQ}\ \delta y' = f\text{P}'\text{Q}'\delta y'$.

That is, the force on any strip such as PQ is proportional to the length P′Q′ of the corresponding strip in the first modulus figure, since the value of f is constant. The total force acting along the beam on one side of the neutral surface is therefore

$$\Sigma_0^y f'PQ\,\delta y' = f\Sigma_0^y P'Q'\delta y'.$$

The right-hand side of this expression is simply the area of one-half of the modulus figure multiplied by f, the stress at the surface. The total force acting along the beam on the other side of the neutral surface is represented by a similar expression, and, since the beam is in equilibrium, it follows that the two forces must be equal and act in opposite directions, and thus the areas of the two parts of the modulus figure must be equal. The two forces form a couple which is the moment of resistance of the beam, and the arm of the couple is the distance between the centres of gravity of the halves of the modulus figure. Let J denote this distance, 2A the total area of the modulus figure and f the fibre stress, then the moment of resistance M is $AJf = Zf$ where Z is the modulus of the section. Or, in words, the greatest moment of resistance at the section is equal to one-half the area of the modulus figure multiplied by the distance between the centres of gravity of the two halves and by the maximum fibre stress; thus it equals the modulus of the section multiplied by the maximum fibre stress.

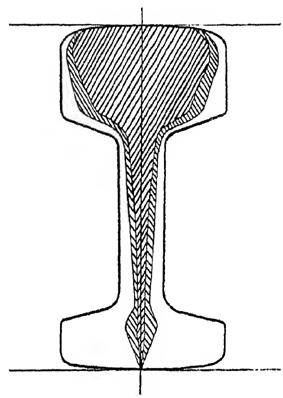


FIG. 64.

It was shown in Art. 40 that $\frac{M}{I} = \frac{f}{y}$ where M is the bending moment acting at a beam section and I the moment of inertia.

Since $M = AJf$

then $\frac{AJ}{I} = \frac{1}{y}$

hence $AJ = \frac{I}{y} =$ the modulus of the section or $I =$ modulus $\times y = Zy$. The relation between the moment of inertia and the modulus can be otherwise easily obtained from a consideration of Figs. 61, 62, or 63. Referring to Fig. 63:

$$I = 2\Sigma_0^y PQ\,y'^2\delta y'$$

but $\frac{PQ}{P'Q'} = \frac{y}{y'}$, hence $I = 2y\Sigma_0^y P'Q'y'\delta y'$

$$= 2yA\frac{J}{2}$$

$$= AJy = Zy.$$

This analysis will apply whether the section is symmetrical about the line XY or not.

Referring now to Figs. 62 and 63:

$$I = \Sigma_0^y PQ\, y'^2 \delta y',$$

which reduces to $I = AJy$.

In this expression A is the area of the whole of the modulus figure and J is the distance of its centre of gravity from XY. It will be evident that this case has no significance in the ordinary theory of the bending of beams, because the line XY is not the neutral axis of the section.

The stages in the method of finding the moment of inertia or modulus for a beam section from the first modulus figure are as follows:—

1. Ascertain the centre of gravity of the section in question. There are many ways of doing this, but the most suitable one in the average case is to cut out the section in cardboard or thin sheet metal and balance it about a knife-edge in two positions. The section will balance about two straight lines whose intersection is the point required.

2. Draw a line through the centre of gravity perpendicular to the plane in which the external loads act; this line will be the neutral axis of the section.

3. Now draw two straight lines parallel to the neutral axis and at equal distances from it on either side; these are the base lines. If the section is symmetrical about the neutral axis, both the lines should be tangent to the section. If the section is unsymmetrical, one of the lines should be tangent to the section on the side where the safe stress approaches the nearer to the allowable working stress, that is, where the beam is weakest. This is essential when the question is one of design or of determining the greatest load that the beam will safely carry, but in questions like that of deflection, where the maximum induced stress is below the safe stress, it is immaterial which of the two lines is drawn tangent to the section.

4. Next draw the first modulus figure.

5. Compute the area of each half of the modulus figure. This can best be done with an Amsler Planimeter, taking an average from three or four circuits. The areas of the two halves should be equal, which gives a check on the accuracy of the work.

6. The positions of the centres of gravity of the halves of the modulus figure are next to be found preferably by cutting out and balancing, and the distance between them in a direction normal to the neutral axis measured. Calling this distance J, A the area of half the modulus figure, y the distance of either of

the base lines from the neutral axis, and Z the modulus, then

$$Z = AJy$$

and the moment of inertia $I = Zy$.

In Fig. 64 is shown the modulus figure drawn for a section of a railway line. Since this section is symmetrical about a vertical line passing through its centre of gravity, only one half of the figures need actually be drawn.

In Fig. 65 both the modulus figures are drawn for a tram-rail section, and in Fig. 66 the first modulus figure is drawn with respect to the neutral axis and the base lines XY and X'Y'. This section is unsymmetrical about its vertical axis, and the

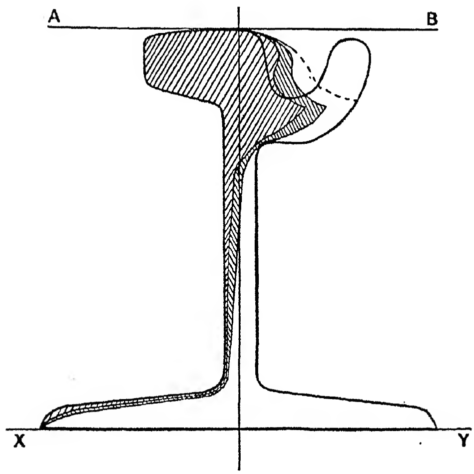


Fig. 65.

figures in Fig. 65 are accordingly constructed by the method shown in Fig. 62.

It is sometimes found convenient, though not necessary, to mass together a separated section. In the above case it is split by the groove in which the flange of the wheel runs.

The massing is done by drawing a number of horizontal lines across the part to be dealt with and transferring the outstanding widths to the solid part, as indicated in Figs. 65 and 66 by the dotted lines. This process does not alter the value of the modulus about the neutral axis parallel to these horizontal lines.

It may be noted that straight lines inclined to the neutral axis and base lines transform into curves in the first modulus figure and that straight lines normal to the neutral axis transform into

inclined straight lines in the first modulus figure and into the curves in the second modulus figure.

Referring to Fig. 66, it is easy to show from the construction

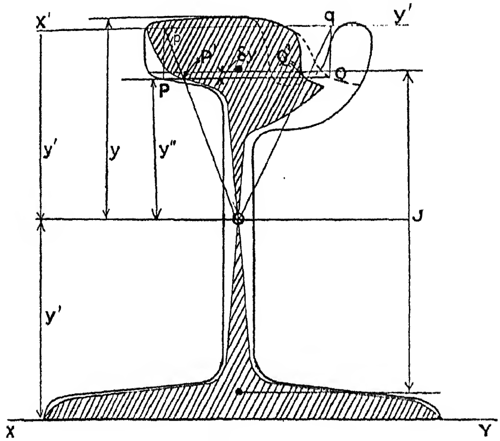


FIG. 66.

of the first modulus figure that the areas of the halves of the figure are equal. The area of the upper half of the figure is

$$\Sigma_0^{y'} P'Q'\delta y = \frac{1}{y'}\Sigma_0^{y'} PQ\, y''\delta y.$$

That is, the moment of the upper half of the original figure about the neutral axis multiplied by $\frac{I}{y'}$. A similar equality can be obtained for the lower half of the figure, and since the moments of

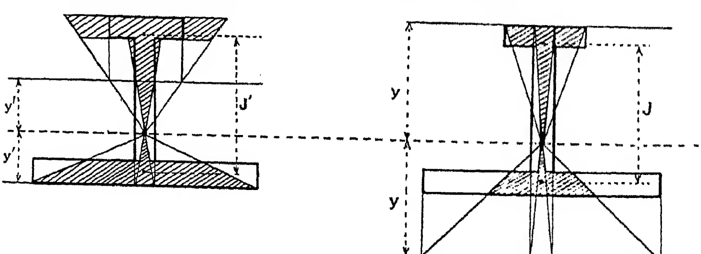


FIG. 67.

either half of the original figure about the neutral axis are equal, it follows that the areas of the halves of the first modulus figure are also equal.

In Fig. 67 the first modulus figure is shown drawn for a section

of cast-iron frame or girder. Since cast iron is only about one quarter as strong in tension as in compression, the section is made unsymmetrical about the neutral axis in order to reduce the induced stress in the tensile side. The section has generally to be designed on the basis of a maximum tensile stress which occurs at the edge of the large flange, and the base lines should accordingly be taken at the distance of this edge from the neutral axis.

The modulus will evidently have a value differing according to which base line is used, but the value of the moment of inertia will of course be the same in either case. Thus in an unsymmetrical section let

$y =$ distance of one base line from neutral axis,
$Z =$ modulus as found from this,
$y' =$ distance of other base lines from neutral axis, and
$Z' =$ the corresponding modulus.

Then the moment of inertia

$$I = Zy = Z'y'$$

from which

$$\frac{Z'}{Z} = \frac{y}{y'}$$

or

$$Z' = Z\frac{y}{y'}$$

Thus Z' can be found from Z by multiplying it by the ratio of the y's. The two pairs of modulus figures giving Z and Z' are shown for the section in Fig. 67.

47. Alternative Graphical Method—There is another graphical method of determining the moment of inertia and hence the modulus of a section with respect to any line in its plane, which is often preferable to those given. In Fig. 68 *a* denotes the area whose moment of inertia is required about the line XY. Draw the line OD normal to XY and make $OC = (OA)^3$ and $OD = (OB)^3$. The construction is repeated and a number of points such as C and D obtained. When the points are joined by a smooth curve a new figure results, which is denoted by *b* in Fig. 68. The moment of inertia of the original figure about XY is equal to the area of the new figure divided by three. To prove this, let the maximum width of the figures in a direction parallel to XY be y. Then the moment of inertia of the original figure about XY is

$$\Sigma_0^y \left\{ \frac{(OB)^3}{3}\delta y - \frac{(OA)^3}{3}\delta y \right\}$$

$$= \frac{1}{3}\Sigma_0^y (OD - OC)\delta y = \frac{1}{3}\Sigma_0^y (CD)\delta y,$$

which is one-third the area of the new figure.

In Fig. 69 half the new figure is shown drawn for a section of aeroplane spar. The position of the centre of gravity of the section is first found and the new figure drawn with respect to the neutral axis. It may be noted that the figure is much larger than the first modulus figure would be, and hence greater accuracy is possible in the determination of its area. The lengths such as OC and OD, that is $(OA)^3$ and $(OB)^3$ respectively, can be conveniently obtained from a table of cubes.

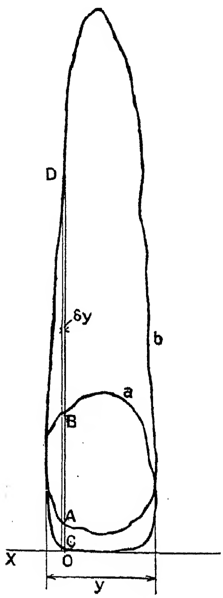


FIG. 68.

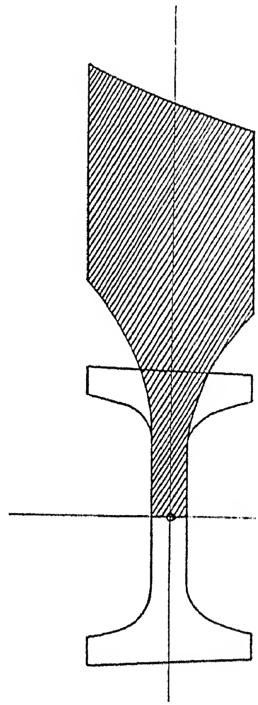


FIG. 69.

48. Ferro-concrete Beams—In the treatment of the bending of ferro-concrete beams the following assumptions will be made :—

1. That the concrete on the reinforced side of the beam withstands no tension, which is all assumed to be taken by the reinforcements. Concrete is weak in tension, and when a ferro-concrete beam is gradually loaded cracks appear on the tension side approximately normal to the neutral surface—if shear is small—at comparatively small loads which extend nearer to the

neutral surface as the load is increased. It should further be noted that when concrete sets it shrinks slightly—which is evidenced, for instance, by its gripping the reinforcements—and hence the concrete is subjected to an initial tension which causes a reduction in the available range of working tensile stress.

2. The concrete is elastic in compression. The relation between stress and strain for concrete in compression is not linear except for small stresses. At the higher stresses the strain increases at a slightly greater rate than the stress. When, however, the stress has been increased along a certain range the relation between the stress and strain within the range after the first application of the load is sensibly linear.

3. That plane normal sections of the beam before bending remain plane after bending. This assumption was considered

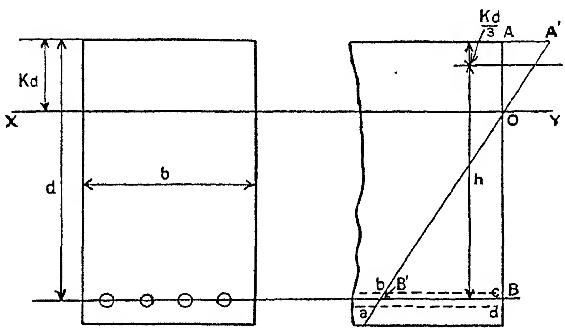


FIG. 70.

in Art. 42. It may be noted that if the concrete is not elastic the assumption does not apply.

The above assumptions are the most usual and place the theory of the bending of ferro-concrete beams on a rational basis. That such assumptions can reasonably be made is borne out, for instance, by experiments conducted by the French Government ("The Properties and Design of Reinforced Concrete," translated by N. Martin, 1912), who obtained values of the modulus of elasticity under light loads varying from $2{\cdot}28 \times 10^6$ lb. per sq. in. to $5{\cdot}69 \times 10^6$ lb. per sq. in., according to quantity of water used in gauging and the method of ramming. The concrete was composed of 6 cwt. of Portland cement, 14·35 cu. ft. of sand, and 28·70 cu. ft. of gravel. In America it is often assumed that the relation between stress and strain is represented by some smooth curve as a parabola.

In Fig. 70 is shown a ferro-concrete beam of rectangular cross-section and neutral axis XY. Since normal plane sections are

assumed to remain plane after bending, the strain is represented to scale by such lengths as AA′, BB′, etc. Let the maximum compressive strain in the concrete which is represented by AA′ be denoted by e_c and the mean tensile strain in the steel represented by BB′ by e_s, then

$$\frac{e_c}{e_s}=\frac{\mathrm{K}d}{d-\mathrm{K}d}=\frac{\mathrm{K}}{1-\mathrm{K}}$$

Now let the compressive stress in the concrete corresponding with e_c be denoted by f_c and the tensile stress in the steel corresponding with e_s by f_s, also let E_c and E_s denote the moduli of elasticity for concrete and steel respectively,

then $$e_c=\frac{f_c}{\mathrm{E}_c}\text{ and }e_s=\frac{f_s}{\mathrm{E}}$$

hence $$\frac{f_c\mathrm{E}_s}{f_s\mathrm{E}_c}=\frac{\mathrm{K}}{1-\mathrm{K}}$$

or $$\frac{f_c}{f_s}=\frac{\mathrm{K}}{n(1-\mathrm{K})} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $$n=\frac{\mathrm{E}_s}{\mathrm{E}_c}$$

Hence if f_c, f_s, and n are given, K, which gives the position of the neutral axis, can be found.

If a = total area of steel the total force in the steel is $f_s a$. The total compressive force in the concrete is proportional to the area OAA′ and is given by $\frac{f_c}{2}b\mathrm{K}d$. These must be equal, hence

$$f_s a=\frac{f_c}{2}b\mathrm{K}d \text{ or } \frac{f_c}{f_s}=\frac{2a}{bd\mathrm{K}}=\frac{2r}{\mathrm{K}} \qquad . \quad . \quad . \quad . \quad . \quad (2)$$

where r is the reinforcement ratio = ratio of area of steel to area of concrete; the area of the concrete is denoted conventionally by bd.

Hence (1) becomes

$$\frac{2r}{\mathrm{K}}=\frac{\mathrm{K}}{n(1-\mathrm{K})}$$

or $$\mathrm{K}=-rn+\sqrt{r^2n^2+2rn} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From which K can be found if the reinforcement ratio r and $n=\frac{\mathrm{E}_s}{\mathrm{E}_c}$ are given.

The total compressive force in the concrete must act through the centre of gravity of the triangle OAA′, that is, at the depth $\frac{\mathrm{K}d}{3}$ from the upper surface of the beam. Also the total tensile

force in the steel must act through the centre of gravity of the trapezium *abcd* and no appreciable error is involved by assuming that it acts along the axis of the reinforcements. The moment of resistance of the beam is accordingly given by $f_s ah$

or
$$\frac{f_c}{2} b\mathrm{K}dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where
$$h = d\left(1 - \frac{\mathrm{K}}{3}\right) = \text{arm of couple.} \quad . \quad . \quad . \quad (5)$$

From the formulæ (1) to (5) and the principles involved in them most problems on the bending of ferro-concrete beams, whether of rectangular cross-section or tee-shaped, can be solved. For example, suppose that f_s, f_c, n, and the bending moment are given and it is desired to design the beam. From equation (1) the value of K can be found and then from equation (3) the value of r, the reinforcement ratio, determined. A value of d the depth must then be assumed and b the breadth worked out from (4) and (5). If the dimensions first determined are unsuitable, other values of d should be tried until the dimensions are satisfactory. The total area of the steel is then given by $a = rbd$, so that a suitable area and number of bars can be decided upon.

The following table gives values of K and r for different values of $\frac{f_c}{f_s}$ when $n = 15$:—

$\frac{f_c}{f_s}$	r (per cent.)	K	$\frac{f_c}{f_s}$	r (per cent.)	K
$\frac{600}{18{,}000}$	·56	·33	$\frac{400}{18{,}000}$	·28	·25
$\frac{600}{16{,}000}$	·68	·36	$\frac{400}{16{,}000}$	·34	·27
$\frac{600}{14{,}000}$	·84	·39	$\frac{400}{14{,}000}$	·43	·30

For a rough approximation K can be taken equal to $\frac{3}{8}$, and hence

$$\frac{\mathrm{K}}{3} = \frac{1}{8} \text{ and } h = \frac{7}{8}d.$$

The economic design of a ferro-concrete beam where the stresses induced in the steel and concrete are the maximum allowable generally results in a beam of large cross-section, which is often undesirable ; for instance, where head room is important. An excess of steel is accordingly often put in and sometimes the beam is reinforced on both tension and compression sides.

Example 1.—A ferro-concrete beam is to rest on supports 12 ft. apart and carry a load of 1,500 lb. per ft. run and is to be reinforced on the tension side only. Taking $f_s = 16{,}000$ lb./sq. in., $f_c = 600$ lb./sq. in., and $n = 15$, design the beam.

From equation (1) $$n(1-\mathrm{K})\frac{f_c}{f_s} = \mathrm{K}.$$

$$\mathrm{K} = \frac{nf_c}{f_s + nf_c}$$

$$= \frac{15 \times 600}{16{,}000 + 15 \times 600} = \frac{0{\cdot}562}{1{\cdot}562} = 0{\cdot}36.$$

From equation (3) $$\frac{2r}{\mathrm{K}} = \frac{\mathrm{K}}{n(1-\mathrm{K})}$$

$$r = \frac{\mathrm{K}^2}{2n(1-\mathrm{K})} = \frac{(0{\cdot}36)^3}{30(1-0{\cdot}36)} = 0{\cdot}00674.$$

Hence a, the total area of the steel $= 0{\cdot}00674\ bd$. The max. bending moment on the beam is

$$\mathrm{M} = \frac{1{,}500 \times 12 \times 144}{8} = 3{\cdot}24 \times 10^5 \text{ lb. in.}$$

From equation (4) : $$\mathrm{M} = \frac{f_c}{2} b\mathrm{K}dh$$

and from equation (5) $$h = d\left(1 - \frac{\mathrm{K}}{3}\right) = 0{\cdot}88d.$$

Hence $$\mathrm{M} = \frac{0{\cdot}88 f_c}{2} b\mathrm{K}d^3$$

which gives $$b = \frac{2\mathrm{M}}{0{\cdot}88 f_c \mathrm{K} d^2} = \frac{3{\cdot}41 \times 10^3}{d^2}$$

If $d = 18$ in., then $b = 10{\cdot}5$ in., and $r = 0{\cdot}00674$.

$a = 1{\cdot}28$ sq. in., say three $\frac{3}{4}$ in. dia. bars or
five $\frac{5}{8}$,, ,, ,,

Example 2.—Design the beam under above conditions, it being given that r the reinforcement ratio $= 1\%$ $f_c = 600$ and $n = 15$.

From equation (3) : $\mathrm{K} = -rn + \sqrt{r^2n^2 + 2rn}$
$= 0{\cdot}417.$

From equation (5): $h = d\left(1 - \frac{K}{3}\right)$

$$= 0{\cdot}861d.$$

and from equation (4):

$$b = \frac{2M}{f_c K d h}$$

$$= \frac{3{\cdot}01 \times 10^3}{d^2}$$

Hence if $d = 17$ in., $b = 10{\cdot}4$ in. It will be noticed that with the new data the cross-sectional dimensions of the beam are slightly reduced.

Beam with Double Reinforcements (Fig. 71)—The same notation

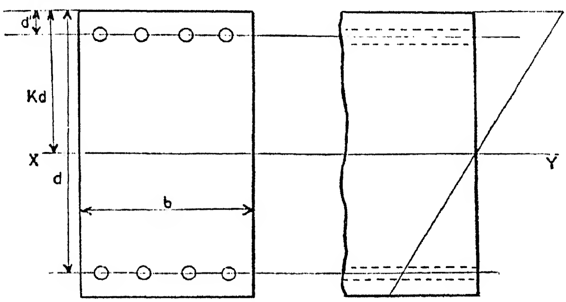


FIG. 71.

is used as for single reinforced beams, but the dashed letters refer to compression reinforcements. For instance, e'_s, f'_s, and r' denote strain, corresponding stress, and the reinforcement ratio for steel in compression.

Since plane normal cross-sections remain plane after bending, then for steel in compression

$$\frac{e_c}{e'_s} = \frac{\frac{f_c}{E_c}}{\frac{f'_s}{E_s}} = \frac{Kd}{(Kd - d')}$$

hence

$$\frac{f_c}{f'_s} = \frac{K}{n\left(K - \frac{d'}{d}\right)} \qquad \text{. . . . . . (6)}$$

This equation corresponds with equation (1) for steel in tension. The total compressive force is $\frac{1}{2} f_c b K d + f'_s a'_s$ and the total

tensile force is $f_s a_s$, and since the beam is in equilibrium these must be equal, hence

$$\frac{1}{2} f_c b \mathrm{K} d + f'_s a'_s = f_s a_s$$

$$\frac{f_c \mathrm{K}}{2} + r' f'_s = r f_s$$

$$\mathrm{K} = 2\left(r \frac{f_s}{f_c} - r' \frac{f'_s}{f_c}\right)$$

Substituting for $\frac{f_s}{f_c}$ and $\frac{f'_s}{f_c}$ from equations (1) and (6)

$$\mathrm{K} = 2\left\{\frac{rn(1-\mathrm{K})}{\mathrm{K}} - \frac{r'n\left(\mathrm{K} - \frac{d'}{d}\right)}{\mathrm{K}}\right\}$$

$$\mathrm{K}^2 + 2n\mathrm{K}(r + r') - 2n\left(r + r'\frac{d'}{d}\right) = 0 \quad . \quad . \quad (7)$$

which corresponds with equation (3).

The moment of resistance M of the beam is given by $\mathrm{M} = f_s a h$ or

$$\mathrm{M} = \frac{1}{2} f_c b \mathrm{K} d\left(d - \frac{\mathrm{K}d}{3}\right) + f'_s a_s (d - d')$$

From equation (6) : $f'_s = \dfrac{n\left(\mathrm{K} - \frac{d'}{d}\right)}{\mathrm{K}} f_c$

hence

$$\mathrm{M} = \frac{1}{2} f_c b \mathrm{K} d^2\left(1 - \frac{\mathrm{K}}{3}\right) + \frac{nfca'_s\left(\mathrm{K} - \frac{d'}{d}\right)(d - d')}{\mathrm{K}} \quad (8)$$

Example 3.—Taking the same bending moment as before, that is, $\mathrm{M} = 324{,}000$ lb. in., design a ferro-concrete beam with double reinforcements.

In this case the dimensions and reinforcement ratios have first to be fixed and then checked, to see if they comply with the conditions. Try a beam 14 in. deep and 10 in. wide. Assume six $\frac{3}{4}$ in. diameter bars in tension which equals 2·65 sq. in. and gives $r = 0{\cdot}0189$; and four $\frac{3}{4}$ in. diameter bars in compression, which equals 1·77 sq. in. and makes $r' = 0{\cdot}0127$. Also assume that axis of compression bars is 2 in. below compression surface of beam, hence $d' = 2$ in. and $d = 14$ in. Take $f_c = 600$ lb. per sq. in. and $n = 15$.

From equation (3): $K^2 + 2nK(r + r') - 2n\left(r + r'\frac{d'}{d}\right) = 0.$

$$K^2 + 0{\cdot}946K - 0{\cdot}621 = 0.$$
$$K = 0{\cdot}446.$$

From equation (1): $f_s = \frac{f_c n(1 - K)}{K}$

$$= 11{,}100 \text{ lb. per sq. in.}$$

From equation (6) $f'_s = \frac{f_c n\left(K - \frac{d'}{d}\right)}{K}$

$$= 6{,}102 \text{ lb. per sq. in.}$$

The position of the compression resultant of concrete is $\frac{Kd}{3}$ from upper surface $= 2{\cdot}09$ in. and the position of compression resultant of steel from upper surface $= 2{\cdot}00$ in.

The moment of resistance of above beam is

$$f_s a_s h = 11{,}100 \times 2{\cdot}65 \times 12$$
$$= 353{,}000 \text{ in. lb.}$$

Hence the above design is suitable.

It may be noted that the allowable stress in compression bars cannot exceed n times the safe allowable compressive stress in the concrete, that is, $15 \times 600 = 8{,}000$ lb./sq. in., or the safe concrete stress would be exceeded.

Tee Beams—The treatment of ferro-concrete tee beams is similar to that given for single reinforced beams of rectangular cross-section. If the neutral axis is either in or at the base of the flange the treatment is the same. If the neutral axis is in the web, which is usually the case, the problem is more involved, and for the sake of simplification certain small quantities are generally omitted. From the assumption that normal plane sections remain plane after the application of the load equation (1) is obtained, that is

$$\frac{f_c}{f_s} = \frac{K}{n(1 - K)} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In considering the longitudinal force in the concrete it is usual to neglect the compression in the flange, and assume that the total compressive force is withstood by the web.

The stress at the top of the web is f_c and at the base it is

$$\frac{f_c(Kd - t)}{Kd} \text{ hence } \frac{1}{2}\left(f_c + \frac{f_c(Kd - t)}{Kd}\right)bt = f_s a_s$$

$$\frac{f_c}{f_s} = \frac{2Kda_s}{(2Kd - t)bt}$$

Substituting for $\frac{f_c}{f_s}$ from (1),

$$\frac{K}{n(1-K)}=\frac{2Kda_s}{(2Kd-t)bt}$$

$$K(2nda_s+2dbt)=2nda_s+bt^2$$

$$K=\frac{nr+\frac{1}{2}\left(\frac{t}{d}\right)^2}{nr+\frac{t}{d}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where r is the reinforcement ratio and $n=\frac{E_s}{E_c}$.

Next assume that the arm of the couple constituting the resisting moment M is equal to $d-\frac{t}{2}$, hence

$$M=f_s a_s\left(d-\frac{t}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

or

$$M=f_c\left(1-\frac{t}{2Kd}\right)\left(d-\frac{t}{2}\right)bt \quad . \quad . \quad . \quad . \quad (11)$$

The error involved by the approximations made in (9), (10), and (11) would in most cases be negligible, but in cases where the

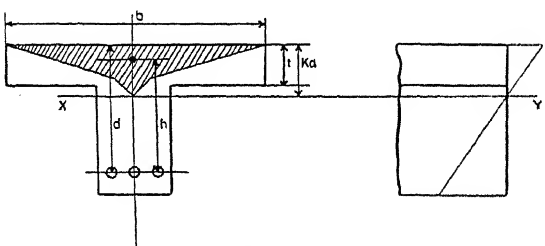


FIG. 72.

neutral axis is some distance below the base of the flange the design should be checked by a more exact method. This is best done by drawing the first modulus figure for the portion of the section in compression as shown in Fig. 72. If A is the area of this figure, then f_cA should be equal to $f_s a_s$ and the moment of resistance M is given by both $M=f_s a_s h$ and $M=f_c Ah$, where h is the distance of the centre of gravity of the modulus figure from the axis of the bars.

Example 4.—Design a reinforced tee beam, having given that $M=324{,}000$ in. lb. $f_c=600$ lb. per sq. in., $f_s=16{,}000$ lb. per sq. in., and $n=15$. Assume $t=4$ in. and $d=12$ in.

From equation (1):

$$K = \frac{n\frac{f_c}{f_s}}{1 + n\frac{f_c}{f_s}} = 0{\cdot}36.$$

From equation (9) $r = 0{\cdot}00671$ and from equation (11) $b = 23{\cdot}7$ in. and $a_s = rbd = 1{\cdot}91$ sq. in. or a_s from $(10) = 2{\cdot}02$ sq. in. Put in three $\frac{15}{16}$ in. diameter bars, the total area of which $= 2{\cdot}07$ sq. in.

EXAMPLES. IV

(1) Find the moment of inertia of a rectangle 8 in. deep and 6 in. wide about a diagonal.

(2) The tram-rail section shown in Fig. 65 is 7 in. deep, the other dimensions being in proportion. It supports a central load of 10 tons over a span of 3 ft. 6 in. Find the max. stress in the material.

(3) A parabola with its axis vertical is 6 in. high and 4 in. wide at the base. Find its area and the distance of the centre of gravity from the base.

(4) A trapezium has a base 6 in. long, the parallel side being 4 in. long and the height 3 in. Find the distance of the centre of gravity from the base and the moment of inertia about the base.

(5) A reinforced concrete beam is 10 in. wide and 18 in. deep to the centre of the reinforcing bars. Find the safe uniformly distributed load on a span of 12 ft. if the beam is simply supported at the ends. Take $f_s = 16{,}000$ lb. per sq. in., $f_c = 600$ lb. per sq. in., and $n = 15$.

(6) If the beam in the above example is additionally reinforced on the compression side with four $\frac{1}{2}$ in. diameter bars 2 in. from the compression face, find the safe uniformly distributed load.

(7) A ferro-concrete beam is to rest on supports 15 ft. apart and to carry a load of 2,000 lb. per ft. run. Find the relation between the breadth b and depth d and also the reinforcement ratio r. Take $f_s = 16{,}000$ lb. per sq. in., $f_c = 600$ lb. per sq. in., and $n = 15$.

(8) A reinforced-concrete tee beam has a cross-piece 2 ft. 0 in. wide and 4 in. thick, the web being 16 in. deep. The reinforcement consists of three $\frac{3}{4}$ in. dia. round bars of high-tensile steel placed 2 in. from the lower face. Find the stress in the steel and the moment of resistance of the section if $f_c = 600$ lb. per sq. in. and $n = 15$.

(9) A flitched beam consists of two timber beams 12 in. deep and 3 in. thick with a steel plate $\frac{1}{2}$ in. thick and 10 in. deep placed symmetrically between them, the whole being firmly bolted together. What central load will the beam support over a span of 12 ft. if the stress in the timber is 1,000 lb. per sq. in.? What is the corresponding maximum stress in the steel? Take the ratio of the moduli as 20.

(10) A reinforced-concrete tee beam has a flange 3 ft. 6 in. wide and $3\frac{1}{2}$ in. deep, the web being 10 in. wide. If the centre of the reinforcement is 16 in. below the top face, calculate its area. Take $f_s = 16{,}000$ lb. per sq. in., $f_c = 600$ lb. per sq. in., and $n = 15$.

(11) Suppose that three beams or planks of the same material are laid side by side across a span of 100 in. and a load of 600 lb. is laid across them at the middle so that they all bend together. The beams are 6 in. wide, but two are 3 in. and one 6 in. deep. What will be the load carried by each beam, and what will be the maximum fibre stress each? (A.M.I.C.E.)

DEFLECTION OF BEAMS CAUSED BY BENDING

49. Elastic Deflections—The amount by which a beam may be distorted from its unloaded form without stressing any of the material beyond its elastic limit is very small in such beams as are used in practice, but it is desirable that the engineer should be in a position to calculate the amount of the deflection when the necessary data are provided. These include: The span or length; the moment of inertia of the section; the elastic quality of the material as expressed by the modulus E, and the deflecting loads.

In order to deduce the general formula for the calculation of the deflection, the simple case of a cantilever loaded at its free end will be considered. The case is shown in Fig. 73, where OB indicates the position of the neutral line or longitudinal axis of the beam before bending and OB′ after bending. The neutral line is referred to the rectangular co-ordinates OX and OY, and Y denotes the deflection of the line at any point distant x from the fixed free end. In Art. 41 it was shown that $\frac{1}{R} = \frac{M}{EI}$, where R is the radius of curvature of the neutral line, M = the bending moment, I = the moment of inertia of the section, and E = Young's Modulus, and to obtain the general formula it is necessary to deduce an expression connecting the curvature $\frac{1}{R}$ with the co-ordinates of x and y of the neutral line. The slope of the line distant x from the free end is ϕ and $\tan \phi = \frac{dy}{dx}$. Differentiate both sides of the equation with respect to the length of the line denoted by s:

$$\frac{d^2y}{dx^2}\frac{dx}{ds} = \sec.^2 \phi \frac{d\phi}{ds}$$

but

$$\frac{dx}{ds} = \cos \phi$$

hence $$\frac{d\phi}{ds}=\frac{\frac{d^2y}{dx^2}}{\sec^3\phi}$$

also $\sec^3\phi=(1+\tan^2\phi)^{\frac{3}{2}}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$

therefore $\frac{d\phi}{ds}=\frac{1}{R}=\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}$ and R may be positive or negative according as $\frac{d^2y}{dx^2}$ is positive or negative.

The deflection of beams in practice is small and the position of the axes OX and OY are chosen so that $\frac{dy}{dx}$ is small, and hence $\left(\frac{dy}{dx}\right)^2$ is of the second order of small quantities and can be neglected.

The formula accordingly reduces to $\frac{1}{R}=\frac{d^2y}{dx^2}$ and substituting for $\frac{1}{R}$ then $\frac{d^2y}{dx^2}=\frac{M}{EI}$.

The formula can also be obtained as follows: Referring again to Fig. 73, the curvature at a section of the beam distant x from the free end is $\frac{1}{R}=\frac{d\phi}{ds}$. Substituting dx for ds the equation becomes $\frac{1}{R}=\frac{d\phi}{dx}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=\frac{M}{EI}$.

If the right-hand side of the equation can be written in terms of x and the equation integrated twice—care being taken to insert the constants of integration—the result will be an equation connecting x and y, from which the deflection y at any point distant x along the beam may be calculated. The assumptions involved in the equation are the same as those made in the ordinary theory of bending, but, in addition, it is assumed that the slope of the beam is everywhere small, so that the square of the slope can be neglected.

Suppose the cantilever (Fig. 73) carries a weight W at its free end. The bending moment at any point distant x from the fixed end is $-W(l-x)$, hence $EI\frac{d^2y}{dx^2}=-W(l-x)$.

The equation can be integrated because E, I, W, and l are

constants. $EI\frac{dy}{dx} = -Wlx + \frac{Wx^2}{2} + C'$, where C′ is the constant of integration. The slope of the beam at the fixed end is 0, so that when $x = 0$ $\frac{dy}{dx} = 0$ and therefore $C' = 0$.

Integrating again, $EI\,y = -\frac{Wlx^2}{2} + \frac{Wx^3}{6} + C''$, when $x = 0$, $y = 0$, and therefore $C'' = 0$

and
$$y = -\frac{Wx^2}{2EI}\left(l - \frac{x}{3}\right).$$

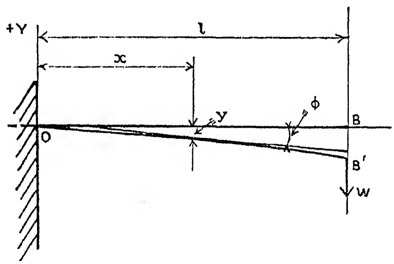


FIG. 73.

This is the equation of the shape of the beam and the deflection Δ at the free end is obtained by putting $x = l$,

hence
$$\Delta = -\frac{Wl^2}{2EI}\left(l - \frac{l}{3}\right)$$
$$= -\frac{Wl^3}{3EI}$$

In the case of a cantilever carrying a uniform load of w units per unit of length the treatment is similar to that just given. Referring to Fig. 73, the bending moment at any point distant x from the fixed end is $-\frac{w(l-x)^2}{2}$.

Hence
$$EI\frac{d^2y}{dx^2} = -\frac{w(l-x)^2}{2} = -\frac{wl^2}{2} + wlx - \frac{wx^2}{2}$$
$$EI\frac{dy}{dx} = -\frac{wl^2x}{2} + \frac{wlx^2}{2} - \frac{wx^3}{6} + C'.$$

Since
$$\frac{dy}{dx} = 0 \text{ when } x = 0,\ C' = 0.$$
$$EI\,y = -\frac{wl^2x^2}{4} + \frac{wlx^3}{6} - \frac{wx^4}{24} + C'',$$

when $x=0$, $y=0$, and hence $C''=0$ and the deflection Δ at the free end is

$$\Delta=-\frac{wl^4}{8EI}=-\frac{Wl^3}{8EI}$$

where $W=wl=$ the total load.

50. Beams Freely Supported at the Ends—In Fig. 74 is shown the case of a beam, AB, freely supported at its ends in a horizontal position and loaded with a concentrated load W at its mid-length. The bending moment at any section distant x from the left-hand support A is $+\frac{W}{2}x$.

Hence
$$EI\frac{d^2y}{dx^2}=+\frac{W}{2}x$$

$$EI\frac{dy}{dx}=\frac{W}{4}x^2+C',$$

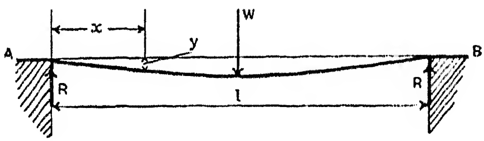


FIG. 74.

when $x=\frac{l}{2}$, $\frac{dy}{dx}=0$, and hence $C'=-\frac{Wl^2}{16}$

$$EI\,y=\frac{Wx^3}{12}-\frac{Wl^2x}{16}+C''$$

when $x=0$, $y=0$, and hence $C''=0$.

The deflection Δ at the mid-length is obtained by putting $x=\frac{l}{2}$, hence $\Delta=-\frac{1}{48}\frac{Wl^3}{EI}$.

If the beam is loaded with a uniform load of w units per unit of length the bending moment at any section distant x from A is

$$+\frac{wl}{2}x-\frac{wx^2}{2}$$

Hence
$$EI\frac{d^2y}{dx^2}=\frac{wlx}{2}-\frac{wx^2}{2}$$

The treatment in this case is similar to the last and the central deflection Δ reduces to $\Delta=-\frac{5}{384}\frac{Wl^3}{EI}$ where $W=wl$, the total load.

The last two results can be simply deduced from the results for cantilevers. For consider the case of the beam supporting

the central load W (Fig. 74). It will be noted that the two halves of the beam can be looked upon as cantilevers with their fixed ends under the load and pressed upwards with loads of $\frac{W}{2}$ at their free ends A and B. The length of one of the two cantilevers is $\frac{l}{2}$, and hence, dropping the sign, the deflection Δ is given by

$$\Delta = \frac{1}{3}\left\{\frac{\frac{W}{2}\left(\frac{l}{2}\right)^3}{EI}\right\} = \frac{1}{48}\,\frac{Wl^3}{EI}.$$

The second case where the beam supports a uniformly distributed load can be similarly treated, but it is first necessary to note that the deflection of a beam at any point caused by a number of loads is equal to the sum of the deflections at the point due to each separate load.

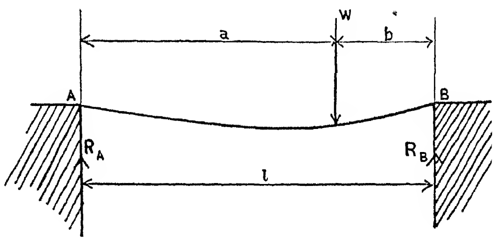


Fig. 75.

The length of each cantilever, as before, is $\frac{l}{2}$ and the ends are pressed upwards with a force of $\frac{wl}{2}$, but in addition they support a uniformly distributed load of w units per unit of length. Hence, considering one cantilever, the deflection

$$\Delta = \frac{1}{EI}\left\{\frac{\frac{wl}{2}\left(\frac{l}{2}\right)^3}{3} - \frac{\frac{wl}{2}\left(\frac{l}{2}\right)^3}{8}\right\}$$

$$= \frac{5}{384}\,\frac{Wl^3}{EI} \text{ where } W = wl.$$

The case of the deflection of a beam freely supported at its ends in a horizontal position and supporting a non-central load will now be considered. This case is shown in Fig. 75, and the reactions at the supports A and B are denoted by R_A and R_B respectively. The position of the load divides the beam into

two unequal lengths denoted by a and b in the figure and the maximum deflection will evidently occur at a point along the longer of these lengths, that is, along the length a. The bending moment at any section between the support A and the load and distant x from A is equal to $+ R_A x$.

Hence
$$EI\frac{d^2y}{dx^2} = + R_A x$$
$$EI\frac{dy}{dx} = \frac{R_A x^2}{2} + C'.$$

If the slope of the beam under the load is denoted by i, then when $x=a$, $\frac{dy}{dx} = i$, and therefore $C' = EIi - \frac{R_A a^2}{2}$

$$EI\frac{dy}{dx} = \frac{R_A x^2}{2} + EIi - \frac{R_A a^2}{2} \quad . \quad . \quad . \quad (1)$$

$$EIy = \frac{R_A x^3}{6} + EIix - \frac{R_A a^2 x}{2} + C'' \quad . \quad . \quad . \quad (2)$$

$y = 0$ when $x = 0$ and hence $C'' = 0$ and the expression for the deflection Δ under the load is obtained by putting $x = a$.

Thus
$$EI\Delta = -\frac{1}{3}R_A a^3 + EIia \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Next consider any section between the support B and the load and distant x from B. The expression for the deflection under the load when deduced with respect to this portion of the beam will be similar to equation (3), but the sign of the slope i will be changed thus:

$$EI\Delta = -\frac{1}{3}R_B b^3 - EIib \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the slope i is eliminated from equations (2) and (3) an expression for the deflection Δ is obtained:

$$EI(\Delta a + \Delta b) = -\frac{1}{3}(R_A a^3 b + R_B b^3 a)$$

$$Ra = \frac{b}{a+b}W \text{ and } Rb = \frac{a}{a+b}W.$$

Hence
$$EI\Delta(a+b) = -\frac{1}{3}ab\left(\frac{ab^2 + a^2 b}{a+b}\right)W$$

$$\Delta = -\frac{Wa^2b^2}{3EI(a+b)}$$

The maximum deflection will occur where $\frac{dy}{dx} = 0$ and its position is determined by equating equation (1) to zero and

substituting the expression for i obtained from (3) and (4). From equations (3) and (4)

$$\mathrm{EI}i = \frac{\mathrm{R_A}a^3 - \mathrm{R_B}b^3}{3(a+b)}$$

and equation (1) becomes

$$\frac{\mathrm{R_A}x^2}{2} - \frac{\mathrm{R_A}a^2}{2} + \frac{\mathrm{R_A}a^3 - \mathrm{R_B}b^3}{3(a+b)} = 0.$$

The solution of this equation gives

$$x = \frac{1}{\sqrt{3}}\sqrt{a^2 + 2ab} = \frac{1}{\sqrt{3}}\sqrt{(a+b)^2 - b^2}$$

When the load is at the mid-length of the beam $b = \frac{1}{2}l$ and $x = \frac{l}{2}$, also when the load is close to the support B $b = 0$ and $x = \frac{1}{\sqrt{3}}\,l$, hence the maximum deflection always occurs within points distant $(0{\cdot}577 - 0{\cdot}5)l$ from the mid-length of the beam, that is, within a region about the mid-length $= 15{\cdot}4$ per cent. of l. Substituting the above value of x in equation (2)

$$y = \frac{\mathrm{W}b(a^2 + 2ab)^{\frac{3}{2}}}{9\sqrt{3}\,\mathrm{EI}(a+b)} = \text{max. deflection.}$$

The maximum deflection of a beam freely supported at its ends and carrying a number of concentrated loads can be found by first deducing the expressions for the slope for each separate load. Since the slope vanishes where the deflection is a maximum, the position is found by adding the expressions together and finding the value of x, which makes the sum vanish. If this value is substituted in the general equation for the deflection, an expression for the maximum deflection will result. This method is usually tedious, and graphical solutions are as a rule preferable.

51. Beams Fixed at Ends—A simple case is shown in Fig. 76. The beam is supported horizontally and can be considered as having its ends exactly fitting into holes in the vertical walls A and B, so that when the beam is bent the slope at the ends cannot alter but the ends are free to slide in a horizontal direction. In order to prevent change of slope at the ends the supports exert couples which in this case are equal and denoted

by M. The bending moment at any section between the left-hand support and the load and distant x from A is

$$Rx+M=\frac{Wx}{2}+M,$$

hence
$$EI\frac{d^2y}{dx^2}=\frac{Wx}{2}+M.$$

Integrating, $EI\frac{dy}{dx}=\frac{Wx^2}{4}+Mx+C'$ when $x=0$,

$$\frac{dy}{dx}=0, \text{ and hence } C'=0.$$

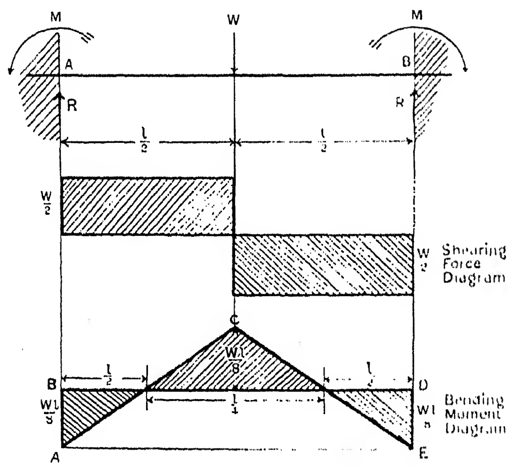


FIG. 76.

Also when $x=\frac{l}{2}$, $\frac{dy}{dx}=0$ and $\frac{Wl^2}{16}+\frac{Ml}{2}=0$, whence $M=-\frac{Wl}{8}$.

Integrating, $EIy=\frac{Wx^3}{12}-\frac{Wlx^2}{16}+C''$, $y=0$ when $x=0$,

hence $C''=0$. The maximum deflection Δ is obtained by putting

$$x=\frac{l}{2} \text{ and } \Delta=\frac{W}{EI}\left(\frac{l^3}{96}-\frac{l^3}{64}\right)=-\frac{Wl^3}{192EI}.$$

In Fig. 76 the shearing-force and bending-moment diagrams are shown drawn for the case. The bending-moment diagram is the sum of the diagrams ACE and ABDE; ACE would be the bending-moment diagram if the beam were freely supported, and ABDE is the diagram for the equal couples M at the ends of the

beam. The line BD is the base line for the bending-moment diagram, and it will be noted that the bending moment changes signs at points distant $\frac{l}{4}$ from either end of the beam. These points where the bending moment vanishes are called points of inflection, and the beam may be looked upon as made up of three distinct parts : two cantilevers of lengths $\frac{l}{4}$ freely supporting at their free ends a beam of length $\frac{l}{2}$ which carries a central load W. The portion of length $\frac{l}{2}$ supporting the load can further be considered as two cantilevers each of length $\frac{l}{4}$ fixed under the load and subjected to an upward thrust $\frac{W}{2}$ at their free ends. The central deflection can therefore be obtained by considering the beam as made up of four cantilevers of lengths $\frac{l}{4}$ each supporting a load of $\frac{W}{2}$ at its free end. The central deflection is given by twice the deflection of any one of the cantilevers or

$$\Delta=2\left\{\frac{\frac{W}{2}\left(\frac{l}{4}\right)^3}{3\,EI}\right\}=\frac{Wl^3}{192\,EI}.$$

The central deflection is also given by the algebraic sum of the deflections due to the central load W assuming the beam to be freely supported and the deflection due to the equal couples M at the ends. The central deflection Δ of a freely supported beam carrying a central load W is $\Delta_1=-\frac{Wl^3}{48\,EI}$. The deflection of a beam subjected to a constant bending moment M can be deduced by integrating the equation $EI\frac{d^2y}{dx^2}=M=-\frac{Wl}{8}$. It may be noted that since M is constant $\frac{d^2y}{dx^2}$ and hence R the radius of curvature is constant, and hence the beam is bent into the shape of an arc of a circle. Referring to Fig. 77,

$$\Delta_2\times 2R=\left(\frac{l}{2}\right)^2, \text{ neglecting terms containing } (\Delta_2)^2,$$

hence $$\Delta_2=\frac{l^2}{8R}.$$

Considering now the signs, R is negative because $\frac{d^2y}{dx^2}$ is negative and also $M = -\frac{Wl}{8}$,

therefore
$$-\frac{1}{R} = \frac{M}{EI} = -\frac{Wl}{8EI}$$

and
$$\Delta_2 = +\frac{Wl^3}{64EI}$$

$$\Delta = \Delta_1 + \Delta_2 = \frac{Wl^3}{EI}\left(\frac{1}{64} - \frac{1}{48}\right)$$
$$= -\frac{1}{192}\frac{Wl^3}{EI}$$

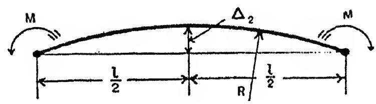


FIG. 77.

The deflection of a beam fixed at the ends as in the last case, but carrying a uniformly distributed load of w units per unit of length, will next be considered. The solution is very similar to the last and is illustrated in Fig. 78. The bending moment at any section distant x from the left-hand support is

$$\frac{wl}{2}x - \frac{wx^2}{2} + M$$

hence
$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2} + M,$$

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + Mx + C'.$$

When $x = 0$ $\frac{dy}{dx} = 0$ and therefore $C' = 0$.

Also when $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$ and $M = -\frac{1}{12}wl^2$,

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wlx^2}{24} + C''.$$

$C'' = 0$ and the central deflection is obtained by putting $x = \frac{l}{2}$

$$y = \frac{wl^4}{EI}\left\{\frac{1}{96} - \frac{1}{384} - \frac{1}{96}\right\} = -\frac{Wl^3}{384\,EI}$$
where $W = wl$.

At the points of inflection $\frac{d^2y}{dx^2} = 0$ and therefore

$$\frac{lx^2}{2} - \frac{x^2}{2} + \frac{l^2}{12} = 0$$

$$6x^2 - 6lx + l^2 = 0$$

$$x = \left.\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right\} l$$

$$= 0{\cdot}211\, l \text{ or } 0{\cdot}789\, l.$$

The shearing-force and bending-moment diagrams are shown drawn in Fig. 78 with base line AB in each case.

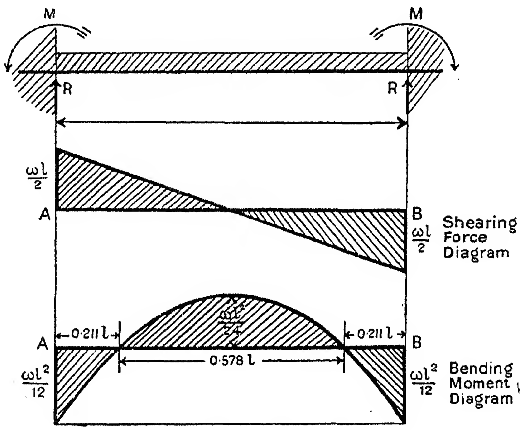


FIG. 78.

The case of a beam fixed at both ends and supporting a non-central load W is shown in Fig. 79. Let M be the total bending moment at any section distant x from the left-hand support, then

$$\text{EI}\,\frac{d^2y}{dx^2} = \text{M}.$$

Integrate this between the limits $x = 0$ and $x = l$,

$$\text{EI}\left(\frac{dy}{dx}\right)_0^l = \int_0^l \text{M}dx = 0.$$

Thus the total area of the bending-moment diagram is zero or the area of the diagram if the beam were freely supported is equal to that for the couples at the ends. Also

$$\text{EI}\int_0^l \frac{d^2y}{dx^2}x\,dx = \int_0^l \text{M}x\,dx$$

$$\text{EI}\,{}_0^l\!\left(x\frac{dy}{dx} - y\right) = \int_0^l \text{M}x\,dx = 0.$$

That is, the moment of the complete bending-moment diagram about the supports is zero. The above analysis will apply whatever the loading, and since the bending-moment diagram for the end couples is always a trapezium, the values of the couples which are the heights of the ends of the trapezium can always be found. The area of the bending-moment diagram for the freely supported beam is $\frac{R_A a^2}{2}+\frac{R_A ab}{2}$ and for the couples it is $\frac{(M_A+M_B)(a+b)}{2}$, hence $R_A a^2+R_A ab=(M_A+M_B)(a+b)$,

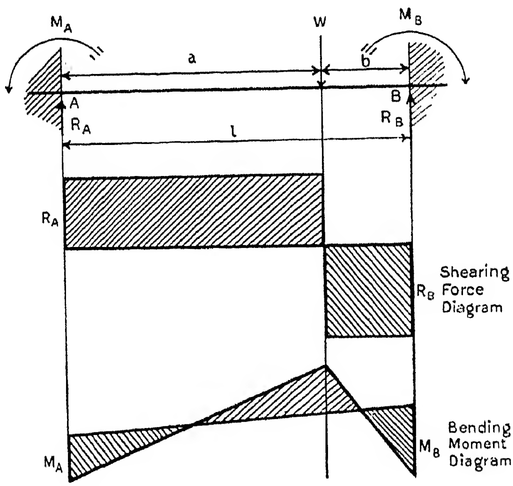


FIG. 79.

$$\text{and since } R_A=\frac{bW}{a+b} \text{ then } \frac{bW(a^2+ab)}{(a+b)^2}=M_A+M_B \quad . \quad . \quad . \quad (1)$$

Also, taking moments about the end A,

$$\frac{R_A a^2}{2}+\frac{R_A ab}{2}\Bigg)\left(\frac{l}{2}+\frac{a-\frac{l}{2}}{3}\right)=\frac{M_A(a+b)l}{6}+\frac{2M_B(a+b)l}{6}$$

$$\frac{2bW(a^2+ab)}{(a+b)^3}(2a+b)=M_A+2M_B \quad . \quad . \quad . \quad (2)$$

Solving equations (1) and (2),

$$M_A=\frac{Wab^2}{l^2} \text{ and } M_B=\frac{Wa^2b}{l^2}$$

The shearing-force and bending-moment diagrams are shown drawn in Fig. 79. In order to find the reactions R_A and R_B take moments about the ends A and B of the beam. About the end A

$$Wa + M_B - R_B l - M_A = 0,$$

and about B

$$Wb + M_A - R_A a - M_B = 0.$$

These equations give

$$R_A = \frac{b^2(3a+b)}{(a+b)^3} W \text{ and } R_B = \frac{a^2(a+3b)}{(a+b)^3} W.$$

The deflection under the load and also the maximum deflection can be found by the method already given, but another method will be used which in this and some other cases is the simpler. Take the origin at the load and let the bending moment at any section distant x to the right of the load be denoted by M. Then

$$EI \frac{d^2y}{dx^2} = M \text{ and } M = R_B(b-x) - M_B.$$

Therefore $$EI \int_0^b \frac{d^2y}{dx^2} x dx = \int_0^b \{R_B(b-x) - M_B\} x dx$$

$$EIy = \left\{ \frac{Wa^2b^3(a+3b)}{6(a+b)^3} - \frac{Wa^2b^3}{2(a+b)^2} \right\}$$

$$y = - \frac{Wa^3b^3}{3EI(a+b)^3}$$

The maximum deflection will occur in the longer portion of the beam between the support and the load, that is, in the length a. To find the distance from A to the point of the maximum deflection take the origin at A. The bending moment at any section distant x from A between A and the load is $R_A x - M_A$,

hence $$\int \frac{d^2y}{dx^2} dx = \int (R_A x - M_A) dx$$

$$\left[\frac{dy}{dx}\right] = \left[\frac{R_A x^2}{2} - M_A x\right]$$

The value of $\frac{dy}{dx}$ is zero where the deflection is a maximum and if the limits in this equation are $x = 0$ and $x =$ the distance of the point of maximum deflection from A the left-hand side of the equation vanishes and $\frac{R_A x^2}{2} - M_A x = 0$,

$$\therefore \ x = \frac{2M_A}{R_A} = \frac{2a}{3a+b}(a+b).$$

If $b=0$, $x=\frac{2}{3}(a+b)$, so that the point of maximum deflection cannot be more than $\frac{1}{6}l$ from the mid-length of the beam.

In order to find an expression for the maximum deflection, take the origin as before at the end A. The bending moment at any section distant x from A and between A and the load is

$$R_A x - M_A,$$

hence
$$EI\int_0^c \frac{d^2y}{dx^2}x\,dx = \int_0^c (R_A x^2 - M_A x)dx$$

where c is the distance of the point of maximum deflection from A

$$-EIy = \frac{R_A c^3}{3} - \frac{M_A c^2}{2}$$

substituting
$$c = \frac{2a(a+b)}{3a+b}$$

$$y = -\frac{2Wa^3b^2}{3EI(3a+b)^2}$$

The positions of the points of inflection can be found from the fact that the value of $\frac{d^2y}{dx^2}$ is zero there. It may also be noted from the bending-moment diagram that the points of inflection are always on either side of the load. Considering that portion of the beam of length a,

$$EI\frac{d^2y}{dx^2} = R_A x - M_A = 0$$

$$\text{and } x = \frac{M_A}{R_A} = \frac{a}{3a+b}(a+b)$$

also the distance of the other point of inflection from B is by symmetry

$$\frac{b}{a+3b}(a+b).$$

The method just described of finding the deflection is generally to be preferred in the cases of cantilevers and beams fixed at one or both ends. For instance, let M be the bending moment at a section of a beam distant x from some origin. Then

$$EI\int \frac{d^2y}{dx^2}x\,dx = \int Mx\,dx$$

$$EI\left[x\frac{dy}{dx} - y\right] = \int Mx\,dx$$

If the origin can be chosen so that the left-hand side of the expression reduces to y, an expression for y is easily deduced, because the right-hand side is simply the moment of the bending-moment diagram about the origin. Let A denote the area of the portion of the bending-moment diagram involved in the expression and $\bar{x}$ the distance of its centre of gravity from the origin,

then $$\int \mathrm{M}x dx = \mathrm{A}\bar{x}$$

and $\mathrm{EI}\left[x\frac{dy}{dx} - y\right] = \mathrm{A}\bar{x}$, which reduces to $\mathrm{EI}y = \mathrm{A}\bar{x}$ if the limits can be suitably chosen.

If the moment of inertia is variable, as is the case for beams of uniform strength or approximating to it, the expression becomes

$$\mathrm{EI}y = \int \frac{\mathrm{M}}{\mathrm{I}} x dx$$
$$= \mathrm{A}_1\bar{x}_1,$$

where A_1 is the area of the $\frac{\mathrm{M}}{\mathrm{I}}$ diagram and $\bar{x}_1$ the distance of its centre of gravity from the origin. If the M and $\frac{\mathrm{M}}{\mathrm{I}}$ diagrams are irregular figures the areas are as a rule best found by means of a planimeter, and the position of the centres of gravity by cutting the figures out in cardboard or thin sheet metal and balancing on a knife-edge.

Example—A vertical timber post 20 ft. long is 12 in. diameter from the fixed end to the mid-length and 6 in. diameter from the mid-length to the free end. Find the deflection at the free end due to a horizontal pull of 1,000 lb. applied 5 ft. from the free end ($\mathrm{E} = 1{,}200{,}000$ lb./sq. in.).

Take the origin at the free end. Then the bending moment between the free end and the load is zero and at any section distant x from the free end and between the fixed end and the load it is $1{,}000\,(x - 60)$. The moment of inertia of the 12 in. diameter portion is $\frac{\pi}{4}(6)^4 = 1{,}020$ in.^{4} units, and of the 6 in. diameter portion $\frac{\pi}{4}(3)^4 = 63{\cdot}8$ in.^{4} units.

Hence $$\mathrm{E}\int_0^{240} \frac{d^2y}{dx^2}\, x dx = -\int_{60}^{120} \frac{1{,}000(x-60)}{63{\cdot}8} x dx - \int_{120}^{240} \frac{1{,}000(x-60)}{1{,}020} x dx$$

$$Ey = -\left[\frac{1{,}000\left(\frac{x^2}{2} - 60x\right)}{63\cdot8}\right]_{60}^{120} - \left[\frac{1{,}000\left(\frac{x^2}{2} - 60x\right)}{1{,}020}\right]_{120}^{240}$$

$$= -\ 42{,}400$$

$$\text{and } y = -\frac{42{,}400}{1{,}200{,}000} = -\ 0\cdot0353 \text{ in.}$$

If the load were 5,000 lb. instead of 1,000 lb., the deflection at the free end would be $0\cdot0353 \times 5 = 0\cdot177$ in.

52. Propped Beams—Consider the case of a beam of length

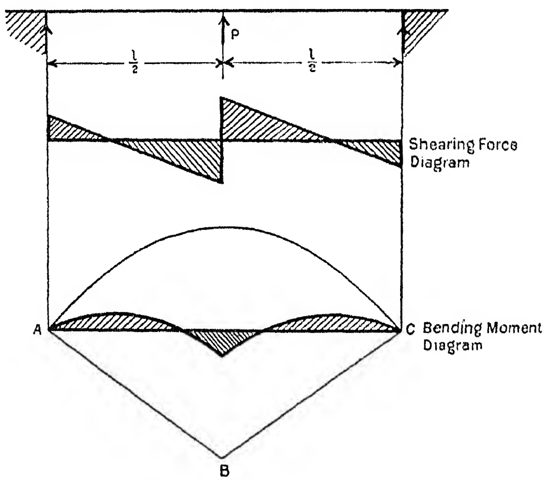


Fig. 80.

l supporting a uniform load of w units per unit of length and resting horizontally on three equally spaced supports at the same level. First assume that the beam rests freely on the two outer supports, a central deflection will result; call this Δ_1. The middle point is now to be lifted up an amount Δ_1, and the force required to lift it will be the pressure P on the prop or central support. The movement is to be considered as the central deflection of a freely supported beam loaded in the middle or $\Delta_1 = \frac{Pl^3}{48EI}$. The value of Δ_1 is given by the relation

$$\Delta_1 = \frac{5wl^4}{384\ EI}$$

Hence
$$\frac{Pl^3}{48EI} = \frac{5wl^4}{384EI}$$

$P = \frac{5}{8}wl = \frac{5}{8}W$ where $W = wl =$ total load, and the pressures on the outer supports are each $\frac{3}{16}W$. The shearing-force and bending-moment diagram for this case are shown drawn in Fig. 80.

The problem of finding the pressure on a prop applied to the free end of a loaded cantilever will next be considered. In Fig. 81, *a* is the bending-moment diagram for the loaded canti-

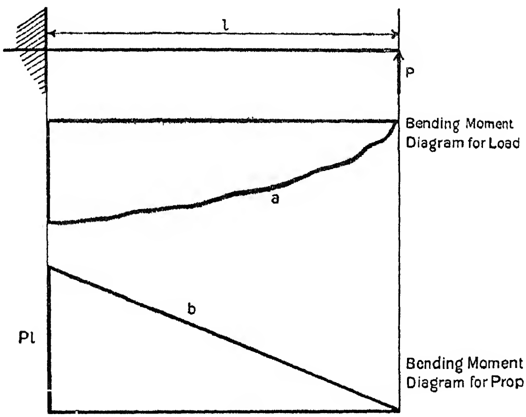


FIG. 81.

lever. Calling the bending moment distant x from the free end M, the deflection at the free end is obtained from

$$EI\int_0^l \frac{d^2y}{dx^2}xdx = \int_0^l Mxdx.$$

Hence if A is the area of the diagram and $\bar{x}$ the distance of its centre of gravity from the free end, $\Delta = -A\bar{x} =$ deflection at free end due to the load. The diagram *b* is the bending-moment diagram for the prop, hence $\delta = B\bar{y} =$ deflection due to the prop acting alone, where B is the area of the diagram for the prop and $\bar{y}$ the distance of its centre of gravity from the free end. $B\bar{y} = \frac{Pl^2}{2} \times \frac{2}{3}l = \frac{1}{3}Pl^3$, and since $\Delta + \delta = 0$, then $\frac{1}{3}Pl^3 - A\bar{x} = 0$, from which P can be obtained.

If the moment of inertia is variable $\Delta_1 = -A_1\bar{x}_1$ where A_1 is

the area of the $\frac{M}{I}$ diagram due to the load and $\bar{x}_1$ the distance of its centre of gravity from the free end. Also $\delta_1 = B_1\bar{y}_1$ where B_1 is the area of the $\frac{M}{I}$ diagram for P, assuming that $P = 1$ and $\bar{y}_1$ is the distance of its centre of gravity from the free end, δ_1 being the upward deflection at the free end caused by unit force at the prop. The value of P is accordingly given by $P = \frac{\Delta_1}{\delta_1}$.

53. Supports not in the same Horizontal Plane—In Fig. 82 is shown a beam freely supported at its ends, but the right-hand support B is below the left A by an amount y' where $\frac{y'}{l}$ is small. Let y be the deflection distant x from A when the supports are at the same level. The deflection at the section when B is lowered an amount y' is

$$\Delta = y + \frac{x}{l}y'.$$

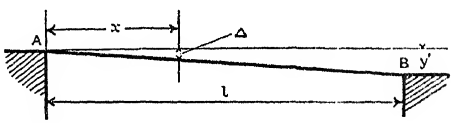


FIG. 82.

The problem can also be solved by twice integrating the equation $EI\frac{d^2y}{dx^2} = M$ where M is the bending moment on the beam at a point distant x from some origin. Integrating twice

$$EIy = \frac{Mx^2}{2} + C'x + C''.$$

For the case shown in Fig. 82 $y = 0$ when $x = 0$ and $y = y'$ when $x = l$, and these conditions determine the values of the constants C' and C''.

54. Work Done in Bending a Beam—When a beam is bent work is done by the bending couples which is stored up in the beam as strain energy. Let M be the bending moment on a beam between points where the slope is θ_1 and θ_2 respectively. Then W', the work done by M between these limits, is

$$W' = \int_{\theta_1}^{\theta_2} \frac{Md\theta}{2}$$

but $$\frac{d\theta}{ds} = \frac{1}{R} = \frac{d\theta}{dx} \text{ since } \theta \text{ is small,}$$

hence $$d\theta = \frac{dx}{R} = \frac{Mdx}{EI}$$

and $W' = \frac{1}{2EI}\int_{x_1}^{x_2} M^2 dx$ where the limits x_1 and x_2 correspond respectively with θ_1 and θ_2.

This equation can be used for finding the deflection of beams and is especially convenient in some cases where the loads are concentrated. For consider the case of a cantilever of length l supporting a load W at its free end. The bending moment M at any point distant x from the fixed end is $W(l-x)$, and if Δ is the deflection at the free end the work done when the load is applied is $\frac{W\Delta}{2}$,

hence $$\frac{W\Delta}{2} = \frac{1}{2EI}\int_0^l \{W(l-x)\}^2 dx.$$

$$W\Delta = \frac{1}{EI}\Big[W^2\Big(l^2x - lx^2 + \frac{x^3}{3}\Big)\Big]_0^l$$

$$\Delta = \frac{Wl^3}{3EI}$$

55. Graphical Method for Finding the Deflection—A graphical method for determining the shape of a beam from the bending-moment diagram will now be considered. In Fig. 83 is shown a short beam of length δx supporting a load of w units per unit of length and acted upon by positive shearing forces and bending moments in accordance with the choice of signs shown in Fig. 40. The load w is + ve downwards and δx is measured + ve to the right.

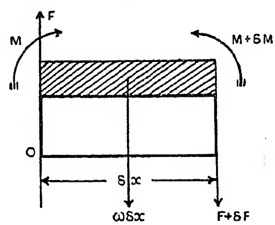


Fig. 83.

First equate the forces acting on the beam

$$w\delta x + F + \delta F - F = 0$$

$$\frac{\delta F}{\delta x} = -w,$$

and as δx becomes smaller and smaller in the limit

$$\frac{dF}{dx} = -w.$$

That is, the slope of the shearing-force diagram at any point is numerically equal to the load on the beam at the point. Now take moments about the point 0.

$$M + (F + \delta F)\delta x + \frac{w\delta x^2}{2} - (M + \delta M) = 0.$$

In the limit $\frac{dM}{dx} = F$. Thus the slope of the bending-moment diagram at any point is equal to the amount of the shear at the point. The above equalities cannot of course be applied at points where the shearing-force and bending-moment diagrams are discontinuous, because $\frac{dF}{dx}$ and $\frac{dM}{dx}$ are indeterminate at such points.

Combining the above relations $\frac{d^2M}{dx^2} = -w$ or $M = -\int\int w dx$, and it was shown in Art. 49 that $EIy = \int\int M dx$, so that the relation between M and w is similar to that between EIy and M. It was shown in Art. 39 how to draw the bending-moment diagram with the aid of the vector polygon. In Fig. 84 the bending-moment diagram is drawn for the continuous load there shown on the beam. The method adopted is to divide the continuous load into sections and treat each section as a concentrated load acting through the centre of gravity of the section. The depth of the first link polygon at any point is proportional to the bending moment on the beam at the point, and it was proved in Art. 39 that if the load scale is 1 in. $= w$ tons and the scale of the distance between each section of load is 1 in. $= s$ in. and the distance of the pole $0'$ from the line representing the load is q in., then the scale of the link polygon is 1 in. vertical depth $= wsq$ ton in. Thus the depth of the link polygon represents to scale the bending moment M on the beam, and hence it also represents to the same scale $-\int\int w dx$ between the proper limits.

Now treat the bending-moment diagram in a similar manner; that is, divide it into sections and treat the area of each section as acting through the centre of gravity of the section. Then draw the second vector and link polygons. Since the depth of the first link polygon represents to scale $-\int\int w dx$, it follows that the depth of the second link polygon represents to scale $\int\int M dx$, that is, EIy.

By a proof similar to that given in Art. 39 it can be shown that if the scale of the area of the bending-moment diagram in the second vector polygon is 1 in. $= n$ units of area of the bending-moment diagram in bending-moment inch units, and the scale of spacing out the sections is 1 in. $= s$ in., and the second polar

distance is q', then the scale of the EIy diagram is 1 in. $= nsq'$ units of EIy. Thus the scale of the deflection y from the second link polygon is 1 in. vertical depth $= \frac{nsq'}{\mathrm{EI}}$ in. of deflection. If the scale of the area of the bending-moment diagram in the second vector polygon is 1 in. $= m$ units of area in sq. in. instead of 1 in. $= n$ units of area in bending-moment inch units, then

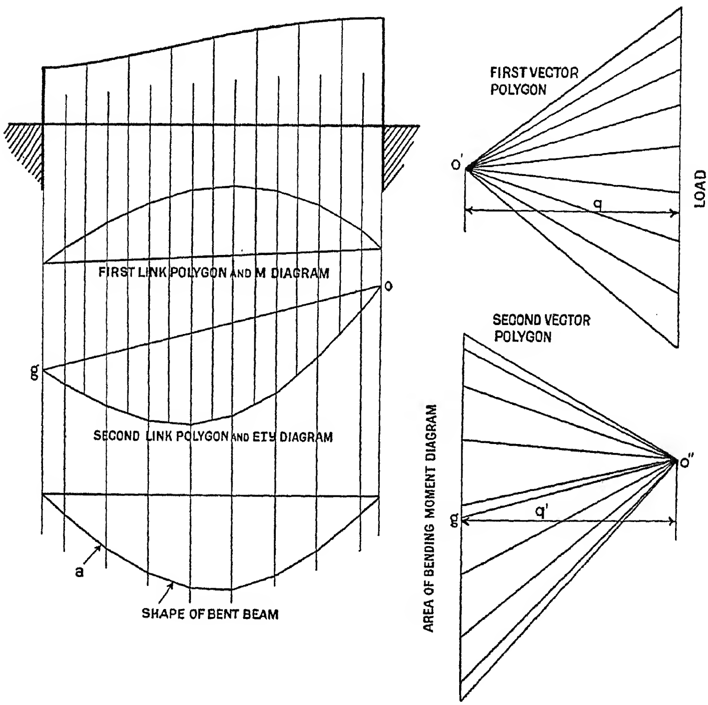


FIG. 84.

since 1 sq. in. of bending-moment diagram is equal to ws^2q ton in.^{2} $n = mws^2q$, hence 1 in. vertical depth is equal to $\frac{mws^3qq'}{\mathrm{EI}}$ in. of deflection. All the dimensions are in inches, and since w is in tons, E must be in tons per sq. in.

The figure a in Fig. 84 is the second link polygon drawn with the closing line go horizontal. A smooth curve drawn tangent to the straight lines which form the figure a represents to scale the shape of the bent beam.

If the moment of inertia I is variable, the shape of the bent beam is obtained by dividing the depths of the figure *a* at a number of points along its length by the corresponding moments of inertia of the beam at the points. A smooth curve drawn tangent to the straight lines forming the new figure will represent to scale the shape of the bent beam and the scale of the deflection will be 1 in. $= \dfrac{mws^3qq'}{E}$ in. of deflection.

It should be noted that if the areas of positive portions of the bending-moment diagrams are represented by downward vectors in the second vector polygon, then the areas of negative portions must be represented by upward vectors.

55A. Theorem of Three Moments—By proper application of what has been given, the bending-moment diagrams for most cases of loaded beams can be drawn, which makes it possible to

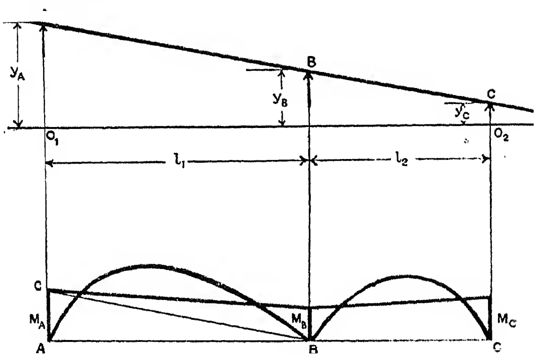


FIG. 85.

calculate the dimensions of the beam to resist the bending moment and also to calculate the deflection at any point desired. In cases of continuous beams, that is, beams either free or fixed at the ends and resting on a number of supports, the determination of the bending-moment diagram is simplified by the use of the Theorem of Three Moments.

In Fig. 85 is shown a beam resting on three supports, A, B, and C, each of which is respectively distant y_A, y_B, and y_C above the horizontal datum XY. First take the origin at the point O_1, and let M_1 be the bending moment at any point between the supports A and B and distant x from O_1. Then

$$\int_0^{l_1} \frac{d^2y}{dx^2} x dx = \int_0^{l_1} M_1 x dx$$

or

$$\left[x\frac{dy}{dx} - y \right]_0^{l_1} = A_1 \bar{x}_1,$$

where A_1 is the total area of the $\frac{M}{EI}$ diagram for the span, l_1 and $\bar{x}_1$ is the distance of its centre of gravity from the vertical through O_1,

thus
$$l_1 i_B - y_B + y_A = A_1\bar{x}_1$$

or
$$i_B = \frac{y_B - y_A}{l_1} + \frac{A_1\bar{x}_1}{l_1},$$

where i_B is the slope of the beam at the support B. Now take the origin at O_2 and measure x positive to the left, then by a similar method

$$i_B = \frac{y_B - y_C}{l_2} + \frac{A_2\bar{x}_2}{l_2}$$

where A_2 is the total area of the bending-moment diagram corresponding with the span l_2, and $\bar{x}_2$ the distance of its centre of gravity from the vertical through O_2. Since x is reckoned positive to the right in the first of these equations and positive to the left in the second, the sign of i_B will be positive in one equation and negative in the other. Hence adding the equations together

$$\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2} = \frac{y_A}{l_1} - y_B\left(\frac{1}{l_1} + \frac{1}{l_2}\right) + \frac{y_c}{l_2} \quad . \quad . \quad . \quad (1)$$

If the beam is fixed at the supports, the complete bending-moment diagram for each span is the sum of two other diagrams; that is, the diagram if the beam were freely supported plus the diagram for the fixing moments. Considering the span l_1, let the area of the bending-moment diagram if the beam were freely supported be denoted by Δ_1 and the distance of its centre of gravity from the vertical through O_1 by δ_1. The area of the diagram for the fixing moments multiplied by the distance of its centre of gravity from the vertical through O_2 can be obtained by splitting the diagram into two triangles by the line BC, and is given by

$$M_A \frac{l_1}{2} \cdot \frac{l_1}{3} + M_B \frac{l_1}{2} \cdot \frac{2l_1}{3}$$

Hence

$$A_1\bar{x}_1 = \frac{1}{EI}\left[\Delta_1\delta_1 + M_A\frac{l^2{}_1}{6} + M_B\frac{l^2{}_1}{3}\right]$$

and similarly

$$A_2\bar{x}_2 = \frac{1}{EI}\left[\Delta_2\delta_2 + M_C\frac{l^2{}_2}{6} + M_B\frac{l^2{}_2}{3}\right]$$

Equation (1) now becomes

$$\left(\Delta_1\frac{\delta_1}{l_1}+\Delta_2\frac{\delta_2}{l_2}\right)+M_A\frac{l_1}{6}+M_B\left(\frac{l_1}{3}+\frac{l_2}{3}\right)+M_C\frac{l_2}{6}$$
$$=EI\left[\frac{y_A}{l_1}-y_B\left(\frac{1}{l_1}+\frac{1}{l_2}\right)+\frac{y_C}{l_2}\right] \quad . \quad . \quad . \quad (2)$$

which is the most general form of the theorem. The bending moments are positive when they tend to bend the beam concave upwards, y_A, y_B, and y_C are reckoned positive upwards, and δ_1 is positive to the right from the vertical through O_1 and δ_2 positive to the left from the vertical through O_2. The differences between the heights of the supports, that is, y_A-y_B and y_B-y_C, are small compared with the respective spans l_1 and l_2.

If there are three supports there will be two spans and one equation similar to (2). Also, if there are four supports there will be three spans and two equations similar to (2), and in general a beam having n supports and $n-1$ spans will yield $n-2$ equations which will involve n unknowns such as M_A, M_B, etc. The other two equations necessary for the determination of the unknowns are furnished by the conditions at the ends of the beam.

If the supports are all at the same level the left-hand side of equation (2) is zero and the equation becomes

$$\left(\Delta_1\frac{\delta_1}{l_1}+\Delta_2\frac{\delta_2}{l_2}\right)+M_A\frac{l_1}{6}+M_B\left(\frac{l_1}{3}+\frac{l_2}{3}\right)+M_C\frac{l_2}{6}=0 \quad . \quad (3)$$

Also if the loading is uniform, the bending-moment diagram for each span of beam, assuming it to be freely supported at its ends, is a parabola. Let the loading be w units per unit length, then

$$\Delta_1=+\frac{2}{3}\;\frac{w_1l_1^2}{8} \text{ and } \delta_1=\frac{l_1}{2}$$

$$\text{Hence } \Delta_1\frac{\delta_1}{l}=+\frac{w_1l_1^3}{24} \text{ and } \Delta_2\frac{\delta_2}{l_2}=+\frac{w_2l_2^3}{24},$$

and equation (3) becomes

$$M_Al_1+2M_B(l_1+l_2)+M_Cl_2=-\frac{1}{4}(w_1l_1^3+w_2l_2^3) \quad . \quad (4)$$

In Fig. 86 is shown a beam resting freely on four equally spaced supports and supporting a uniform load of w units per unit of length. Considering first the spans AB and BC, the theorem of three moments gives

$$M_Al+4M_Bl+M_Cl=-\frac{1}{2}wl^3$$

and for the spans DC and CB

$$M_D l + 4M_C l + M_B l = -\frac{1}{2} wl^3.$$

Since $$M_A = M_D = 0$$

then $$4M_B + M_C = -\frac{1}{2} wl^2$$

and $$4M_C + M_B = -\frac{1}{2} wl^2$$

therefore $$M_B = M_C = -\frac{wl^2}{10}.$$

To find the reactions take moments about the supports.

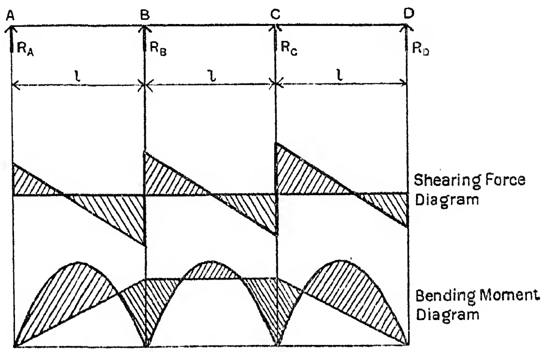


FIG. 86.

Taking moments about the support B, considering the left-hand side only, and calling clockwise moments positive,

$$M_B + M_A + R_A l - \frac{wl^2}{2} = 0$$

$$\frac{wl^2}{10} + 0 + R_A l - \frac{wl^2}{2} = 0$$

hence $$R_A = \frac{4}{10} wl$$

and by symmetry $$R_D = \frac{4}{10} wl$$

and $$R_B = R_C = \frac{1}{2}\left(3 - \frac{8}{10}\right) wl = \frac{11}{10} wl.$$

The shearing-force and bending-moment diagrams are shown drawn in Fig. 86.

Equation (1) now becomes

$$\left(\Delta_1\frac{\delta_1}{l_1}+\Delta_2\frac{\delta_2}{l_2}\right)+M_A\frac{l_1}{6}+M_B\left(\frac{l_1}{3}+\frac{l_2}{3}\right)+M_C\frac{l_2}{6}$$
$$=EI\left[\frac{y_A}{l_1}-y_B\left(\frac{1}{l_1}+\frac{1}{l_2}\right)+\frac{y_C}{l_2}\right] \quad . \quad . \quad . \quad . \quad (2)$$

which is the most general form of the theorem. The bending moments are positive when they tend to bend the beam concave upwards, y_A, y_B, and y_C are reckoned positive upwards, and δ_1 is positive to the right from the vertical through O_1 and δ_2 positive to the left from the vertical through O_2. The differences between the heights of the supports, that is, y_A-y_B and y_B-y_C, are small compared with the respective spans l_1 and l_2.

If there are three supports there will be two spans and one equation similar to (2). Also, if there are four supports there will be three spans and two equations similar to (2), and in general a beam having n supports and $n-1$ spans will yield $n-2$ equations which will involve n unknowns such as M_A, M_B, etc. The other two equations necessary for the determination of the unknowns are furnished by the conditions at the ends of the beam.

If the supports are all at the same level the left-hand side of equation (2) is zero and the equation becomes

$$\left(\Delta_1\frac{\delta_1}{l_1}+\Delta_2\frac{\delta_2}{l_2}\right)+M_A\frac{l_1}{6}+M_B\left(\frac{l_1}{3}+\frac{l_2}{3}\right)+M_C\frac{l_2}{6}=0 \quad . \quad (3)$$

Also if the loading is uniform, the bending-moment diagram for each span of beam, assuming it to be freely supported at its ends, is a parabola. Let the loading be w units per unit length, then

$$\Delta_1=+\frac{2}{3}\ \frac{w_1l_1^2}{8} \text{ and } \delta_1=\frac{l_1}{2}$$

Hence $\Delta_1\frac{\delta_1}{l}=+\frac{w_1l_1^3}{24}$ and $\Delta_2\frac{\delta_2}{l_2}=+\frac{w_2l_2^3}{24}$,

and equation (3) becomes

$$M_Al_1+2M_B(l_1+l_2)+M_Cl_2=-\frac{1}{4}(w_1l_1^3+w_2l_2^3) \quad . \quad (4)$$

In Fig. 86 is shown a beam resting freely on four equally spaced supports and supporting a uniform load of w units per unit of length. Considering first the spans AB and BC, the theorem of three moments gives

$$M_Al+4M_Bl+M_Cl=-\frac{1}{2}wl^3$$

and for the spans DC and CB

$$M_D l + 4M_C l + M_B l = -\frac{1}{2} w l^3.$$

Since $$M_A = M_D = 0$$

then $$4M_B + M_C = -\frac{1}{2} w l^2$$

and $$4M_C + M_B = -\frac{1}{2} w l^2$$

therefore $$M_B = M_C = -\frac{wl^2}{10}.$$

To find the reactions take moments about the supports.

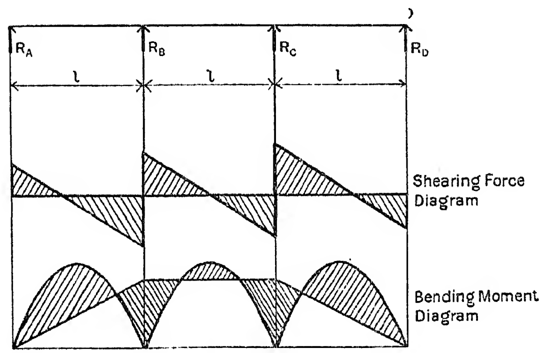


FIG. 86.

Taking moments about the support B, considering the left-hand side only, and calling clockwise moments positive,

$$M_B + M_A + R_A l - \frac{wl^2}{2} = 0$$

$$\frac{wl^2}{10} + 0 + R_A l - \frac{wl^2}{2} = 0$$

hence $$R_A = \frac{4}{10} wl$$

and by symmetry $$R_D = \frac{4}{10} wl$$

and $$R_B = R_C = \frac{1}{2}\left(3 - \frac{8}{10}\right) wl = \frac{11}{10} wl.$$

The shearing-force and bending-moment diagrams are shown drawn in Fig. 86.

The case of a beam of length $2l$ fixed at the ends, supported at its mid-length and carrying a uniform load of w units per unit of length over the left-hand span, is shown in Fig. 87. Let M be the bending moment at any point along the left-hand span distant x from B, then

$$\int_0^l \frac{d^2y}{dx^2}\,x\,dx = \int_0^l \frac{Mx}{EI}\,dx$$

$$\Big[x\frac{dy}{dx} - y\Big]_0^l = \int_0^l \frac{Mx}{EI}\,dx.$$

The left-hand side of the expression vanishes, hence

$$\int_0^l Mx\,dx = 0$$

$$\frac{2}{3}\,\frac{wl^3}{8} \times \frac{l}{2} + \frac{M_B l^2}{6} + \frac{M_A l^2}{3} = 0.$$

$$M_B + 2M_A = -\frac{wl^2}{4} \quad \ldots\ldots \quad (1)$$

and by symmetry

$$M_D + 2M_C = 0 \quad \ldots\ldots \quad (2)$$

The theorem of three moments gives

$$M_A l + 4M_B l + M_C l = -\frac{1}{4}wl^3$$

or

$$M_A + 4M_B + M_C = -\frac{1}{4}wl^2 \quad (3)$$

From the equations (1), (2), and (3) the values of M_A, M_B, and M_C can be obtained.

From equation (2) $M_C = -\frac{M_B}{2}$ and equation (3) becomes

$$2M_A + 7M_B = -\frac{1}{2}wl^2 \quad \ldots\ldots \quad (4)$$

From equations (1) and (4)

$$M_B = \cdot\ \frac{}{24}\,wl^2$$

$$M_A = -\frac{5}{48}\,wl^2$$

also

$$M_C = +\frac{1}{48}\,wl^2.$$

To find the reactions R_A take moments about the support B

for the left-hand side only. Hence, calling clockwise movements positive,

$$M_B + M_A + R_A l - \frac{wl^2}{2} = 0$$

$$\frac{1}{24} wl - \frac{5}{48} wl + R_A - \frac{wl}{2} = 0$$

$$R_A = \frac{27}{48} wl$$

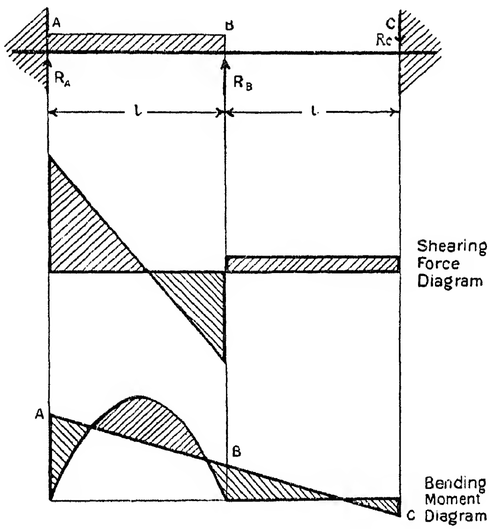


FIG. 87.

Now take moments about the support B and consider the right-hand side only:

$$M_B + M_C + R_C l = 0$$

$$-\frac{1}{24} wl - \frac{1}{48} wl = -R_C$$

$$R_C = \frac{3}{48} wl$$

and since clock-hand-wise moments have been taken positive, R_C must act downwards.

To find the reaction R_B take moments about the left-hand support A,

$$wl^2 + 2lR_C + M_A + M_C + R_B l = 0$$

$$\frac{wl}{2} + \frac{6}{48}wl - \frac{5}{48}wl - \frac{1}{48}wl = -R_B.$$

Hence $R_B = -\frac{24}{48}wl$ and acts upwards.

As a check $R_A + R_B + R_C$ should equal wl, which is the case. The shearing force and bending-moment diagrams are shown drawn in Fig. 87. The line ABC is the base line for the bending-moment diagram.

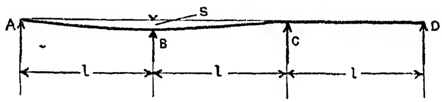


FIG. 88.

A beam resting on four equally spaced supports, A, B, C, D, and carrying a uniform load of w units per unit of length, with the support B lower than the others by an amount s, is shown in Fig. 88. Since $M_A = M_D = 0$ the theorem of three moments gives for the spans A, B, C

$$\frac{wl^3}{24} + \frac{wl^3}{24} + \frac{2}{3}M_B l + \frac{1}{6}M_C l = EI\left(\frac{2s}{l}\right)$$

and for the spans B, C, D

$$\frac{wl^3}{24} + \frac{wl^3}{24} + \frac{1}{6}M_B l + \frac{2}{3}M_C l = -EI\left(\frac{2s}{l}\right).$$

These equations reduce to

$$4M_B + M_C = -\frac{wl^2}{2} + \frac{12EIs}{l^2}$$

$$M_B + 4M_C = -\frac{wl^2}{2} - \frac{6EIs}{l^2}$$

Hence

$$M_B = -\frac{wl^2}{10} + \frac{24}{15}\frac{EIs}{l^2}$$

$$M_C = -\frac{wl^2}{10} - \frac{36}{15}\frac{EIs}{l^2}$$

If the value of s is known, M_B and M_C can be obtained, and hence the values of the reaction R_A, R_B, R_C, and R_D.

In continuous beams the greatest bending moment usually occurs over the supports, whereas in discontinuous beams it occurs about the middle of the span. The weight of the beam

itself is usually a large proportion of the total weight to be carried, and since in a continuous beam the heavier portions are in the vicinity of the supports, the bending moments due to its own weight are less than is the case with a discontinuous beam. A serious objection to the use of continuous beams is that if the supports sink relative to each other by only a small amount, the stresses in the beam or some of its members may be considerably altered and so cause failure. The objection can be partially overcome by hinging the beams at the points of inflexion.

56. Deflection of Ferro-concrete Beams—The deflection of ferro-concrete beams can be determined by the methods given, but it is necessary to find an expression for the moment of inertia I in terms of either the steel or the concrete. Referring to Fig. 71, the moment of inertia about the neutral axis XY in terms of the concrete, assuming that the concrete withstands no tension, is $I_C = \frac{b(Kd)^3}{3} + na_s d^2(1-K)^2$, where a_s is the area of the steel and $n = \frac{E_s}{E_c}$. The deflection can then be obtained from the formula $E_c I_c \frac{d^2y}{dx^2} = M$. If I_s is the moment of inertia in term of the steel, then $I_s = \frac{I_c}{n}$ and the formula for the deflection becomes $E_s I_s \frac{d^2y}{dx^2} = M$. The same argument applies to doubly reinforced beams and to tee beams.

An approximate method, given by E. White,* is to assume that the portion of the beam above the neutral axis—that is, the compression-resisting portion—is removed and replaced by steel bars having the same total area (see Fig. 89). The moment of inertia I of the area of the bars about the neutral axis is sensibly given by $I = 2a'_s\{d(1-K)\}^2$, where a_s is the total area of the steel in tension or compression. The deflection can then be obtained from the formula $E_s I \frac{d^2y}{dx^2} = M$, where I has the above value and E_s is Young's Modulus for steel. It is, of course, assumed that the concrete withstands no tension.

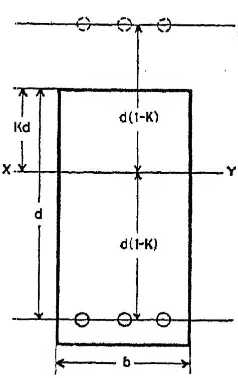


FIG. 89.

* "Engineering Record," November 9, 1907.

FROM SPOONER'S "MACHINE DESIGN"

BENDING MOMENTS, SHEARING FORCES, AND DEFLECTIONS OF BEAMS

Beam of uniform cross section, case 5 excepted.	Greatest shearing force.	Greatest bending moment.	Relative strength.	Deflection, δ.	Relative deflection.
1. Cantilever load at end .	W	WL	1	$\frac{WL^3}{3EI}$	128
2. Cantilever uniformly distributed load . . .	W	$\frac{WL}{2}$	2	$\frac{WL^3}{8EI}$	48
3. Beam fixed at both ends, load at one end, as in a wheel arm	W	$\frac{WL}{2}$	2	$\frac{WL^3}{12EI}$	32
Cantilever, load uniformly decreasing from fixed end to free end .	W	$\frac{WL}{3}$	3	$\frac{WL^3}{15EI}$	25·6
Simple beam, load at centre, breadth uniformly decreasing from middle to point at ends	$\frac{W}{2}$	$\frac{WL}{4}$	Uniform	$\frac{WL^3}{32EI}$	12
Simple beam, load at centre	$\frac{W}{2}$	$\frac{WL}{4}$	4	$\frac{WL^3}{48EI}$	8
Simple beam, load uniformly distributed .	$\frac{W}{2}$	$\frac{WL}{8}$	8	$\frac{5WL^3}{384EI}$	5
Beam, fixed at one end, supported at other, load at centre.*	$\frac{5}{16}$W $\frac{11}{16}$W	$\frac{3WL}{16}$	$\frac{16}{3}$	$\frac{7WL^3}{768EI}$	3·5
Beam, fixed at one end, supported at other, distributed load . .	$\frac{3}{8}$W $\frac{5}{8}$W	$\frac{WL}{8}$	8	$\frac{WL^3}{192EI}$	2
Beam, fixed at both ends, load at centre . . .	$\frac{W}{2}$	$\frac{WL}{8}$	8	$\frac{WL^3}{192EI}$	2
Beam, fixed at both ends, load uniformly distributed (G.B.M. at ends)	$\frac{W}{2}$	$\frac{WL}{12}$	12	$\frac{WL^3}{384EI}$	1
Simple beam, loaded x ft. from one end and y ft. from the other † . .	—	$\frac{Wxy}{x+y}$	—	$\frac{WL^3}{3EI} \times \frac{x^2}{L^2} \times \frac{y^2}{L^2}$	—
Ends equally overhanging two supports, A and Ba, distance x, load on each end = W, span = L	—	Between A and B, BM at A and B = Wx	—	$\frac{WL^3}{8EI} \times \frac{x}{L}$ at centre	—

* In this case the deflection given occurs under the weight, but the maximum deflection $= \sqrt{\frac{1}{5}} \frac{WL^3}{48EI}$ is at a part x feet from the end opposite the fixed end, where $x = L\sqrt{\frac{1}{5}}$.

† In this case the deflection given occurs under the weight, but the maximum deflection is at a part x_1 feet from a support, where $x_1 = x\sqrt{\frac{1}{3} + \frac{2y}{3x}}$.

Example (see Art. 48, example [1])—A singly reinforced concrete beam rests on supports 12 ft. apart and supports a uniform load of 1,500 lb. per foot run. The breadth of the beam is 10·5 in. and the depth from the upper surface to the centre of the reinforcements is 18 in. The total area of the steel is 1·28 sq. in. and the distance of the neutral axis from the upper surface 6·48 in. Find the approximate central deflection, taking Young's Modulus $= 30 \times 10^6$ lb. per sq. in.

The moment of inertia

$$I = 2 \times 1{\cdot}28\{18 - 6{\cdot}48\}^2 = 340\ (\text{in.})^4.$$

The central deflection

$$\delta = \frac{5}{384}\,\frac{wl^3}{EI}$$

$$= \frac{5 \times 18{,}000 \times (144)^3}{384 \times 30 \times 10^6 \times 340} = 0{\cdot}069\ \text{in.}$$

In the preceding table are given the greatest shearing forces, bending moments, and deflections of beams supported in various ways.

EXAMPLES. V

(1) A continuous beam rests on four supports at the same level and carries a uniform load of w units per unit of length. The supports divide the beam into three lengths, a, b, and a respectively, the length b being midway between the two lengths a. Find the ratio of b to a, so that the neutral axis of the beam shall be horizontal over the intermediate supports.

(2) A beam of length $2l$ is fixed at both ends and supported in the middle to the level of the ends. If it carries a load W at a distance $\frac{l}{2}$ from one end, find the bending moment and reaction at the supports. Also draw the bending-moment and shearing-force diagrams.

(3) A monoplane weighs 2,000 lb. Each of the wings has a span of 20 ft. The front span of the wings, which in slow flight may take the entire load, is stayed by four wire cables attached 8 ft. and 17 ft. from the body and leading to a point on the chassis 5 ft. vertically below the root of the wings. Determine the maximum bending moment in the spars and the tension in each of the cables. (Victoria.)

(4) A beam is firmly built into a wall at one end and rests freely at the other end on a vertical column whose centre line is distant 8 ft. from the wall. The beam supports a wall whose weight, added to that of the beam itself, is equivalent to a uniformly distributed load of 3,200 lb. per foot run of the beam. Find (a) the total load supported by the column, (b) the bending moment and shear force at the section of the beam adjoining the wall, (c) the position of the point of zero bending moment. (S. and A.)

(5) A plate web girder is to have a span of 120 ft. What depth

would you make the central cross-section if the coefficient of stiffness (that is, the ratio of deflection in centre to length of span) is to be $\frac{1}{1200}$? The girder is to support a uniformly distributed load, the working stress in the metal is to be 6 tons per sq. in., and the modulus of elasticity E may be taken as 11,160 tons per sq. in. (S. and A.)

(6) A girder is supported on three piers. Each of the spans is 20 ft. and the central prop is half an inch below the others. If the total load is half a ton per foot run, the moment of inertia of the section 2,600 (in.)^4, and Young's Modulus 30×10^6 lb. per sq. in., what is the load on the central prop? What would be the effect of a load four times greater or four times smaller?

(7) A rectangular concrete beam is simply supported over a span of 20 ft. The beam is 12 in. $\times$ 19 in. and is reinforced by five $\frac{5}{8}$ in. bars placed 17 in. below the surface. What will be the central deflection produced by a load of 500 lb. per foot run? (Take $E_s=30\times 10^6$ and $E_c=2\times 10^6$ lb. per sq. in.). (Victoria.)

(8) A beam of span l is fixed horizontally at both ends. Two equal loads, W, are placed at equal distances h from the ends of the beam. Prove that the greatest deflection of the beam is equal to $\frac{Wh^2}{24EI}(3l-4h)$ and that the greatest bending moment at the centre of the beam is equal to $\frac{Wh^2}{l}$. (B.Sc., Lond.)

(9) The total load on the axle of a truck is 6 tons. The wheels are 6 ft. apart and the two axle boxes 5 ft. apart. Draw the curve of bending moment on the axle. (Victoria.)

(10) Find the height of the prop relative to the end supports in a centrally propped beam with an evenly distributed load when the load on the prop is equal to that on the supports.

CHAPTER VI

SHEAR STRESS IN BEAMS—DEFLECTION DUE TO SHEAR

57. Shear Stress in Beams—In addition to the direct stresses dealt with so far, arising from the bending moments acting on a beam, there are also shear stresses caused by the shearing forces. In the more ordinary cases these act on vertical and horizontal surfaces. It is not difficult to see how these occur. In Fig. 90 (*a*) is shown the side view of a solid rectangular beam of elastic material resting freely on supports and showing little deflection. If now four pieces of the same material, the same length and breadth, but one quarter the thickness, be

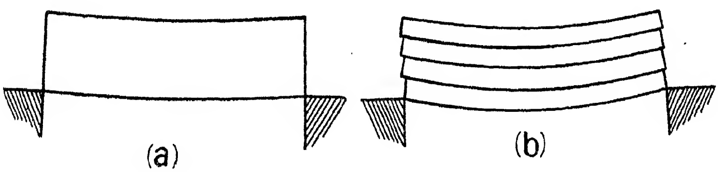


FIG. 90.

arranged on the supports to form a beam having the same cross-sectional area, it will sag down to a comparatively much greater extent, as shown in Fig. 90 (*b*). This is because the beam is now acting as four separate beams of one quarter the depth of the solid beam, and sliding is possible along the dividing surfaces. It may be noted that the tendency to slide is greatest at the supports and zero midway between them, which agrees with the shearing-force diagram for a uniformly distributed load (as is the present case).

In order to find the intensity of the shear stress over any horizontal section, consider the general conditions shown in Fig. 91. Here is seen a length of the beam in question, the neutral axis is denoted by XY, and AABB is a vertical slice of which the upper portion above the horizontal surface LL′ is the piece under consideration. Let x be the distance of the plane AA from the left-hand support and δx the thickness of the slice. Also let the bending moment on the beam at AA be M and at

BB be $M+\delta M$. Thus the bending moment is greater at BB, which causes the compressive force induced by bending to be greater on BL′ than on AL, so that there is a tendency for the piece ALL′B to slide from right to left and develop shear stresses on the horizontal surface LL′. Considering the horizontal forces on ALL′B, the greater—P—is opposed by the smaller—P′—and the balance is maintained by the total shearing force on LL′. Let the thickness of a horizontal strip of the portion ALL′B distant y' from the neutral surface be $\delta y'$ and its width z'. Also let the stress on the side of the strip terminated by the plane AA be

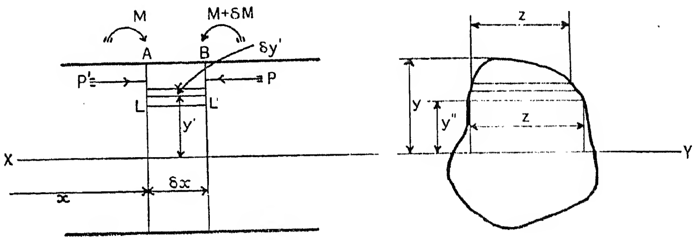


FIG. 91.

denoted by f' and the stress on the side terminated by the plane BB by $f'+\delta f'$. Then

$$P' = \Sigma^{v}_{v''} f'z'\delta y'$$

also
$$P = \Sigma^{v}_{v''} (f'+\delta f')z'\delta y'.$$

Hence
$$P - P' = \Sigma^{v}_{v''} \delta f'z'\delta y'$$

but
$$\frac{M}{I} = \frac{f}{y} \text{ and } \frac{M+\delta M}{I} = \frac{f+\delta f}{y},$$

where f is the stress on the upper surface AB distant y from the neutral axis and I is the moment of inertia of the section. Hence

$$\frac{\delta M}{I} = \frac{\delta f}{y} = \frac{\delta f'}{y'}.$$

and
$$-P' = \Sigma^{v}_{v''} \frac{\delta f}{y} y'z'\delta y'$$

$$= \Sigma^{v}_{v''} \frac{\delta M}{I} y'z'\delta y'.$$

Let s denote the average shearing stress on the surface LL′, then

$$s \times z \times \delta x = \frac{\delta M}{I} \Sigma^{v}_{v''} y'z'\delta y',$$

and in the limit $s = \frac{dM}{dx} \times \frac{1}{IZ}\int_{y''}^{y} y'z'dy'$. It was shown in Art. 55 that $\frac{dM}{dx} = F$, where F is the shearing force at the section, and

therefore $$s = \frac{F}{IZ}\int_{y''}^{y} y'z'dy'.$$

If A is the area of the portion of a normal section above LL′ and $\bar{x}$ the distance of its centre of gravity from the neutral axis,

then $$s = \frac{F}{IZ} A\bar{x}.$$

It should be noted that, although the total shearing force on the neutral surface is greater than on any other parallel surface, the shear stress is not necessarily a maximum. This will be evident on considering Art. 59 and Fig. 95. If the cross-section of the beam is not uniform the above analysis will apply because in the limit δx—the thickness of the slice—becomes infinitesimal, so that the dimensions of the sections which terminate the slice become ultimately identical.

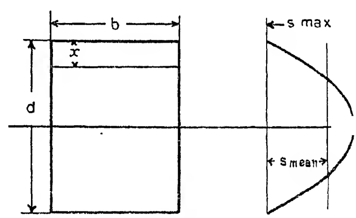


Fig. 92.

The case of a beam of rectangular cross-section of breadth b and depth d is shown in Fig. 92.

$$A = bx \text{ and } \bar{x} = \frac{1}{2}(d - x).$$

Hence $$s = \frac{F}{Ib}\ \frac{bx}{2}(d - x)$$

$$= \frac{6F}{bd^3}(d \cdot x - x^2).$$

The relation between s and x is evidently parabolic. When $x = 0$ $s = 0$ and when $x = \frac{d}{2}$ $s = \frac{3}{2}\frac{F}{bd}$. The mean intensity of the shear stress over the section is $\frac{F}{bd}$ and hence the maximum exceeds the mean by 50 per cent. The parabola showing the relation between s and x is drawn in Fig. 92.

The case of a beam of circular cross-section is shown in Fig. 93.

$$A\bar{x} = \int_{y}^{a} z'y'dy'$$

$$z' = 2a\cos\theta,\ y' = a\sin\theta,\ dy' = a\cos\theta d\theta.$$

Hence
$$A\bar{x} = \int_{\phi}^{\frac{\pi}{2}} 2a^3 \sin\theta \cos^2\theta d\theta$$

$$= \frac{2}{3} a^3 \cos^3\phi$$

$$s = \frac{F}{Iz} A\bar{x} = \frac{4F}{\pi a^4 2a\cos\phi} \times \frac{2}{3} a^3 \cos^3\phi$$

$$= \frac{4F}{3\pi a^2}\cos^2\phi$$

$$= \frac{4F}{3\pi a^2}\left\{1 - \left(\frac{y}{a}\right)^2\right\}$$

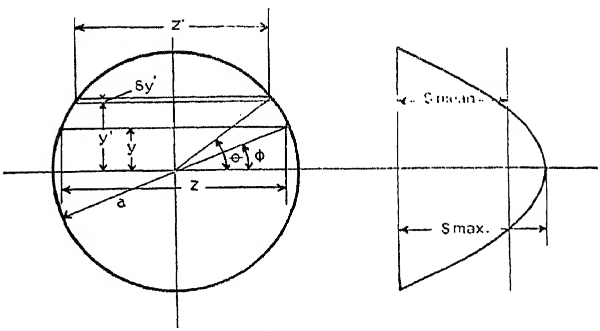


Fig. 93.

At the neutral axis $y = 0$, hence $s = \frac{4}{3}\frac{F}{\pi a^2}$ which is $\frac{4}{3}$ times the mean intensity of the shear stress over the whole section. The parabola connecting s and y is shown drawn in Fig. 94.

It should be noted that the shear stress across the planes parallel to the neutral plane is not uniform, but varies from one side of the beam to the other, being a maximum in the middle. In the case of a beam of circular cross-section the maximum shear stress on the neutral plane is 1·45 times the mean over the whole section. In the case of a beam of rectangular cross-section whose breadth is great compared with the depth, the maximum shear stress across a plane may exceed the mean by as much as 25 per cent., but in beams of usual dimensions the difference is not more than 5 to 10 per cent. and can be neglected.

58. Beam of I Section with Sharp Corners—Refer to Fig. 94 and consider the portion of the flange above the horizontal line distant y from the neutral axis XY.

$$A\bar{x} = B\left(\frac{D}{2} - y\right) \times \frac{1}{2}\left(\frac{D}{2} + y\right)$$
$$= \frac{1}{2}B\left(\frac{D^2}{4} - y^2\right)$$

and $s_F = \frac{F}{2I}\left(\frac{D^2}{4} - y^2\right)$ equals shear stress in the flange.

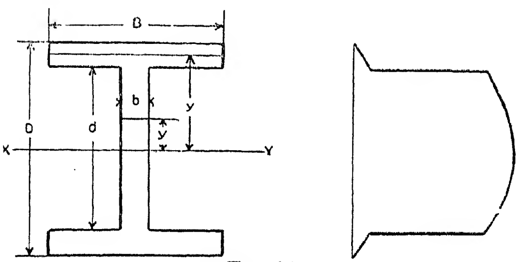


FIG. 94.

Now consider the shear stress in the web along the plane distant y from the neutral plane.

$$A\bar{x} = \frac{1}{2}B(D - d) \times \frac{1}{4}(D + d) + b\left(\frac{d}{2} - y\right) \times \frac{1}{2}\left(\frac{d}{2} + y\right)$$
$$= \frac{1}{8}B(D^2 - d^2) + \frac{1}{2}b\left(\frac{d^2}{4} - y^2\right)$$

and $s_W = \frac{F}{I}\left\{\frac{B}{8b}(D^2 - d^2) + \frac{d^2}{8} - \frac{y^2}{2}\right\}$ = sheer stress in web.

At the junction with the flange $y = \frac{d}{2}$ and $s_W = \frac{B}{8b}\ \frac{F}{I}(D^2 - d^2)$, also $s_F = \frac{F}{8I}(D^2 - d^2)$. Thus the shear stress just inside the web is $\frac{B}{b}$ times that just inside the flange. The curves connecting s_W, s_F, and y are parabolas and are shown drawn in Fig. 94. It is seen from these curves that the shear stress in the web is much greater than in the flanges, and it is often assumed that the flanges resist the whole of the bending moment, whilst the web resists the whole of the shearing force.

Example—A girder 6 ft. deep is freely supported over a span of 70 ft. and carries a uniformly distributed load of 2 tons per

ft. run. Assuming a safe stress of 7 tons per sq. in. tensile or compressive in the flanges, 2 tons per sq. in. shear stress in the web, 5 tons per sq. in. shear stress in the rivets, and 8 tons per sq. in. bearing stress on the rivets, design the girder.

The max. bending moment $=\frac{2\times 70^2}{8}=1{,}225$ ton ft. The effective area of one flange, including the portions of the two angle irons riveted to it which fix it to the web, should be $\frac{1{,}225\times 12}{6\times 12\times 7}=29{\cdot}2$ sq. in.

Use two $4\times 4\times\frac{1}{2}$ in. angle irons and $\frac{7}{8}$ in. rivets. Effective area of the portions of the two angles which are riveted to the flange $=\left(8-2\times\frac{7}{8}\right)\times\frac{1}{2}=3{\cdot}125$ in.

Hence effective area of plates in one flange should be $29{\cdot}2-3{\cdot}125=26$ sq. in. Assume flanges 24 in. wide. Effective area $=\left(24-2\times\frac{7}{8}\right)=22{\cdot}25$ sq. in. and thickness $=\frac{26}{22{\cdot}25}=1{\cdot}17$. Use two plates, each $\frac{5}{8}$ in. thick, for each flange.

Max. shear $=\frac{70\times 2}{2}=70$ tons.

Hence thickness of web $=\frac{70}{72\times 2}=\frac{1}{2}$ in.

Shear per ft. in a vertical or horizontal section of web $=\frac{70}{6}=11\frac{2}{3}$ tons.

The rivets are in double shear and the resistance to shear of one rivet $=2\times\frac{\pi}{4}\left(\frac{7}{8}\right)^2\times 5=6{\cdot}0$ tons.

Safe bearing strength on one rivet $=\frac{7}{8}\times\frac{1}{2}\times 8=3{\cdot}5$ tons.

Hence number of rivets per ft. $=\frac{70}{6\times 3{\cdot}5}=3{\cdot}33$ and pitch $=\frac{12}{3{\cdot}33}=3{\cdot}6$ in., say $3\frac{1}{2}$ in.

For the purpose of finding the mean shear stress at the neutral axis of a beam the formula $s=\frac{F}{Iz}A\bar{x}$ can be written $s=\frac{F}{A'k^2z}\times\frac{A'\bar{x}}{2}=\frac{F\bar{x}}{2k^2z}$ where k^2 is the radius of gyration of the section about the neutral axis and A′ is the total area of the cross-section. In the case of circular tubular section of thickness

t, small compared with the mean radius r, $\bar{x} = \frac{r}{2\pi}$, $k^2 = \frac{r^2}{4}$, $z = 2t$, hence $s = \frac{F}{\pi r t}$ and the mean shear stress over the whole section $\frac{F}{2\pi r t}$, which is one-half the above.

59. Graphical Method for Finding Shear Stresses in Beams—In cases of beams whose cross-sections are bounded by irregular curves, the method given can be used, but it has as a rule to be applied graphically. In Fig. 95 is shown the normal section of a beam of irregular cross-section and neutral axis XY. The shaded figure is the first modulus figure of the section drawn with respect to the base lines AB. Let A be the area of the section above the line ab and $\bar{x}$ the distance of its centre of gravity from XY. Then $A\bar{x} = A'y''$ where A′ is the area of the modulus

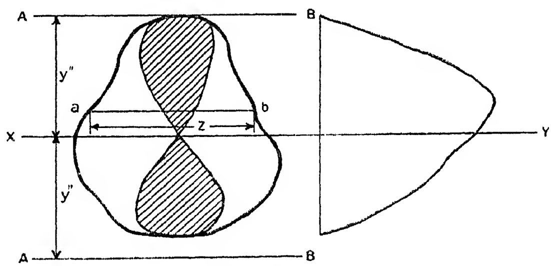


FIG. 95.

figure above the line ab. The quantity $\frac{F}{Iz}A\bar{x}$, corresponding with the plane whose end view is ab, is accordingly given by $\frac{F}{I}\frac{A'y''}{z}$, where z is the width of the section. When the values of $\frac{A'}{z}$, corresponding with several planes such as ab, are plotted, the result is the diagram shown in Fig. 95, so that the height of the diagram at any point multiplied by $\frac{Fy''}{I}$ is the intensity of the shear stress on the corresponding plane in the beam where the external shear on the beam is denoted by F and I is the constant moment of inertia.

If the cross-section is uniform, one diagram similar to that shown in Fig. 95 will suffice for the calculation of the shear stress anywhere along the beam. If the cross-section is variable,

a different diagram will generally be required to correspond with every section where the shear stress is desired.

60. Deflection of Beams due to Shear—In order to find the deflection of beams due to shear the expression for the internal shear-strain energy will be determined and equated to the work done by the external loads to produce the shear deflection. In Fig. 96 AA, BB are normal sections of a beam distant δx apart, the plane AA being distant x from some origin. The rectangle *abcd* is the side view of a strip of the beam parallel to the neutral surface XY and distant y from it. When the shear stresses are acting the rectangle will be distorted into a trapezium and the sides *da* and *bc* will each become inclined to their original directions by an amount α. Since the values of δx and δy are very small, the shear stresses on the surfaces whose end views are the sides of the trapezium can be considered constant and are

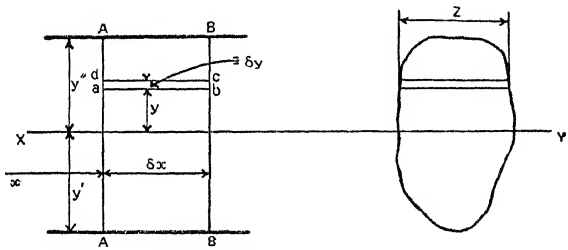


FIG. 96.

denoted by s. If G is the modulus of rigidity for the material of the beam $\frac{s}{\alpha} = \text{G}$ (see Art. 5) and the work necessary to change the shape of the strip is $\frac{s\alpha}{2} z\delta x\delta y$, which is equal to $\frac{1}{2}\frac{s^2}{\text{G}} z\delta x\delta y$, where z is the width of the strip. The energy $\delta\text{W}'$ stored in the length δx of the beam bounded by the planes AA and BB is accordingly

$$\delta\text{W}' = \frac{\delta x}{2\text{G}}\int_{y'}^{y''} s^2 z dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the term under the integral sign is constant, that is, if the cross-section of the beam and the external shearing force are constant, the work done on a length l of the beam is given by

$$\text{W} = \frac{l}{2\text{G}}\int_{y'}^{y''} s^2 \delta z dy.$$

Consider the case of a beam of rectangular cross-section, breadth

b, depth d, and length l, and subjected to a constant shearing force F,

$$s=\frac{F}{Iz}A\bar{x}=\frac{6F}{bd^3}\left(\frac{d^2}{4}-y^2\right)$$

where s is the shear stress along any plane distant y from the neutral plane (see Art. 57). Hence

$$W'=\frac{18lF^2}{bGd^6}\int_{-\frac{d}{2}}^{+\frac{d}{2}}\left(\frac{d^2}{4}-y^2\right)^2 dy$$

$$\frac{18lF^2}{bGd^6}\quad\frac{d^5}{16}-\frac{d^5}{24}+\frac{d^5}{80}$$

$$=\frac{3}{5}\left(\frac{lF^2}{bdG}\right)$$

For a cantilever of length l supporting a weight W at its free end the shear energy stored in the beam is equal to $\frac{W\delta'}{2}$ where δ' is the shear deflection. Hence

$$\frac{W\delta'}{2}=\frac{3}{5}\left(\frac{lW^2}{bGd}\right)$$

$$\delta'=\frac{6}{5}\frac{W'l}{bdG}=\frac{6}{5}\frac{Wl}{AG}$$

where A is the area of the cross-section. A beam of length l, freely supported at its ends and carrying weight W at its mid-length, can be treated as two cantilevers of length $\frac{l}{2}$ each supporting a load of $\frac{W}{2}$. Considering one cantilever $\frac{W\delta'}{4}=\frac{6lW^2}{40bdG}$

hence $$\frac{3}{10}\frac{Wl}{bdG}=\frac{3}{10}\frac{Wl}{AG}$$

If the beam is fixed at its ends and carries a weight W at its mid-length it can be considered as made up of four cantilevers each of length $\frac{l}{4}$ and supporting a weight of $\frac{W}{2}$ at its free end. The deflection of one cantilever will be $\frac{\delta'}{2}$ where δ' is the deflection at the mid-length of the beam. Hence

$$\frac{W\delta'}{8}=\frac{6lW^2}{80bdG}$$

$$=\frac{3}{10}\frac{Wl}{AG}$$

The deflection Δ due to bending is $\frac{1}{192}\frac{Wl^3}{EI}$, hence the total deflection is

$$\frac{3}{10}\frac{Wl}{bdG}+\frac{1}{192}\frac{Wl^3}{EI}$$

$$=\frac{Wl^3}{4E}\left\{\frac{6}{5}\frac{E}{bdGl^2}+\frac{1}{48I}\right.$$

$$=\frac{Wl^3}{192EI}\left\{\frac{24}{5}\frac{E}{G}\left(\frac{d}{l}\right)^2+1\right.$$

The value of $\frac{E}{G}$ is usually about $\frac{5}{2}$ and the expression becomes $\frac{Wl}{192EI}\ 12\left(\frac{d}{l}\right)^2+1\Big\}$. If $\left(\frac{d}{l}\right)^2$ is small, so that the quantity in the brackets is sensibly unity, the deflection reduces to $\frac{Wl^3}{192EI}$. Hence if the depth of the beam is small compared with the length, the deflection due to shear can be neglected.

The total deflection of a cantilever of rectangular cross-section and length l and carrying a weight W at its free end is

$$\frac{Wl^3}{3EI}\left\{\frac{3}{4}\left(\frac{d}{l}\right)^2+1\right\}$$

and for a freely supported beam of rectangular cross-section carrying a weight W at its mid-length it is

$$\frac{Wl^3}{48EI}\left\{3\left(\frac{d}{l}\right)^2+1\right\}$$

assuming in both cases that $\frac{E}{G}=\frac{5}{2}$. It will be noted that if the ratio of the depth to the length is small, the expressions approximate to those for the deflections due to bending alone.

For a beam of circular cross-section and radius a the shear strain stored in a length l is (see Fig. 94 and Art. 57)

$$W'=\frac{l}{2G}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{4F}{3\pi a^2}\cos^2\phi\right)^2(2a^2\cos^2\phi d\phi)$$

$$\frac{32lF^2}{9G\pi^2a^2}\int^{\frac{\pi}{2}}\cos^6\phi d\phi$$

$$\frac{5}{9}\frac{F^2l}{\pi Ga^2}$$

In the case of a freely supported beam of length l carrying a load W at its mid-length

$$\frac{1}{2}W\delta' = \frac{5}{9}\,\frac{W^2 l}{4\pi G a^2}$$

$$\delta' = \frac{5}{18}\,\frac{Wl}{\pi G a^2} = \frac{5}{18}\,\frac{Wl}{AG}$$

where A is the area of the cross-section and the total deflection is

$$\frac{1}{48}\,\frac{Wl^3}{EI} + \frac{5}{18}\,\frac{Wl}{A'G}$$

$$= \frac{Wl^3}{48EI}\left\{\frac{25}{12}\left(\frac{d}{l}\right)^2 + 1\right\} \text{ assuming } \frac{E}{G} = \frac{5}{2}$$

Here again it will be noted that if $\frac{d}{l}$ is small the ratio of the shear deflection to the bending deflection is a quantity of the second order and the shear deflection may be neglected.

When the load is distributed the shearing force on the beam is not uniform. Equation (1) is

$$\delta W' = \frac{\delta x}{2G}\int_{y'}^{y''} s^2 z\,dy.$$

Substituting $s = \frac{F}{Iz}A\bar{x}$, this becomes

$$\delta W' = \left(\frac{1}{2GI^2}\int_{y'}^{y''}\frac{A^2\bar{x}^2}{z}\,dy\right)F^2\delta x \tag{2}$$

If the cross-section of the beam is uniform, the quantity in the bracket is constant. Hence, denoting it by K,

$$\delta W' = KF^2\delta x.$$

In the case of a cantilever of length l supporting a continuous load of w units per unit of length, let the deflection at distances x and x' from the free end be y and y' respectively where $x > x'$. Then if the beam supports w units of load per unit of length, the shear at a section distant x' from the free end is wx' and

$$\frac{1}{2}w\int_0^x (y' - y)\,dx' = Kw^2\int_0^x x'^2\,dx'$$

$$-\frac{1}{2}w\int_0^x \frac{dy}{dx}\,dx' = Kw^2x^2$$

$$\frac{dy}{dx} = -2Kwx$$

and $y = -Kwx^2 + B$ where B is a constant. When $x = l$ $y = 0$

and therefore $B = Kwl^2$. Hence $y = Kw(l^2 - x^2)$. Also when $x = 0$, $y = \delta'$, whence $\delta' = Kwl^2$.

For a beam of rectangular cross-section it has been shown that $K = \frac{3}{5bdG}$ and therefore $\delta' = \frac{3Wl}{5AG}$ where $W = wl$ the total load, and A is the area of the cross-section. The result can be obtained in another manner. Let the deflection be y at a distance x from the free end. Then the slope is $-\frac{dy}{dx}$ and $\frac{dy}{dx} = -\frac{F}{AG}$ where F is the shearing force on the beam at the section assumed constant over the section. Since $F = wx$, $y = -\frac{wx^2}{2AG} + C$ where C is a constant and therefore $y = \frac{w}{2GA}(l^2 - x^2)$. At the free end $x = 0$ and the deflection is δ', therefore

$$\delta' = \frac{wl^2}{2AG}.$$

On considering the result for the concentrated load, it will be seen that the shear is not constant over the section. This result should therefore be multiplied by $\frac{6}{5}$, hence

$$\delta' = \frac{3Wl}{5AG}.$$

It should be noted that the theory given on shear in beams is only approximate. It applies fairly well to long beams of nearly square rectangular cross-section or very similar shaped sections, and to narrow beams. The error may be appreciable for other shapes of cross-section, including the circular, and also for very short beams. The problem is somewhat similar to that of the torsion of non-circular prisms, and is treated in books on elasticity.

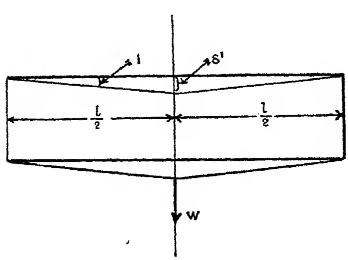


Fig. 97.

61. Approximate Method for I Beam—The shear deflection of an I beam can be found on the assumption that the web withstands the whole of the shear stress uniformly distributed over it. Referring to Fig. 97, the angle $i = \frac{\delta'}{\frac{l}{2}}$ where δ' is the

shear deflection, hence $\delta' = \frac{il}{2}$. If s is the average shear stress in the web $\frac{s}{i} = G$ where G is the Modulus of Rigidity and $\delta' = \frac{sl}{2G}$ but $s = \frac{W}{2A_w}$ where A_w = area of web, hence $\delta' = \frac{lW}{4A_wG}$. The deflection due to bending is $\Delta = \frac{Wl^3}{48EI}$.

Assume that the web withstands the whole of the bending stresses, then $I = \frac{A_f d^2}{2}$, where A_f = area of one flange and d is the depth of the beam.

Hence $\Delta = \frac{Wl^3}{24EA_f d^2}$. Also $\frac{\delta'}{\Delta} = \frac{6E}{G} \frac{A_f}{A_w}\left(\frac{d}{l}\right)^2$.

Example—A 10 in. × 5 in. × 30 lb. per ft. length rolled-steel joist has a web 0·36 in. thick and 8 in. deep, and the mean thickness of each flange is 0·552 in. Find the ratio of the shear to the bending deflection when (1) the length is ten times the depth, and (2) when it is twenty times the depth. The area of the web = 8 × 0·36 = 2·65 sq. in. and the area of one flange = 2·76 sq. in., so that $\frac{A_f}{A_w} = \frac{2 \cdot 76}{2 \cdot 65} = 1 \cdot 04$.

Taking $\frac{E}{G} = \frac{5}{2}$ then $\frac{6E}{G} = 15$ and $\frac{\delta'}{\Delta} = 15 \cdot 5 \left(\frac{d}{l}\right)^2$.

When $l = 10d$ $\frac{\delta'}{\Delta} = \frac{15 \cdot 5}{100} = 15 \cdot 5$ per cent.

and when $l = 20d$ $\frac{\delta'}{\Delta} = \frac{15 \cdot 5}{400} = 3 \cdot 9$ per cent.

In the case of a beam of rectangular cross-section breadth b, depth d, and length l, and supporting a concentrated load W in the middle, the total deflection Δ has been shown to be

$$\Delta = \frac{Wl^3}{48EI}\left\{1 + 3\left(\frac{d}{l}\right)^2\right\},$$ assuming that the ratio $\frac{E}{G} = \frac{5}{2}$.

When calculating the value of Young's Modulus from the results of deflection experiments the deflection due to shear is sometimes neglected, that is, E is taken as $\frac{Wl^3}{48\Delta I}$ where Δ is the measured deflection, whereas it should be $E = \frac{Wl^3}{48\Delta I}\left\{1 + 3\left(\frac{d}{l}\right)^2\right\}$

the percentage error being $\left\{\dfrac{3\left(\dfrac{d}{l}\right)^2}{1+3\left(\dfrac{d}{l}\right)^2}\right\}\times 100$. If this is calculated for several values of $\dfrac{l}{d}$ the following quantities are obtained :—

	percentage error $= \left\{\dfrac{3\left(\dfrac{d}{l}\right)^2}{1+3\left(\dfrac{d}{l}\right)^2}\right\}\times 100$
	15·8
	7·70
	4·48
10	2·91
20	0·74
50	0·12
100	0·03

It is clear how rapidly the error increases as the ratio $\dfrac{l}{d}$ diminishes.

62. Graphical Method for Finding the Shear Deflection —The formula (1) for the shear-strain energy sorted in a short length δx of a beam subjected to a shearing load F is

$$\delta W' = \frac{\delta x}{2G}\int_{y'}^{y''} s^2 z\,dy,$$

where y' and y'' are the distances of the extreme fibres from the neutral surface. It was shown in Art. 57 that

$$s = \frac{F}{Iz}\times A\bar{x} = \frac{FA'Y}{Iz},$$

where Y is the distance of either base line of the first modulus figure from the neutral axis and A′ the smaller portion of the area of the modulus figure below or above the plane of width z, parallel to the neutral plane along which the shear stress is s. Hence if the cross-section is uniform, the total energy stored in the beam is

$$W' = \frac{Y^2}{2GI^2}\left\{\int_{y'}^{y''}\frac{A'^2}{z}\,dy\right\}\left\{\int_0^l F^2dx\right\}$$

The method of calculating this expression will be explained by referring to Fig. 98. Here is shown a view of the cross-section of a beam, the shaded figure being the first modulus figure. Let the area of this figure above the plane whose end

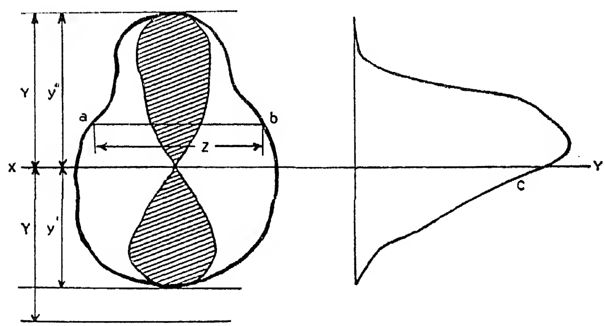


Fig. 98.

view is ab be A′, then this area squared and divided by z is equal to $\frac{A'^2}{z}$.

The values of $\frac{A'^2}{z}$ for a number of other planes such as ab are now obtained and a figure drawn having the values as ordinates. This figure is denoted by c in Fig. 98. The area of this figure will be the quantity $\int_{y'}^{y''}\frac{A'^2}{z}dy = A''$.

Hence $W' = \frac{A''Y^2}{2GI^2}\int_0^l F^2dx$, so that if the value of F in terms of W, the load or loads on the beam and of x is known, W′ can be found. The shear deflection can then be determined by equating W′ to the work done by W in producing the shear deflection.

Let the linear scale of the original cross-section of the beam

be 1 in. $= l$ in. Then 1 sq. in. area of the first modulus figure $= l^2$ units of A′ and

$$\frac{(1 \text{ sq. in. area of the first modulus figure})^2}{z} = \frac{l^4}{l} = l^3 \text{ units of } \frac{A'^2}{z}.$$

Also 1 sq. in. area of the figure (*c*) $= l^4$ units of

$$\int_{y'}^{y''} \frac{A'^2}{z} dy = l^4 \text{ units of } A''.$$

Hence the area of the figure (*c*) multiplied by l^4 by $\frac{Y^2}{2GI^2}$ and by $\int_0^l F^2 dx$ is the shear-strain energy stored in the beam.

63. Shear Deflection of Tram Rail Section by Calculation and Experiment—The investigation given for finding the shear deflection is based on certain assumptions. In addition to the assumption, considered in Art. 57, that the shear stress across planes parallel to the neutral plane is constant, a further assumption is involved when finding the deflection caused by a shearing force which is variable, that is, when the shearing force is represented by an expression of the first or higher order. The error introduced by these assumptions is small and in most practical cases can be safely neglected. For the more exact theory the reader is referred to the work of St. Venant, some of which is translated in Todhunter and Pearson's "History of the Theory of Elasticity." The following case of the comparison of the calculated and experimental * deflection for a tram rail section will illustrate the degree of accuracy of the approximate theory.

A view of the section in question is given in Fig. 99. The section is divided into fourteen horizontal strips each $\frac{1}{2}$ in. wide. The intensities of the shear stresses along the planes whose end views are the lines which define the sections were calculated by the expression already given. This is $s = \frac{FA\bar{X}}{IZ}$, where F is the shearing force on the beam, I the moment of inertia, A the area of the section above or below the plane of width Z where the shear stress is *s*. The values of $A\bar{X}$ and Z in the above fraction were measured for the area corresponding with each horizontal line, and are given in Table I opposite. The values of $\frac{A\bar{X}}{Z}$ are given in the fifth column and are also plotted as horizontal ordinates

* A great part of the following is taken from an article by W. C. Popplewell in "The Engineer," September 24, 1909.

from the vertical base line on the diagram (*b*) shown on the left-hand side of the rail. In this diagram the curve drawn through the ends of the ordinates indicates the variation of shear stress over the section, each ordinate representing $\frac{A\bar{X}}{Z}$.

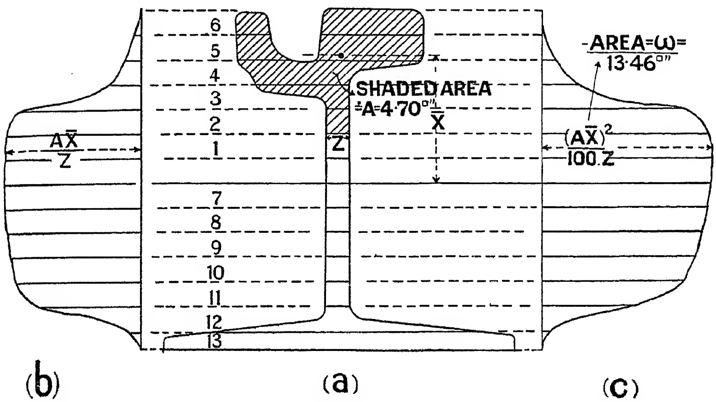


Fig. 99.

TABLE I

Area above line. Number	A	$\bar{X}$	Z	$\frac{A\bar{X}}{Z}$	$\left(\frac{A\bar{X}}{Z}\right)^2 Z$	
1	4·92	2·52	0·46	26·95	333·8	Above XY
2	4·70	2·60	0·46	26·56	323·9	
3	4·47	2·66	0·48	24·68	292·2	
4	3·94	2·75	1·85	5·86	63·5	
5	2·78	3·02	2·98	2·82	23·5	
6	1·35	3·27	2·65	1·67	7·4	
7	4·64	2·64	0·46	26·63	326·5	Below XY
8	4·41	2·74	0·46	26·27	317·5	
9	4·18	2·84	0·46	25·81	307·0	
10	3·95	2·94	0·46	25·25	293·0	
11	3·72	3·02	0·46	24·42	274·6	
12	3·46	3·08	0·56	19·03	166·5	
13	2·48	3·20	5·75	1·38	10·9	

The rail was centrally loaded, and calling the load W and δ' the shear deflection

$$\frac{W\delta'}{2} = \frac{W^2 l}{8GI^2}\int\frac{(A\bar{X})^2}{Z}dy \text{ and } \delta' = \frac{Wl}{4GI^2}\int\frac{(A\bar{X})^2}{Z}dy.$$

Taking $\frac{E}{G}=\frac{5}{2}\left(\text{that is, Poisson's Ratio}=\frac{1}{4}\right)$

$$\delta'=\frac{5}{8}\frac{Wl}{EI^2}\int\frac{(A\bar{X})^2}{Z}dy.$$

On the right-hand side of the rail section is drawn a second diagram (*c*) from the figures in the sixth column of Table I. Here the ordinates represent $\frac{(A\bar{X})^2}{Z}$ to a scale of $\frac{1}{100}$, while the vertical abscissæ are full size. The enclosed area (ω) of this diagram is $\omega=\frac{1}{100}\int\frac{A\bar{X}}{Z}dy$, hence $\delta'=\frac{62{\cdot}5\,Wl\omega}{EI^2}$.

Young's Modulus E was found by experiments on a sample cut from the rail to be 28,400,000 lb./sq. in. W was 3,510 lb., I was 72·0 in.[4] units, and $\omega=13{\cdot}46$ sq. in. Hence

$$\delta'=5{\cdot}75\times Wl\times 10^{-9}.$$

The bending deflection $\Delta'=\frac{Wl^3}{48EI}=1{\cdot}02\ Wl^3\times 10^{-11}$.

In Table II are tabulated the calculated values of δ' and Δ' and also their sum $\delta'+\Delta'=\Delta$. In the fifth column are the corresponding deflections Δ as obtained directly by experiment.

TABLE II

Span.	Deflections of Calculation.			Direct deflection by experiment.
Feet.	Inches.			Inches.
	δ'	Δ'	Δ	Δ
5	0·0077	0·0012	0·0089	0·0090
6	0·0133	0·0014	0·0147	0·0145
7	0·0212	0·0017	0·0229	0·0230
8	0·0316	0·0019	0·0335	0·0330
9	0·0450	0·0022	0·0472	0·0470
10	0·0619	0·0024	0·0643	0·0640
11	0·0824	0·0026	0·0850	0·0845
12	0·1065	0·0029	0·1094	0·1090
13	0·1355	0·0032	0·1387	0·1390

In finding the quantities A and $\bar{X}$ the areas A were computed with a planimeter from a carefully drawn section and the positions of the centres of gravity of the areas beyond each line by cutting out in cardboard and balancing. As a matter of fact, the area was drawn on cardboard and then cut away through (1), through

(2), and then through (3), and so on until the whole had been dealt with. Where great accuracy is desired, this method of obtaining values of $A\bar{X}$ is generally to be preferred to that of determining them from the first modulus figure.

The deflection experiments were made on an Amsler Laffon oil-pressure machine in the Materials Testing Laboratory of the Manchester College of Technology. In this machine the span may be varied at will. A lever deflection meter was used to measure the deflections capable of reading to $\frac{1}{2,000}$th in.

64. Direction of Principal Stress in Beams—When a beam is subjected to bending alone the principal stresses act along the direction of its length. When in addition shear is present, the lines of action of the principal stresses are inclined. In Fig. 100 is shown the side view of a small triangular prism, ABC, situated in the tension side of a beam where it is acted upon by positive bending moments and shearing forces, the side AB of the prism coinciding with a normal cross-section of the beam. The shear stress acting on the prism is denoted by s and the tensile stress induced by bending by p. The problem is to find the magnitude and direction of r, which is one of the principal stresses. Resolving horizontally :

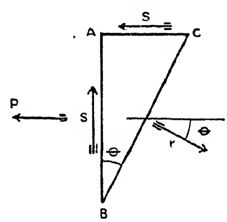


FIG. 100.

$$p.\mathrm{AB} + \mathrm{s.AC} = r.\mathrm{BC} \cos \theta$$
$$p \cos \theta + \mathrm{s} \sin \theta = r \cos \theta$$
$$(r - p) \cos \theta = \mathrm{s} \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Resolving vertically :

$$r.\mathrm{BC} \sin \theta = \mathrm{s.AB}$$
$$r \sin \theta = \overset{s}{\cancel{y}} \cos \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and $$\tan \theta = \frac{\cancel{y}\,s}{}$$

The product of (1) and (2) gives :

$$r^2 - rp - s^2 = 0$$

$$r = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + s^2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating r from (1) and (2) :

$$p \cos \theta \sin \theta = s(\cos^2 \theta - \sin^2 \theta)$$

$$\frac{p}{2} \sin 2\theta = s \cos 2\theta$$

$$\tan 2\theta = \frac{2s}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From equations (3) and (4) the magnitude and direction of the principal stresses at any point in the beam can be found if p and s are known. It should be noted that the directions at any point are mutually perpendicular and that one value of r is tensile and the other compressive.

In Fig. 101 is shown a side view of a freely supported beam which supports loads W at points one-third of its length from either end. In the portion of the beam between the loads $s = 0$, and therefore $r = p$ or 0. Also $\tan 2\theta = 0$, hence $\theta = 0$ or $\frac{\pi}{2}$. At the upper and lower surfaces of the beam between either support and the nearest load $s = 0$, hence $r = p$ or 0 and $\theta = 0$ or $\frac{\pi}{2}$. At the neutral axis between a support and the nearest load $p = 0$, therefore $r = \pm s$ and $\tan 2\theta = \infty$ or $\theta = \frac{\pi}{4}$. In Fig. 101 a few of the lines showing the directions of the principal

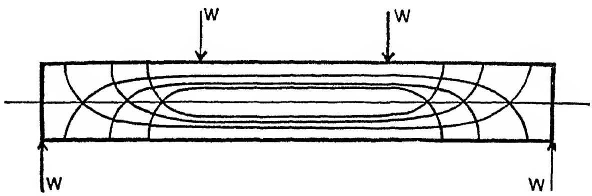


FIG. 101.

stresses are drawn. If the deflection is small and the breadth of the beam not great compared with its depth, these lines are sensibly the side views of cylindrical surfaces which cut each other orthogonally and normal to which the stress is wholly normal. For materials weak in tension, such as concrete, failure generally begins by cracking along surfaces which approximate to those normal to which the principal tensile stresses act.

65. Shear Stress in Ferro-concrete Beams—In Fig. 102 is shown a view of a ferro-concrete beam of rectangular cross-section reinforced on the tensile side. The shear stress on the compression side, that is, the side above the neutral plane, is obtained by a method similar to that used for homogeneous beams. The normal planes AA and BB (Fig. 102) define a slice of the beam of thickness δx, the plane AA being distant x from some origin. The bending moment at BB is greater than at AA, and hence the compressive stress induced by bending is greater on BL′ by the amount δP, which tends to cause the piece ALL′B to slide from right to left. Let f be the compressive stress on

the plane AA at distance y from the neutral axis and let $f+\delta f$ be the corresponding stress on the plane BB. Then

$$\delta P = \Sigma_{y'}^{Kd} \delta f b \delta y$$

but

$$\frac{f_c}{Kd} = \frac{f}{y} \text{ and } M = \frac{f_c b K d h}{2}$$

where h is the vertical distance between the resultant compressive force in the concrete and the axis of the bars, and f_c is the

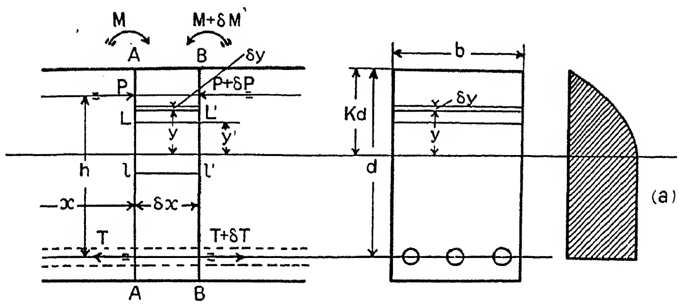


FIG. 102.

maximum compressive stress in the concrete. Hence

$$f = \frac{2My}{bh(Kd)^2}$$

also

$$\delta f = \frac{2\delta My}{bh(Kd)^2}$$

and if s is the shear stress on the plane LL′ then

$$\delta P = s\delta x b = \Sigma_{y'}^{Kd} \frac{2\delta My}{h(Kd)^2}\delta y$$

and in the limit

$$s = \frac{F}{bh(Kd)^2}\int_{y'}^{Kd} y dy.$$

This may also be written

$$s = \frac{F}{h(bKd)^2}A\bar{X}$$

where A is the area of the section above the line parallel to the neutral axis where the shear stress is s, and $\bar{X}$ the distance of the centre of gravity of the area from the neutral axis. At the neutral axis $A = bKd$ and $\bar{X} = \frac{Kd}{2}$, hence $s = \frac{F}{hb}$.

In considering the tension side it will be assumed that the

concrete withstands no tension in the direction of the length of the beam. Let T be total tension in the bars at the plane AA and $T+\delta T$ the tension at the plane BB, then $h\delta T = \delta M$. If s is the shear stress across any plane ll' below the neutral axis, $\delta T = s\delta x b$.

Hence $$s = \frac{\delta M}{\delta x h b} = \frac{F}{hb}$$

The diagram (*a*), Fig. 103, shows the variation of the shear over the section, the portion of the diagram above the neutral axis being parabolic.

Now let q be the shear stress between the surface of the bars and the concrete and O the total perimeter of the bars. Then

$$\delta T = Oq\delta x.$$

Hence $$q = \frac{F}{hO}$$

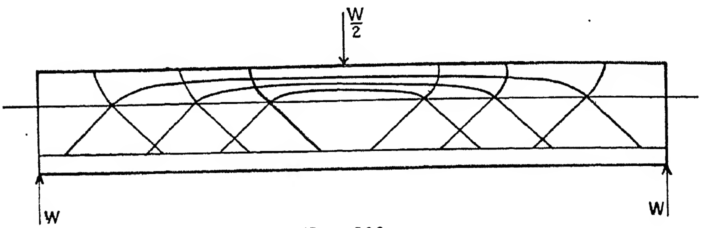


FIG. 103.

It may be noted that since a shear stress produces direct tensile and compressive stresses on planes inclined to the direction of the shear stress, it follows that if the concrete were incapable of withstanding tensile stress, shear stresses could not exist. In Fig. 103 is shown a view of a freely supported beam carrying a concentrated load W at its mid-length. The curves drawn on the side of the beam are a few of the lines of action of the principal stresses. The curves for the portion of the beam above the neutral axis are similar to those for a homogeneous beam. Those below are drawn on the assumption that a pure shear alone exists in the concrete, direct tension in the direction of the length of the beam being assumed zero.

EXAMPLES. VI

(1) A rectangular wooden beam 12 in. deep and 6 in. wide carries a continuous load of 1 ton per foot run and is simply supported. The breaking strength of the wood is 6, 3, and 0·3 tons per sq. in. tension, compression, and shear respectively. What span will give a factor of safety of 4 ?

(2) A beam 10 in. square rests on two supports and carries a central load of 10 tons. Find the direction and magnitude of the principal stresses at a point 25 in. from a support and 3 in. below upper surface.

(3) Find from the data given below the ratio of the deflection due to shearing and bending stresses respectively in a rolled-steel joist when used as a cantilever. The joist is 12 in. deep, the flanges are 6 in. wide, the thickness of the web is $\frac{1}{2}$ in., and of the flanges $\frac{7}{8}$ in. The length of the cantilever is 5 ft., and it is loaded at the free end. Take Young's Modulus and the Modulus of Rigidity as 30×10^6 and 12×10^6 lb. per sq. in. respectively. (S. and A.)

(4) A timber beam 12 in. square rested on supports in a testing machine and was centrally loaded. When the load reached 21 tons the beam split along the neutral axis. Calculate the approximate shear stress.

(5) A girder 20 in. deep has flanges $7\frac{1}{2}$ in. wide and 1 in. thick and a web $\frac{13}{32}$ in. thick. The greatest moment of inertia is 1,650 in. units and the total shear over the section is 80 tons. Show by a diagram the intensity of shear stress at all points of the section.

(6) Find the greatest shear stress at a section of a girder at which the total shear is 15 tons; the overall depth is 8 in., flanges 6 in. $\times$ 0·61 in., web 0·44 in. thick. (B.Sc., Lond.)

CHAPTER VII

STRESS IN COMPRESSION MEMBERS

INSTANCES in practice where material is subjected to simple compression loads are very numerous. Compression members of framed structures, isolated pillars, brickwork, masonry, concrete, earth foundations, bedplates and machinery frames, cast-iron tubbing in mine shafts, piston rods, and connecting rods represent a few cases. In any one of these the stress may be uniform or variable. The member may be a short pillar, which is practically a slab on end with very little tendency to bend under load; or it may be a strut or a pillar in which the ratio of length to cross dimensions is sufficiently great to make it necessary to reckon with the possibility of an appreciable amount of bending taking place along with crushing; or again it may be a thin pillar where there is a great deal of bending and little crushing. Compression members naturally divide themselves into the above three classes. The first and second are of the most frequent occurrence. Thin pillars seldom occur in practice, but it is necessary to investigate their relations between load and stress as a stage in similar investigations relating to pillars of practical dimensions.

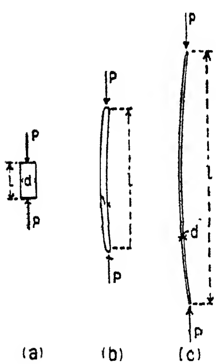


FIG. 104.

The application of an axial tensile load to a bar is a stable process. On the other hand, a compression load tends to make the bar unstable, and if it is not quite straight, tends further to increase the defect and to destruction by flexure. For this reason, compression loads are avoided as far as possible on long structural members.

In Fig. 104 are shown views of the types of three pillars which have been mentioned, namely, (*a*) the short pillar in which there is no visible bending, and if the load is axial there is a uniform compressive stress on sections normal to its geometrical axis; (*c*) the long pillar in which bending takes a large part in the

ultimate failure ; and (*b*) the medium-length pillar of engineering practice in which the conditions are somewhere between those of (*a*) and (*c*).

66. Load and Stress in Short Pillars—In Fig. 105 is shown in greater detail (*a*) Fig. 104. Here *l*, *m*, *o*, *p* is the side view of a short pillar or prism of irregular cross-section, the shape of the cross-section being shown below. The dotted line represents the geometrical axis and a load, P, is shown acting along a second dotted line which does not coincide with the axis. If P acts along the geometric axis the compressive stress normal to every section will be uniform and equal to $\frac{P}{A}$ where A is the area of the section. If, however, P acts nearer to the side F, the intensity of the stress will increase towards F and diminish towards the opposite side at G.

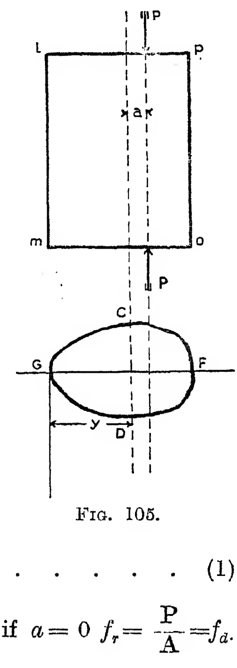


Fig. 105.

The resultant stress f_r at any point is made up of two stresses, one the direct compressive stress f_d and the other a stress f_b, due to the action of the bending moment which arises from the fact that P acts at a distance *a* from the neutral axis CD. The resultant stress f_r is given by the relation $f_r = f_d \pm f_b$.

But $f_d = \frac{P}{A}$ and $f_b = \frac{My}{I} = \frac{Pa.y}{Ak^2}$, where M = P*a* is the bending moment on the pillar, *y* is distance from the neutral axis of the point where f_b is required, A is the area of the section, and *k* the radius of gyration about the neutral axis CD. Hence

$$f_r = \frac{P}{A}\left(1 \pm \frac{ay}{k^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this expression it will be noted that if $a = 0$ $f_r = \frac{P}{A} = f_d$. If $\frac{ay}{k^2} = 1$, that is, if $a = \frac{k^2}{y}$, f_r vanishes, and if *a* is greater than $\frac{k^2}{y}$ there will be tensile stresses on the section, which in most practical cases have to be specially avoided. Thus it appears that for no tensile stress the load line must not be allowed

to stray beyond certain limits from the axis of the prism. In the case of a circular cross-section of radius r, $a = \frac{r^2}{4r} = \frac{r}{4}$.

In the case of a rectangular cross-section suppose the load acts at a point in the portion ABCD (Fig. 106) whose co-ordinates are x and y. Then the bending will be unsymmetrical, and the stress at any point of the section whose co-ordinates are x', y' is

$$f_r = \frac{P}{A} + \frac{12yy'}{bd^3} + \frac{12xx'}{bd^3}$$

$$= \frac{P}{A}\left\{1 + \frac{12yy'}{b^2} + \frac{12xx'}{d^2}\right\}$$ where A is the area of the cross-section.

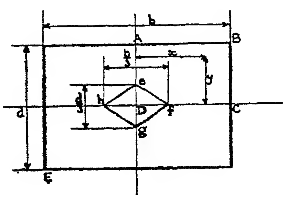


FIG. 106.

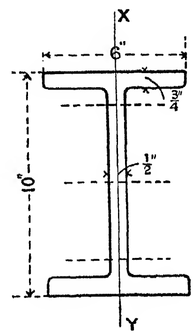


FIG. 107.

The least value of this is obtained by putting $x' = -\frac{u}{2}$ and $y' = -\frac{b}{2}$ and is accordingly at the corner E. Making this substitution, $f_r = \frac{}{A}\Big\{1 - \frac{6y}{b} \quad 6x$, which is evidently zero when $y = -\frac{bx}{d} + \frac{b}{6}$. This is the equation to the straight line ef, and if the above argument be applied to the other quarters of the rectangular section it will appear that for no tensile stresses on the section the load must act within the diamond-shaped figure $efgh$ with diagonals of lengths $\frac{b}{3}$ and $\frac{d}{3}$.

A typical girder section is shown in Fig. 107, and the limits along the line XY for no tension are indicated by the dotted lines. It is instructive to note that in this section, where the area is spread well away from the centre, more latitude is possible than if the section were more concentrated.

The resultant stress anywhere on a section is given by

$f_r = \frac{P}{A}\left(I \pm \frac{ay}{k^2}\right)$ and the uniform stress by $\frac{P}{A}$, so that the difference per cent. is $\frac{ay}{k^2} \times 100$.

In the table below this expression is evaluated for various values of a, which in the case of the girder are measured in the direction of the line XY.

a	Difference per cent.		
	Circle (diameter d).	Rectangle (depth d).	Girder (depth d).
0·05d	40	30	—
0·10d	80	60	30
0·20d	160	120	60
0·30d	240	180	90
0·40d	320	240	120
0·50d	400	300	150

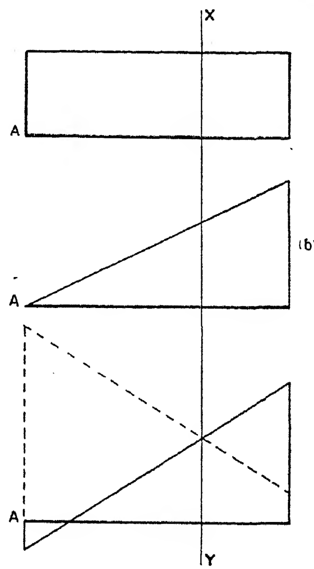


FIG. 108.

67. Graphic Representations of Stress in a Short Pillar—The stress variations described above are shown in Fig. 108 (a), (b), and (c). The line XY represents the geometrical axis of the pillar and also the neutral axis of the sections. The height of each diagram from the base lines AB represents the stress corresponding with the three types of loading. Diagram (a) shows the distribution of stress when the load is applied along the geometrical axis and the stress is uniform across the section. Diagram (b) shows the distribution when the load is applied at such a distance from the axis as to make the stress zero on one side of the section, whilst the full-line diagram at c represents the case where the load is applied at a slightly greater distance from the axis than in (b), so that there is tensile stress on one side of the section. It will be noted that in each case the stress at the neutral axis is the same and that the change in stress on the

two sides of the section caused by eccentricity of loading is not the same unless the section is symmetrical about the line XY. If in case (c) the eccentricity were on the side towards A instead of towards B, and of equal amount, the diagram would be as shown by the dotted lines at (c) and there would be no tensile stress on the section.

The stress variations can be shown experimentally by measuring the change in the lengths of the opposite sides of the pillar where they are cut by the plane containing the axis of the pillar and the line of loading. A convenient method of doing this is by a Marten's Extensometer. A blade, B (Fig. 109), has one end bent over and sharpened to a knife edge, whilst the other end has a V-shaped notch cut into it. The blade is laid on the side

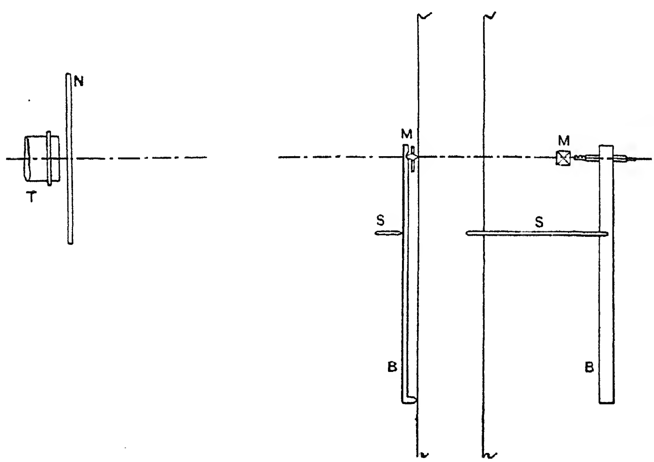


FIG. 109.

of the pillar and a diamond-shaped piece of steel is placed with one edge at the bottom of the V and the opposite edge on the pillar. The length of the blade is in the direction of the length of the pillar, and it will be evident that when the side of the pillar alters in length the diamond piece will tilt slightly. It is by measuring this tilt that the change in length is determined.

The tilt is measured by a reflection method. A small mirror, M, is fixed to one end of the diamond piece and a scale, N, and telescope, T, are arranged so that the scale is reflected by the mirror along the telescope. The telescope is fitted with a cross wire, and on looking through it when the mirror is tilting the scale and cross wire appear to move with respect to each other. If the distance of the mirror from the scale is known and the

width of the diamond piece, the change in length of the specimen can be calculated corresponding with the tilt of the mirror as registered by the scale and telescope. In the actual instrument the scale is placed about 47 in. from the mirrors and one division on the scale represents $\frac{1}{1,000}$th in. change in length of the specimen.

A similar arrangement of blade, mirror, etc., is arranged on the opposite side of the specimen, and the two blades are held in position by means of the spring S. If the changes in length corresponding with a certain load are unequal on the two sides, there is bending taking place. The usual procedure is to increase

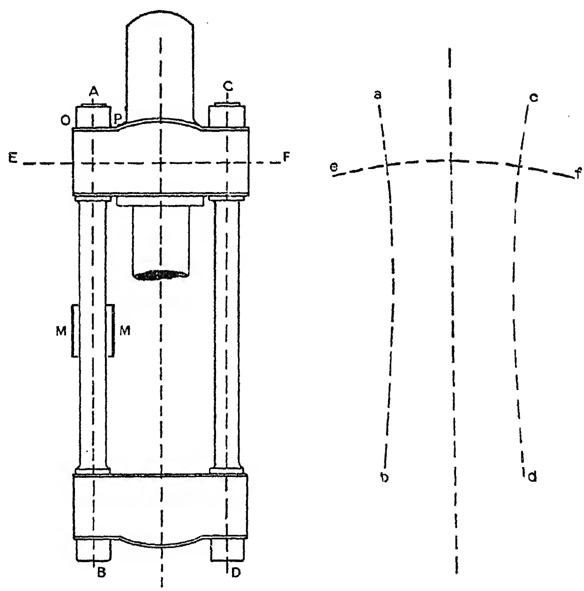


Fig. 110.

the load by small equal amounts and read after every increase. The loads and readings are then plotted, and from the difference in the slopes of the two resulting lines the amount of bending can be ascertained. Below the elastic limit the lines will be straight and represent to scale the actual stress in the material. The proportions of the instrument are such when set up that the error introduced by plane scales is negligible.

An instance of the application of the above method is illustrated in Fig. 110, which represents a large compression-testing machine or hydraulic press such as is installed in the Materials

Testing Laboratory of the Manchester College of Technology. When the machine is in use the cross-heads act as beams supported by the pillars AB and CD. Considering the upper cross-head, a horizontal straight line such as EF before the pressure is applied becomes slightly curved, as shown by the dotted line *ef*. This causes the pressure on the nuts at P to be greater than at O, and there is thus an eccentric tensile load on the pillars. That such is the case is indicated by fixing a pair of Marten's blades and mirrors, MM, to the pillars. When the pressure is applied the extension on the inside is found to be appreciably greater than on the outside, and hence the centre lines of the pillars become curved, as shown by the dotted lines *ab* and *cd*.

68. Mode of Failure of Short Pillars—When a short pillar of brittle material is loaded to such an extent that collapse follows, the failure may take the form of slide on a surface or surfaces inclined to the axis, or if the strength in the lateral direction is low, it may split into a number of prisms, whereas if the material is soft, the piece generally fails by crumpling into small pieces or even powder. The first of the three types is the most normal one: the most familiar examples are compression-test specimens of cast iron, hard steel, stone, and concrete.

In this type failure generally commences along surfaces whose normals are inclined at 45 deg. with the axis of the prism, but when actual slide begins the angle of shear is changed, owing to the friction of movement between the two sliding surfaces. This was considered in Art. 34, and it was shown that if the angle between the normal to the shear plane and the axis of the prism is θ and the angle of friction ϕ, then $\theta = \frac{\pi}{4} + \frac{\phi}{2}$. In this equation θ can be obtained by experiment, and hence ϕ can be calculated, and also μ, the coefficient of friction, since $\mu = \tan \phi$.

In the case of short ductile pillars of mild steel and wrought iron, failure occurs, not as a rule by shearing, but by lateral bulging. This is best seen to manifest itself in the case of cylindrical pieces, and can be noticed as soon as the compression-yield point has been passed. In some cases the shortening can be continued until the piece is pressed down flat in the form of a circular plate, but cracks usually appear in the side before this stage is reached.

It should be noticed that in the case of wrought iron and ductile steel the passage of the yield point is followed by a shortening of the cylinder and a uniform expansion for the greater part of the length. The portions beyond this parallel part near the ends become roughly conical, partly owing to the effect of friction between the platens of the machine and the specimen in retarding the spreading tendency. The form

afterwards merges into the barren shape, but for a considerable range of load there is a perfectly parallel region.

The relation between the stress p and proportional strain e for tension, compression, or torsion of ductile materials when the yield point has been passed is of the form $\log p = \mathrm{K} \log e + \log \mathrm{C}$, where K and C are constant. This equation is usually written $p = \mathrm{C}e^{\mathrm{K}}$ and values of C and K for compression for annealed soft steel and wrought iron are given below.

	C	K
Annealed soft steel	63·11	0·251
,, wrought iron . . .	49·68	0·178*

The experiments to determine these constants were performed on an Amsler-Laffon oil-pressure machine. The load was increased by equal amounts, and after every increase the diameter of the specimen was measured and also the distance between a pair of scratches on the sides placed so far from the ends as to be in the permanently parallel region. The measurements were made with a micrometer microscope.

The above figures provide a relation between stress and proportional strain for mild steel and wrought iron when in the approximately plastic state.

It is interesting to note what the relation between the nominal stress and the proportional strain would be if the material were perfectly plastic. If P be the load or nominal stress and A_1 the corresponding area, then for plasticity $\frac{\mathrm{P}}{\mathrm{A}_1} = \text{constant}$. But $\mathrm{A}l = \mathrm{A}_1 l_1$ where A and l are the area and length of the unloaded specimen, hence

$$\frac{\mathrm{P}l_1}{\mathrm{A}l} = \text{constant}$$

and

$$\mathrm{P} - \frac{\mathrm{P}l_1}{l} = \mathrm{P} - \text{constant}.$$

$$\frac{\mathrm{P}(l - l_1)}{l} = \mathrm{P} - \text{constant}$$

or

$$\mathrm{P}(1 - e) = \text{constant},$$

where e is the proportional strain. The curve showing the

* W. C. Popplewell, "Manchester Lit. and Phil. Soc. Proc.," Vol. XLIX, Part V.

relation between P and e is shown in Fig. 111. It is a rectangular hyperbola with asymtotes XO and XZ. It will be noted that as e approaches unity the nominal stress P approaches infinity.

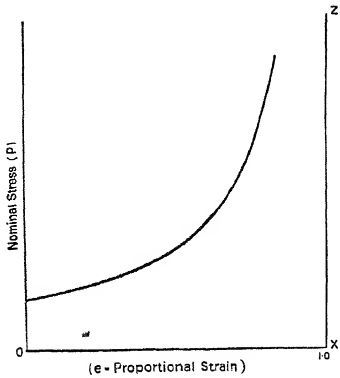


FIG. 111.

69. Loads carried by Long Pillars, Euler's Formula—When in practice a load sufficiently large is applied to a long pillar in the direction of its length it causes it to bend and ultimately to collapse by flexure. If the pillar were perfectly straight and the load could be applied perfectly axially there would be no bending moment on it, and the load could be increased until failure occurred by direct crushing. The pillar, however, becomes unstable at certain critical loads and if it be disturbed when the critical load has just been passed it will fail by buckling. Since in practice ideal conditions cannot be attained, bending will ensue as soon as the load is applied and failure will occur before or in the region of the first critical load.

In Fig. 112 (a) is shown a long pillar hinged at its ends and supporting a vertically applied load, P. When the first critical load is just passed the pillar will tend to take the form shown, and if disturbed its mid-point will move outwards by an amount δ, which increases very rapidly as the critical load is exceeded. The bending moment M on the pillar at a point distant x from the upper end is Py. The relation between x, y, and M is

$$\frac{M}{EI} = \pm \frac{d^2y}{dx^2}.$$

Calling M negative when it bends the strut concave towards its initial position, the equation becomes $\frac{Py}{EI} = -\frac{d^2y}{dx^2}$.

Integrating, $\frac{P}{EI}\frac{y^2}{2} = -\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + C$, where C is a constant.

When $\qquad \frac{dy}{dx} = 0 \quad y = \delta$ and $C = \frac{P}{EI}\frac{\delta^2}{2}$

$$\therefore \quad \frac{P}{EI}(\delta^2 - y^2) = \left(\frac{dy}{dx}\right)^2 \text{ or}$$

$$\frac{\perp}{EI}dx = \frac{dy}{\sqrt{\delta^2 - y^2}}$$

Integrating again,

$$\frac{}{\mathrm{EI}}\, x = \sin^{-1}\frac{y}{\delta} + \mathrm{A},$$

where A is a constant. When $x = 0, y = 0$ $\therefore$ $\mathrm{A} = 0$ and

$$= \delta \sin\left(x\sqrt{\frac{\mathrm{P}}{\mathrm{EI}}}\right) \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I})$$

When $$x = \frac{\cdot}{2} \quad y =$$

hence $$1 = \sin\left(\frac{l}{2} \qquad \overline{\frac{\mathrm{P}}{\mathrm{EI}}}\right)$$

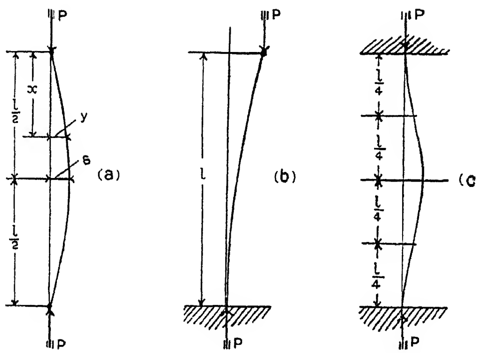


FIG. 112.

The first solution is

$$\frac{\pi}{2} \quad \frac{l}{2}\sqrt{\frac{\mathrm{P}}{\mathrm{EI}}}$$

or $$\mathrm{P} = \frac{\pi^2\mathrm{EI}}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{II})$$

Of these equations (I) is the equation to the sinusoid or centre line of the pillar when loaded, (II) is Euler's equation, which gives the buckling load P of a long pillar of length l, the least moment of inertia of whose cross-section is I and the elastic modulus of the material, E. Equation (II) refers to a long pillar whose ends are pressed upon by the load P in such a manner that they are quite free to turn through small angles. Such a condition is referred to as "free ends," and is approximated to in engineering practice when hinges are employed, an important case being an engine connecting rod. All pin-jointed struts

come in the same category, but, although the ends may turn about the centre lines of their pin holes, friction tends in some degree to retard angular movement.

The case of a strut fixed at one end and free at the other is shown in Fig. 112 (*b*). When buckling begins the strut takes the shape of the upper half of that shown at (*a*) and the buckling load is obtained by putting $\frac{l}{2}$ for l in equation (II), thus—

$$P = \frac{4\pi^2 EI}{l^2}$$

At (*c*) is shown a strut fixed at both ends. The curve splits up naturally into three parts, the middle part is equivalent to a free pillar of length $\frac{l}{2}$ and the end portions are equal to one-half the above, or $\frac{l}{4}$. The buckling load is obtained by putting $2l$ for l in equation (II), thus—

$$P = \frac{\pi^2 EI}{4l^2}$$

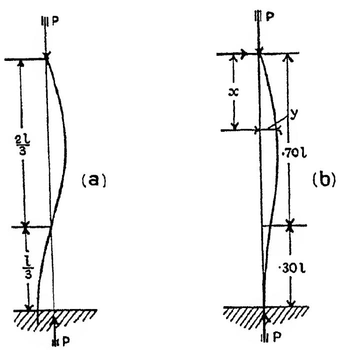


Fig. 113.

In Fig. 113 (*a*) is shown a strut fixed at one end and free at the other. A vertical line drawn from the top of the strut cuts it at the point of inflexion distant $\frac{1}{3}l$ from the bottom. The upper portion is thus equivalent to a free strut of length $\frac{2}{3}l$ and the buckling load is obtained by putting $\frac{2}{3}l$ for l in equation (II).

Hence
$$P = \frac{9\pi^2 EI}{4l^2}.$$

This buckling load is the second critical load for case (*b*) of Fig. 112, so that the strut would fail before it was reached. A more general case is where the free end is constrained to move in a vertical line which is tangent to the fixed end. In order to make this possible a horizontal force, H, must act at the free end, as shown in Fig. 113 (*b*).

The bending moment at any section distant x from the top

end is $Py - Hx$, hence $-EI\frac{d^2y}{dx^2} = Py - Hx$. Let $y = y' + u$, where $y' = \frac{Hx}{P}$.

Then $\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2}$ and the equation becomes $-EI\frac{d^2u}{dx^2} = Pu$ or $\frac{d^2u}{dx^2} = -a^2u$, where $a = \sqrt{\frac{1}{EI}}$.

Integrating, $\left(\frac{du}{dx}\right)^2 = -a^2u^2 + b^2$ where b is a constant, hence

$$dx = \frac{du}{\sqrt{b^2 - a^2u^2}}.$$

Integrating again,

$$c + x = \frac{1}{a}\sin^{-1}\frac{au}{b}$$

or $u = y - y' = d\sin(ax + e)$, where c, d, and e are constants.
Hence $y = \frac{H}{P}x + d\sin(ax + e)$ and $\frac{dy}{dx} = \frac{H}{P} + da\cos(ax + e)$.
When $x = 0$, $y = 0$, therefore $d\sin e = 0$ or $e = 0$.
Also when $x = l$, $y = 0$ and $\frac{dy}{dx} = 0$, and therefore

$$\frac{H}{P}l + d\sin(al + e) = 0$$

and

$$\frac{H}{-} + da\cos(al + e) = 0.$$

From the last two equations $\tan(al + e) = al$, and since $e = 0$ $\tan al = al$. The solution of this gives

$$al = 4{\cdot}493 = l\sqrt{\frac{P}{EI}}$$

This may be written $P = \frac{2{\cdot}045^2\pi EI}{l^2}$

or sensibly $\frac{2\pi^2 EI}{l^2}$

At the point of inflexion the bending moment vanishes and hence $Hx = Py$ and the equation

$$y = \frac{H}{\;} \cdot d\sin(ax + e)$$

becomes $\sin(ax) = \pi$

or $$x = \sqrt{EI}\ \pi$$

also $$l = \sqrt{2{\cdot}045EI}\ \cdot \pi$$

hence $$\frac{x}{l} = \overline{\sqrt{2{\cdot}045}} = 0{\cdot}70,$$

so that the point of inflexion is $0{\cdot}70\ l$ from the top of the strut.

Consideration of the end conditions is very necessary both in engineering structures and testing. In testing it is not difficult to satisfy the free conditions by using carefully placed knife-edges or rollers; in practice it is difficult to eliminate friction. The fixed-end condition is nearly approximated to by riveting in framed structures, and some reinforced concrete pillars can be considered as fixed at the ends.

70.—Pillars of Medium Length—In very short pillars bending is ignored and ultimate failure is assumed to take place in one of the usual ways, by shearing or by plastic deformation. In long pillars it is only the elastic conditions which have to be reckoned with which may cause collapse by flexure. In columns of medium length such as are used in engineering practice, failure is in part due to crushing of the material, partly to the flexure of the complete strut, and in some cases to local failures of the ends. The strut formulæ of Gordon, Rankine, Johnson, and Ritter (among many others) are partly empirical and partly rational, and take into account both the direct crushing and the flexure. Gordon based the formula which he recommended on the results of a long series of experiments carried out by Hodgkinson.

Gordon's formula is interesting chiefly because it is the origin of some of the more modern formulæ and is $P = \dfrac{Af}{1 + c\left(\dfrac{l}{d}\right)^2}$, where P is the buckling load, A the area of the cross-section, f the stress near the yield stress or breaking stress in the case of brittle materials, l the length, d the smallest cross dimension, and c an empirical constant. Values of c were obtained from the experiments for columns having all the principal sections. The columns tested were mostly of cast iron and the derived constants are not applicable to the struts of modern steel.

Apart from the material, another objection to Gordon's formula lies in the fact that a different constant c is required for each section. This was overcome by Rankine by substituting the least radius of gyration k for d. With this alteration the formula becomes

$P = \dfrac{Af}{1+a\left(\dfrac{l}{k}\right)^2}$, where a is the constant and the other quantities are as defined above.

Rankine's formula is similar to that derived from a consideration of the equation $\dfrac{1}{P}=\dfrac{1}{P_c}+\dfrac{1}{P_e}$, where P is the crippling load for a strut of any length l, $P_c=\dfrac{f}{A}$ is the crushing load when the strut is very short, and $P_e=\dfrac{\pi^2 EI}{l^2}$ is the buckling load when the strut is long and has free ends. It will be seen that if the strut is long P_e will be small and $\dfrac{1}{P_e}$ will be great compared with $\dfrac{1}{P_c}$ and flexure will be the main factor in the failure. On the other hand, if the strut is very short P_e will be great and $\dfrac{1}{P_e}$ will be small compared with $\dfrac{1}{P_c}$, hence sensibly $P = P_c$.

On substituting values for P_e and P_c the equation becomes

$$\frac{1}{P}=\frac{1}{Af}+\frac{l^2}{\pi^2 EAk^2}$$

hence

$$P=\frac{Af}{1+\dfrac{f}{\pi^2 E}\left(\dfrac{l}{k}\right)^2}$$

The values of f and a generally used and also of $\dfrac{f}{\pi^2 E}$ taking $E = 30 \times 10^6$ lb. per sq. in. for wrought iron and steel, 15×10^6 lb. per sq. in. for cast iron, and $1{\cdot}5 \times 10^6$ lb. per sq. in. for timber, are given in the table on following page.

These values of a and $\dfrac{f}{\pi^2 E}$ are for free ends. If one end is fixed and the other free, the constants should be multiplied by 4; if both ends are fixed, they should be divided by 4. If one end is fixed and the free end is constrained to move in a line tangent to the fixed end, the constants should be divided by 2.

The last formula when written in the form $P=\dfrac{Af}{1+\dfrac{mf}{4\pi^2 E}\left(\dfrac{l}{k}\right)^2}$ is Ritter's formula. The value of m is evidently 16 if one end is fixed and the other free and 4 if both ends are free. Its value

Material.	f (lb. per sq in.).	Empirical constant a	$\frac{f}{\pi^2 E}$
Wrought Iron . .	36,000	$\frac{1}{9,000}$	$\frac{1}{8,210}$
Mild Steel . . .	48,000	$\frac{1}{7,500}$	$\frac{1}{6,180}$
Hard Steel . . .	70,000	$\frac{1}{5,000}$	$\frac{1}{4,230}$
Cast Iron. . . .	80,000	$\frac{1}{1,600}$	$\frac{1}{1,850}$
Timber	7,200	$\frac{1}{750}$	$\frac{1}{2,050}$

for other end conditions are in proportion to the numbers given above.

It may be noticed that the value of the constant c in Gordon's formula is connected with the constant a in Rankine's by the relation $c = a\left(\frac{d}{k}\right)^2$.

The three formulæ given are of one type, and of the three Rankine's is generally used, the value of the empirical constant a depending on the nature of the material.

The values of f and a can be determined experimentally as follows: Suppose, for example, the constants are required for mild steel struts 2 in. wide and $\frac{1}{2}$ in. thick. At least twelve struts should be prepared, varying in length from about 4 in. to 4 ft., and these should be buckled in the testing machine. This will yield a series of values of the buckling loads P, the corresponding values of $\frac{l}{k}$ being calculated from the dimensions of the struts. Now Rankine's formula can be written $\frac{1}{P} = \frac{1}{fA} + \frac{a}{fA}\left(\frac{l}{k}\right)^2$, so that, if the values of $\frac{1}{P}$ as ordinate are plotted against values of $\left(\frac{l}{k}\right)^2$ as abscissa, the points will lie

approximately on a straight line and the intercept on the ordinate of $\frac{1}{P}$ is evidently $\frac{1}{fA}$, whilst the slope of the line is $\frac{a}{fA}$, so that both the value of f and a are determined. If the constants are required for free ends, the ends of the struts need only bear against the flat platens of the machine. If for fixed ends, the ends of the struts should be gripped in special fittings which can be secured to the platens.

Johnson's Parabolic Formula—This differs from most of the others in use. The diminution of the load which can be carried as the length of the strut is increased is not effected by increasing a denominator which includes a function of the radius of gyration k, but by subtracting an increasing quantity which contains this. Professor Johnson devised this formula from the published results of two long series of tests on pillars, one by Considére and the other by Tetmajer. The formula is $\frac{P}{A}=f-B\left(\frac{l}{k}\right)^2$, where P is the crippling load, A the cross-sectional area, f the compressive yield stress in the case of ductile materials and the ultimate compressive stress in brittle materials, and B is an experimental constant. Johnson's values for f and B are given in the following table :—

Material.	Condition of Ends.	$\frac{l}{k}$ not greater than	f lb. per sq. in.	B
Wrought Iron.	Pin (hinged)	170	34,000	0·67
	Flat	210	34,000	0·43
Mild Steel .	Pin (hinged)	150	42,000	0·97
	Flat	190	42,000	0·62
Cast Iron . .	Rounded	70	60,000	6·25
	Flat	120	60,000	2·25
White Pine .	Flat	60	2,500	0·6

The constant B in Johnson's formula can be given a value which will make the curve connecting $\frac{P}{A}$ with $\frac{l}{k}$ tangential to Euler's curve. If this be so, then

$$\frac{P}{A}=p=\pi^2E\left(\frac{l}{k}\right)^{-2}=f-B\left(\frac{l}{k}\right)^2$$

and

$$\frac{dp}{d\left(\frac{l}{k}\right)}=-2\pi^2E\left(\frac{l}{k}\right)^{-3}=-2B\left(\frac{l}{k}\right)$$

from which $$B = \frac{f^2}{4\pi^2 E}$$

The empirical values of B in the above table are approximately equal to $\frac{f^2}{64E}$ for free ends and $\frac{f^2}{100E}$ for fixed or flat ends. In order that the curve connecting p and $\frac{l}{k}$ obtained with the empirical constants should be tangential to Euler's curve Johnson modified Euler's for free ends and fixed ends to $p = \frac{16Ek^2}{l^2}$ and $p = \frac{25Ek^2}{l^2}$ respectively. The difference between the empirical and algebraic values of B is owing to friction in the case of free ends and the insecurity of fastenings in the case of fixed or flat ends.

The curves from the Euler, Rankine, Ritter and Johnson formulæ are plotted in Fig. 114 for a case where both ends are free. The safe value of f has been taken as 5 tons per sq. in. and $\frac{l}{k}$ varies from 0 to 240.

If a strut is loaded in a testing machine where the force can be controlled, the effects are the following. As the load increases from zero the initial deflection, of which there is always a certain amount, is increased by the centre line taking the form of a sinusoid according to Euler's law. This deflection from the straight is small in amount and at first entirely elastic, and when the load is removed it disappears. When the elastic limit at the most highly stressed part of the column has been reached and passed, there commences a certain amount of permanent set on removal of the load. Increase of deflection continues until the arm of the bending moment is so large that the strut has no longer strength to resist flexure and sudden crippling takes place. The crippling load may be less in amount than the maximum. With certain types of testing machines this point is not as a rule appreciated.

Choice of Formula—In the United States it seems to be recognized that the Johnson formula is all round the most reliable within certain limits. When the results of the Tetmajer and Considère experiments are plotted from the same co-ordinates as points from the Johnson and Euler formulas, it is found that the Johnson curve very closely follows the experimental points between the limits $\frac{f}{\frac{P}{A}} = 1{\cdot}0$ and about $0{\cdot}5$. About this lower

limit the Johnson curve leaves the points and the Euler comes into them. It is therefore taken that the Johnson formula is the most accurate within the first range mentioned, and that the Euler should be used from about 0·5 downwards. In doing this the designer knows that he is making use of actual experimental results on modern columns of almost every possible shape and size. In addition, the two formulæ mentioned are more suitable for office work and rapid computation than that of Rankine.

Whether Rankine's or Johnson's formula is used the value of the stress f, which can be taken as the yield stress in ductile materials and the breaking stress in brittle materials, should be obtained from tests of the material in question. The safe load is usually taken as about one-sixth of the crippling load. Average values of f for the more common materials are given below.

f lb. per sq. in.				
Wrought Iron.	Mild Steel.	Harder Steel.	Cast Iron.	Timber (average).
34,000 to 36,000	40,000 to 46,000	Up to 70,000	60,000 to 80,000	6,000

The modulus of elasticity for wrought iron and steel may be taken as 30,000,000 lb. per sq. in., for cast iron as 15,000,000 lb. per sq. in., and for timber as 1,500,000 lb. per sq. in.

It is to be concluded that for mild-steel struts where $\frac{l}{k}$ is greater than about 150 it is safe to use the Euler equation, which is the most convenient where cross dimensions are desired for given loads. Below this limit the Johnson formula is the best.

As an example of the practical use of strut formulæ, the American Bridge Company used the following, where p is the safe stress in lb. per sq. in. of section.

$$P = \frac{15,000}{1 + \frac{1}{13,500}\left(\frac{l}{k}\right)^2}$$

Here the material was mild steel and the ends were free.

Example—A hollow mild-steel pillar with two pin ends is 30 ft. long and 10 in. external diameter and carries a load of 80,000 lb. If the factor of safety is six, find the thickness of the metal from Rankine's and Johnson's formulæ.

Let r be the internal diameter in inches, the moment of inertia $I=\frac{\pi}{4}(5^4-r^4)$, and the cross-sectional area $A=\pi(5^2-r^2)$, hence the radius of gyration squared $=k^2=\frac{5^2+r^2}{4}$. Rankine's formula for mild steel is

$$P=\frac{48{,}000A}{1+\frac{1}{7{,}500}\left(\frac{l}{k}\right)^2} \text{ hence}$$

$$6\times 80{,}000=\frac{48{,}000\times\pi(25-r^2)}{1+\frac{360\times 360\times 4}{7{,}500(25+r^2)}}$$

$$3{\cdot}19=\frac{625-r^4}{94{\cdot}1+r^2}$$

$$r^4=3{\cdot}19r^2-325=0$$

and $$r=\sqrt{16{\cdot}5}=4{\cdot}06 \text{ in.}$$

Thickness of metal $=5-4{\cdot}06=0{\cdot}94$ in.

Johnson's formula for mild steel is

$$\frac{P}{A}=48{,}000-\frac{(48{,}000)^2}{64E}\left(\frac{l}{k}\right)^2$$

$$\frac{6\times 80{,}000}{48{,}000\times\pi\times(25-r^2)}=1-25\times 10^{-6}\,\frac{360\times 360\times 4}{25+r^2}$$

$$3{\cdot}19(25+r^2)=625-r^4-12{\cdot}95(25-r^2)$$

$$r^4-9{\cdot}77r^2-221{\cdot}5=0$$

$$r=\sqrt{20{\cdot}6}=4{\cdot}53$$

and thickness of metal $=5-4{\cdot}53=0{\cdot}47$ in.

Using the constant $\frac{f^2}{4\pi^2E}$ instead of $\frac{f^2}{64E}$ the thickness comes out to be $t=0{\cdot}77$ in.

It will be noted that the thickness given by Johnson's formula is the smaller. On reference to Fig. 114 it will be seen that within the working range of Johnson's formula the safe stress given by the formula for a given value of $\frac{l}{k}$ is greater than that given by Rankine's, and hence the cross-sectional dimensions of a strut calculated from Johnson's formula to support a given load will be less. The thickness given by Rankine's formula approaches to that given by Johnson's, by taking $a=\frac{1}{13{,}500}$

instead of 7,500, as was done by the American Bridge Company. The formula becomes

$$p=\frac{48{,}000}{1+\frac{1}{13{,}500}\left(\frac{l}{k}\right)^2}\quad\text{and gives}$$

$$r=\sqrt{19\cdot0}=4\cdot36,\ \text{hence}$$

$$t=5-4\cdot36=0\cdot64\ \text{in.}$$

71. Local Failures of Struts—Failures of the individual parts of built-up struts may occur, but local failure in the more general sense is found in the case of flat-ended pillars. In concrete and reinforced-concrete columns it frequently happens

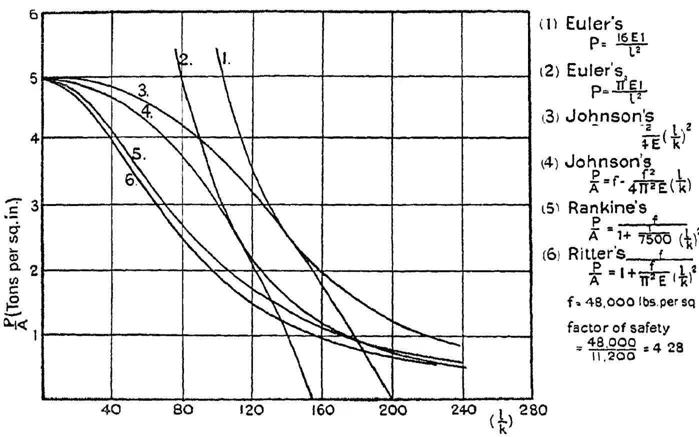


FIG. 114.

that before the load has become large enough to cause crippling in the middle the material near the ends gives way by shearing. This can be provided for by enlarging the ends or by surrounding them by steel caps or sheathing. This is always done in the case of reinforced-concrete piles which are subjected to impact. Reinforced-concrete pit props have also been observed to fail by shearing or crushing at the ends. The ends may be strengthened if desired by placing metal rings round them to prevent spreading of the concrete. Failure also often occurs at the ends of hardened-steel tubular struts such as are used in aeroplane construction, especially where the ends are brazed and so may have become softened.

If a short piece of a tubular strut with thin walls is compressed, failure will begin if the walls are thin enough by their folding or

crinkling such that the total load divided by the cross-sectional area is less than the elastic-limit stress of the material. An example is shown in Plate II (*a*). This has been investigated by the authors * for high-tensile steel tubing, and it was found that the proportional limit stress f in tons per sq. in. for annealed tubes was given by $f = 300\frac{t}{r}$, where t is the thickness and r the mean radius in inches. This formula was found to hold up to $\frac{t}{r} = 0{\cdot}1$. For greater values the load at the proportional limit divided by the cross-sectional area was equal to the elastic limit of the material, which was about 30 tons per sq. in.

When designing a strut of any shape of cross-section with thin walls such that failure might occur by crinkling, it is important to find the stress at which crinkling begins and use it in strut formulæ instead of the elastic limit or yield stress of the material. The crinkling stress can usually be found by loading a short piece of the strut in a testing machine and noting the load at which incipient crinkles are observed. For a high-tensile steel tube where $\frac{t}{r} = \frac{1}{10}$ or over, f, the elastic-limit stress, = 30 tons per sq. in., but if $\frac{t}{r} = \frac{1}{20}$, say, then f, the crinkling stress, = 15 tons per sq. in. and Johnson's formula becomes for pin ends

$$p = 15 - \frac{(15)^2}{64E}\left(\frac{l}{k}\right)^2$$

and Ritter's

$$p = \frac{15}{1 + \frac{m15}{4\pi^2 E}\left(\frac{l}{k}\right)^2}$$

where $m = 4$ for free ends. The crinkling stress for mild-steel tubes † is given by the formula $f = \frac{t}{r}$ up to $\frac{t}{r} = 0{\cdot}06$. For greater values of $\frac{t}{r}$ failure begins when the load divided by the cross-sectional area is equal to the elastic limit of the material.

It should be noted that it is usual to use the yield stress for ductile materials and the maximum stress for brittle materials in strut formulæ. If the proportional limit stress is used, which

* "Failure of Short Tubular Struts of High Tensile Steel," "Min. Proc. Inst. C.E.," vol. cciii.

† Barling and Webb, "Aeronautical Journal," Oct., 1918.

will be the elastic-limit stress in most cases but may be the crinkling stress in the case of struts with thin walls, a smaller factor of safety can be employed.

72. Eccentrically Loaded Struts—In Fig. 115 is shown a strut fixed at its lower end and supporting a load P, which is applied at a distance b from the upper end of the axis. The deflection of the upper end is denoted by δ. The bending moment at any section distant x from the fixed end is $P(\delta+b-y)$, hence $EI\left(\frac{d^2y}{dx^2}\right)=P(\delta+b-y)$ or $\frac{d^2y}{dx^2}+m^2y=m^2(\delta+b)$, where $m^2=\frac{P}{EI}$.

The solution of this equation by a method similar to that used for the axial load is

$$y=(\delta+b)(1-\cos mx).$$

When $x=l$ $y=\delta$, hence

$$\delta=b(\sec ml-1).$$

If $ml=0$ $\delta=0$, and for every other value of ml, that is, of $\sqrt{\frac{P}{EI}}\times l$, δ has a value.

The bending moment M at the foot of the column is

$$M=P(\delta+b)=Pb\sec ml,$$

and if there were no deflection, that is, if the struts were very short, so that δ equalled 0, the bending moment at the foot would be Pb. Hence the bending moment is increased by flexure in the ratio $\dfrac{\sec\sqrt{\frac{P}{EI}}\times l}{1}$.

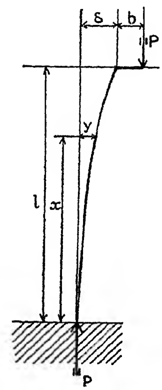


FIG. 115.

Let y' be the distance of the extreme fibres on the compression side from the neutral axis of the cross-sections. The greatest compressive stress will be $p=\frac{P}{A}+\frac{My'}{I}$, where A is the cross-sectional area and I is the moment of inertia of the area about its neutral axis. That is, $p=\frac{P}{A}\left(1+\frac{by'}{k^2}\sec ml\right)$, where k is the radius of gyration about the neutral axis. It may be noted that when $ml=\frac{\pi}{2}$, that is, when $P=\frac{\pi^2EI}{4l^2}$, p is infinite. This is Euler's equation for a strut fixed at one end and free at the other.

For a strut free at both ends write $\frac{l}{2}$ for l, and the equation becomes

$$p=\frac{P}{A}\left(1+\frac{by'}{k^2}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\right) \quad . \quad (1)$$

If P—the load on the strut, the dimensions, and the modulus E are given—p, the greatest compressive stress, can be calculated. If, on the other hand, p is the safe allowable compressive stress, the value of the safe load P can only be obtained by trial or graphically. If the expression

$$Y=\frac{P}{A}\left(1+\frac{by'}{k^2}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\right)-p$$

is plotted for different values of P as abscissa, then the abscissa of the curve corresponding with the ordinate $Y=0$ is the required value of P.

In order to design a strut of given length l and to support a load P at a given eccentricity b, it is usually best first to assume that the strut is short, so that there is no appreciable bending moment due to flexure. Formula (1) then becomes

$$p=\frac{P}{A}\left(1+\frac{by'}{k^2}\right)$$

from which a suitable shape and size of cross-section can be obtained by trial. Suppose that a rectangular cross-section is decided upon and that the dimensions given by the last equation are breadth a and depth in the plane of bending d. Owing to flexure, these dimensions must be increased. Let the correct breadth be a and the correct depth be cd, where c is a constant. Then equation (1) becomes

$$p=\frac{P}{acd}\left(1+\frac{6b}{cd}\sec\frac{l}{2}\sqrt{\frac{12P}{Eb(cd)^3}}\right)$$

The value of c can be obtained by evaluating the right-hand side for several values of c and thus finding, either by trial or graphically, the value which makes the expression equal to p.

The above procedure can be adopted for any shape of section. First design the section on the assumption that there is no flexure. The necessary increase in the area of the section to withstand the increased bending moment due to flexure can be obtained by putting cd in equation (1) for one of the dimensions d, where c is a constant, and finding the value of c either by trial or graphically. If the new value of d makes the shape of the section uneconomical or out of proportion, alter the original

dimensions and find the new value of c. After a few trials a suitable section will be obtained.

The function $\sec \frac{l}{2}\sqrt{\frac{P}{EI}}$ can be written $\sec \frac{\pi}{2}\sqrt{\frac{P}{Pe}}$, where $Pe=\frac{\pi^2 EI}{l^2}$, and Professor Perry * has shown that this is sensibly equal to $\frac{1\cdot 2}{1-\frac{P}{Pe}}$, where 1·2 refers to values of $\frac{P}{Pe}$ from 0·5 to 0·9. If $\frac{P}{Pe}=0\cdot 2$ the constant is 1·05. The solution of equation (1) is simplified by making the above substitution and becomes

$$p=\frac{P}{A}\left(1+\frac{by'}{k^2}\cdot\frac{1\cdot 2}{1-\frac{Pl^2}{\pi^2 EI}}\right)$$

or

$$\left(1-\frac{Pl^2}{\pi^2 EAk^2}\right)\left(\frac{Ap}{P}-1\right)=\frac{1\cdot 2\, by'}{k^2}$$

It is often convenient to evaluate the desired unknown from this equation and verify its value by substituting in equation (1).

In the case of materials weaker in tension than compression the ultimate tensile strength may be the governing factor in fixing the size and shape of the section. Cast iron is typical of such materials, its breaking tensile strength being about 10 tons per sq. in., whilst its maximum strength in compression is four or five times this. The greatest tensile stress p in the material of the strut will be $p=\frac{My''}{I}-\frac{P}{A}$, where y'' is the maximum distance of the surface of the strut on the tensile side from the neutral axis. The corresponding equation to equation (1) is

$$p=\frac{P}{A}\left(\frac{by''}{k^2}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}-1\right)$$

and, making Perry's substitution,

$$\left(1-\frac{Pl^2}{\pi^2 EAk^2}\right)\left(\frac{Ap}{P}+1\right)=\frac{1\cdot 2\, by''}{k^2}$$

These equations, which involve the tensile stress, are dealt with in the same manner as those involving the compressive stress.

Example 1—A vertical mild-steel strut 4 in. diameter is fixed at its lower end and supports a vertical load of 10 tons at its

* "The Engineer," December 10 and 24, 1886.

upper end, which acts at a point $\frac{1}{2}$ in. from the centre. Find the greatest compressive stress in the strut, (1) if it is 1 ft. long, and (2) if it is 15 ft. long. Also find for case (2) the deflection of the upper end of the strut from its initial position. Take $E = 30 \times 10^6$ lb. per sq. in. For case (1) the maximum compressive stress p is given by $p = \frac{P}{A}\left(1 + \frac{by'}{k^2}\right)$, hence

$$p = \frac{10}{12{\cdot}57}\left(1 + \frac{\frac{1}{2} \times 2}{1}\right) = 1{\cdot}59 \text{ tons per sq. in.}$$

For case (2) $p = \frac{P}{A}\left(1 + \frac{by'}{k^2} \sec l\sqrt{\frac{P}{EI}}\right)$

$$\frac{10}{12{\cdot}57}\left(1 + \frac{1}{2} \times 2 \sec 180 \sqrt{\frac{10 \times 2{,}240}{30 \times 10^6 \times 12{\cdot}57}}\right)$$

$$\frac{10}{12{\cdot}57}(1 + 5{\cdot}80) = 5{\cdot}41 \text{ tons per sq. in.}$$

The deflection δ is given by

$$\delta = b\left(\sec l\sqrt{\frac{P}{EI}} - 1\right)$$

$$= \frac{1}{2}\left(\sec 180 \sqrt{\frac{10 \times 2{,}240}{30 \times 10^6 \times 12{\cdot}57}} - 1\right)$$

$$= \frac{1}{2}(5{\cdot}80 - 1) = 2{\cdot}40 \text{ in.}$$

Example 2—A cast-iron pillar is 40 ft. long, with free ends and of hollow square section 10 in. square, the metal being 1 in. thick. It supports a load which acts on a line parallel to a pair of sides and 2·9 in. from the centre of the square. If the allowable safe tensile stress in the metal is 2 tons per sq. in., find the safe load and also the max. compressive stress. ($E = 15 \times 10^6$ lb. per sq. in.)

The area A of the section is

$$20 + 18 = 38 \text{ sq. in.}$$

The moment of inertia I equals

$$\frac{10{,}000 - 81 \times 81}{12} = 285{\cdot}8 \text{ (in.)}^4.$$

Hence k^2 the radius of gyration squared equals

$$\frac{285{\cdot}8}{38} = 7{\cdot}52 \text{ (in.)}^2.$$

The safe load P is given by the formula

$$p = \frac{P}{A}\left(\frac{by''}{k^2} \sec \frac{l}{2} \sqrt{\frac{P}{EI}} - 1\right)$$

$$2 = \frac{P}{38}\left(\frac{2 \cdot 9 \times 5}{7 \cdot 52} \sec 240 \sqrt{\frac{P \times 2{,}240}{15 \times 10^6 \times 286}} - 1\right)$$

$$2 = \frac{P}{38}(1 \cdot 93 \sec 0 \cdot 173\sqrt{P} - 1)$$

By trial P is found to be equal to 31·5 tons.

The max. compressive stress is

$$p = \frac{31 \cdot 5}{38}(1 + 1 \cdot 93 \sec 0 \cdot 173\sqrt{P})$$

$= 3 \cdot 66$ tons per sq. in., which is well within the safe limit.

73. Struts with Lateral Loads—These are the same as beams subjected to end thrusts. A simple case is shown in Fig. 116. The free ends of the beam are in the same horizontal line and are subjected to an end thrust P. In addition, the beam supports a load W at its mid-point, and the problem is to find the bending moment, M_0, at the mid-length of the beam. The bending moment at any section distant x from the centre of the line joining the two ends is $-\frac{W}{2}\left(\frac{l}{2} - x\right) - Py$, the bending moment being reckoned negative when it bends the strut concave towards its initial position. Hence

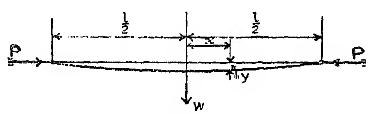


FIG. 116.

$$\frac{W}{2}\left(\frac{l}{2} - x\right) + Py = -EI\frac{d^2y}{dx^2}$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = -\frac{W}{2EI}\left(\frac{l}{2} - x\right)$$

The solution of this equation is

$$y = \frac{W}{2P}\sqrt{\frac{EI}{P}}\tan\frac{l}{2}\sqrt{\frac{P}{EI}} - \frac{Wl}{4P}$$

and

$$M_0 = -\frac{W}{2}\sqrt{\frac{EI}{P}}\tan\frac{l}{2}\sqrt{\frac{P}{EI}}$$

where y_0 and M_0 are the values of y and M when $x = 0$. Hence

$$M_0 = \frac{Wl}{4} + \frac{Wl^3}{48EI}P\left[1 + \frac{\pi^2}{10}\frac{P}{P_e} + \frac{M\pi^4}{20{,}160}\left(\frac{P}{P_e}\right)^2 + \dots\right]$$

It will be noted that the coefficients of $\frac{P}{P_e}$ are nearly unity. If the ratio of $\frac{P}{P_e}$, that is, if the load P on the strut divided by

P_e, the buckling load, calculated by Euler's formula, is small, the maximum bending moment is approximately given by $\frac{Wl}{4} + P\delta$. Here $\frac{Wl}{4}$ is the maximum bending moment due to the lateral load W and $P\delta$ the maximum bending moment due to the end thrust P where δ is the central deflection $= \frac{Wl^3}{48EI}$. The above case, in addition to others, has been solved by Prof. Morley,* but in many cases that occur in practice graphical methods have to be adopted. Such a method has been given by Dr. C. H. Lander,† and will be described by reference to Fig. 117. The freely supported beam AB carries the loads W_1, W_2, W_3, and W_4, and is subjected to an end thrust P. The method consists in first finding the shape of the bent beam due to the lateral loads only, which was described in Art. 55. This is done by drawing the bending-moment diagram for the loads on the beam which is represented by C, D, E, F, G, H in Fig. 117. Now treat the diagram as if it were a curve showing load intensity at any point of the beam and draw the bending-moment diagram for this load-intensity curve. The ordinates of the resulting figure divided by EI will give the deflection to scale due to the lateral system acting alone.

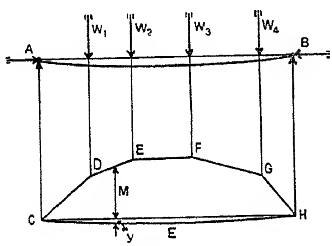


Fig. 117.

The shape of the bent beam or primary deflection diagram is represented by the line CEH in Fig. 117.

Let the ordinates of the bending-moment diagram due to the lateral loads be denoted by M and of the primary-deflection diagram by y. Then $M + Py$ would be the total bending moment on the beam at any section if the ultimate shape corresponded with the primary-deflection diagram, but it does not do so. A second approximation to the correct result is obtained by treating the secondary bending-moment diagram whose ordinates are Py in the same manner as the bending-moment diagram for the lateral loads. The secondary deflection so produced, $= y_1$, is now added to y, and the total bending moment becomes $M + P(y + y_1)$. Also, by treating the tertiary bending-moment diagram whose ordinates are $P(y + y_1)$ in a similar manner, another deflection curve with ordinates y_2 is obtained and

* "Phil. Mag.," June, 1908.
† "Phil. Mag.," January, 1914.

the total bending moment at any section is more accurately $M + P(y + y_2)$. By continuing the process, a curve is finally arrived at from which the total bending moment at any section can be obtained with the accuracy possible in graphical work. It is not usually necessary to proceed as far as the tertiary diagram—in many cases the second is not needed—and in most cases the total bending moment is accordingly given by $M \pm P(y+y_1)$, where the positive and negative signs refer to an end thrust or pull of amount P respectively. If the end load acts at a distance b from the longitudinal axis the equation becomes $M \pm P(b + y + y_1)$. The method can also be applied to beams with fixed ends and to continuous beams.

74. Ferro-concrete Pillars—The stresses in the steel and concrete of a short ferro-concrete pillar can be calculated if it be granted that both the materials shorten under the load in the same proportion. Let a_s be the total area of the longitudinal reinforcements and A the area of a normal cross-section of the pillar. Then $\frac{a_s}{A} = r$, the reinforcement ratio. If a_c is the area of the concrete, then $a_c = A - a_s = A(1 - r)$. Also let E_s and E_c denote Young's Modulus for steel and concrete respectively and e the strain which is the same for both. Then f_s, the stress in the steel, is given by $f_s = eE_s$ and the stress in the concrete is $f_c = eE_c$, hence $\frac{f_s}{f_c} = \frac{E_s}{E_c} = n$. The load carried by the steel $= f_s a_s$ and by the concrete $= f_c a_c$, hence the load P on the pillar is $P = f_s a_s + f_c a_c$. Substituting for f_s, a_s, and a_c, this becomes $P = f_c A\{1 + r(n - 1)\}$.

Allowing an average safe value of f_c of 600 lb. per sq. in., the stress in the steel will not exceed $f_c = 600n$, and if $n = 15$, $f_s = 600 \times 15 = 8{,}000$ lb. per sq. in. Mild steel can safely withstand a stress of 16,000 lb. per sq. in. and special steels much higher stresses, and hence it is desirable to increase the compressive strength of the concrete. This can be done by placing wire binding round the longitudinal reinforcements. The binding may be spiral or in separate hoops and the compressive strength of the concrete is increased by an amount depending upon the shape of the hoop, the cross-sectional area of the wire, and the longitudinal spacing or pitch. Empirical formulæ giving the increased allowable concrete stress are given in the London County Council regulations relating to reinforced concrete. The increased value of f_c thus obtained can be used in the formulæ just given.

Example—A reinforced-concrete column is 6 in. square and has four steel reinforcing bars $\frac{3}{4}$ in. diameter. The modulus of

elasticity of the steel is 30×10^6 lb. per sq. in. and it is found that a length of 12 in. is shortened 0·0032 in. by the application of a load of 13 tons. Find the modulus of elasticity of the concrete.

$$\frac{f_s}{f_c} = \frac{E_s}{E_c} = n, \text{ hence } f_c = \frac{f_s}{n}$$

Also the total load is given by

$$P = f_s a_s + f_c a_c = f_s a_s + \frac{f_s a_c}{n}$$

$$n = \frac{f_s a_c}{P - f_s a_s}$$

$$= \frac{0{\cdot}000267 \times 30 \times 10^6 \times (36 - 1{\cdot}77)}{13 \times 2{,}240 - 0{\cdot}000267 \times 30 \times 10^6 \times 1{\cdot}77}$$

$$= 17{\cdot}2, \text{ hence } E_c = \frac{30 \times 10^6}{17{\cdot}2} = 1{\cdot}74 \times 10^6 \text{ lb. per sq. in.}$$

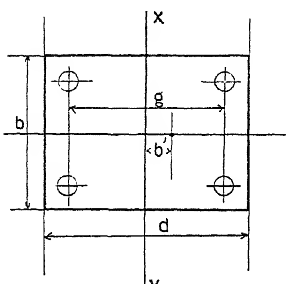


Fig. 118.

In the treatment of ferro-concrete pillars where the load is eccentric or the pillar is long so that flexure is involved, it is best to reduce the section to an equivalent concrete or steel section. In Fig. 118 is shown a normal section of a short ferro-concrete pillar and the load P acts in a direction parallel to the axis of the pillar and distant b' from it. It will first be assumed that the eccentricity is not sufficiently great to cause tension on the section, so that bending will ensue about the symmetrical axis XY. The section is reduced to an equivalent concrete section by assuming the steel to be replaced by concrete of n times its area, where $n = \frac{E_s}{E_c}$. The equivalent concrete area A'_c of the pillar will then be $A'_c = bd + (n - 1)a_s$, where a_s is the total area of the steel. Also the equivalent concrete moment of inertia I'_c about the axis XY will be sensibly

$$I'_c = \frac{bd^3}{12} + (n - 1)a_s\left(\frac{g}{2}\right)^2$$

Hence the maximum and minimum compressive stress f_c in the concrete is

$$f_c = \frac{P}{A'_c} \pm \frac{Md}{2I'_c}$$

Let the maximum and minimum values of f_c from the last equation be denoted by f'_c and f''_c respectively ; then the corresponding stresses f'_s and f''_s in the steel will be

$$f'_s = \frac{nf'_c g}{d} \text{ and } f''_s = \frac{nf''_c g}{d}$$

If $f'_c = 0$ then $\dfrac{\mathrm{P}}{\mathrm{A}'_c} = \dfrac{\mathrm{M}d}{2\mathrm{I}'_c}$ and since $\mathrm{M} = \mathrm{P}b'$, $b' = \dfrac{2\mathrm{I}'_c}{d\mathrm{A}'_c}$ and if b' exceeds this amount there will be tension in the concrete. Substituting the values for I'_c and A'_c, the equation becomes

$$b' = \left\{ \frac{\dfrac{bd^3}{12} + (n-1)a_s\left(\dfrac{g}{2}\right)^2}{b'd + (n-1)a_s} \right\} \frac{2}{d},$$

or

$$b' = \frac{\dfrac{d}{6} + \dfrac{(n-1)a_s}{bd}\left(\dfrac{g^2}{2d}\right)}{1 + \dfrac{(n-1)a_s}{bd}}$$

Hence if $g = \dfrac{d}{\sqrt{3}}$, $b' = \dfrac{d}{6}$, but since g is usually nearly equal to d, the value of b' is greater than this, so that, if the load acts within the middle third, there will be compression only on the section.

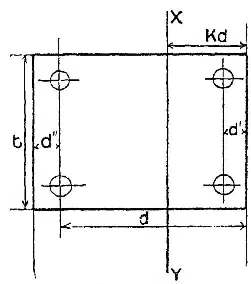


FIG. 119.

In cases where the value of b' is greater than that given by the above equation, then, since the concrete does not withstand tension, a different procedure must be adopted, and in what follows it is assumed that the neutral axis is in the same position as if the pillar were subjected to pure bending only. In Fig. 119 is shown a section of a ferro-concrete pillar where $\mathrm{K}d$ is the distance of the neutral axis from the compression side of the concrete determined as for a ferro-concrete beam, that is, on the assumption that there are no tensile stresses in the concrete. The equivalent steel area A''_s of the section is

$$\mathrm{A}''_s = \frac{b(d+d'')}{n} + \frac{(n-1)}{n}(a_s + a'_s),$$

where a_s and a'_s are the total areas of the steel in tension and compression respectively. The equivalent steel moment of inertia I''_s of the section about the axis XY is sensibly

$$\mathrm{I}''_s = a_s(d - \mathrm{K}d)^2 + \frac{(n-1)a'_s}{n}(\mathrm{K}d - d')^2 + \frac{b(\mathrm{K}d)^3}{3n}$$

Hence the compressive and tensile stresses f'_s and f''_s respectively in the steel are

$$f'_s=\frac{P}{A''_s}+\frac{M(Kd-d')}{I''_s} \text{ and } f''_s=\frac{Md(1-K)}{I''_s}-\frac{P}{A''_s}$$

and the compressive stress in the concrete is

$$f'_c=\frac{f'_s\times Kd}{n(Kd-d')}$$

It should be noted that the neutral axis passes through the centre of gravity of the equivalent section and in the case of tension on the section the neutral axis is the nearer to the load. Hence for no tension the bending moment M is equal to Pb', and if tension is present $M=P(b'-x)$ where x is the distance between the two axes.

These equations can also be written in terms of the equivalent concrete area and moment of inertia of the section. Let A''_c and I''_c denote these respectively, then $A''_c=nA''_s$ and $I''_c=nI''_s$. The expressions given for the moments of inertia neglect the moment of inertia of the area of the steel about the axis through its centre of gravity and parallel to the neutral axis of the section.

In the case of medium-length and long pillars the same formulæ given in this chapter for homogeneous material can be used, provided the proper values of the moment of inertia and area of the section are substituted. Hence, for free ends, Euler's formula becomes

$$P=\frac{\pi^2 E_s I'_s}{l^2}$$

Rankine's becomes

$$P=\frac{f_s A'_s}{1+\dfrac{l^2 A'_s}{7{,}500\;I'_s}}$$

and Johnson's

$$\frac{P}{A'_s}=f_s-\frac{f_s^2}{64E_s}\,\frac{l^2A'_s}{I'_s}$$

where A'_s and I'_s are the equivalent steel area and moment of inertia respectively, the position of the neutral axis and the value of I'_s being calculated on the assumption that there is no tension on the section.

Similar substitutions can be made in the equations for eccentric loading of long pillars, care being taken to substitute the proper values of the equivalent area and moment of inertia.

The preceding treatment can be modified to suit cases where

the concrete outside the hooped core is neglected or where the pillar is not of rectangular cross-section.

The proportions of ferro-concrete members should conform with the London County Council regulations relating to ferro-concrete, which also include empirical values to facilitate their design.

EXAMPLES. VII

(1) A vertical post 24 ft. in height supports at its upper end a horizontal arm projecting 6 ft. from the post. Find the horizontal and vertical displacements of the free end of the horizontal arm when a load of 6,000 lb. is suspended from it. Young's Modulus for post and arm $= 28 \times 10^6$ lb. per sq. in. Moment of inertia of post $= 412$ and of arm $= 360$ in.^{4} units. Neglect direct compression of post. (B.Sc., Lond.)

(2) A cast-iron beam 10 in. deep and 6 in. wide has a bar of steel 2 in. thick and 4 in. wide laid symmetrically along its upper surface and fixed firmly to it. Find the maximum stress in steel and cast iron if the difference in temperature is 60° F. Coefficient of thermal expansion for steel equals 12×10^6 per degree C. Young's Modulus for steel and cast iron equals 30×10^6 and 12×10^6 lb. per sq. in. respectively.

(3) A steel strut of circular cross-section 5 ft. long with hinged ends is subjected to a total compressive load of 30,000 lb., the line of action of which is eccentric to the geometrical axis of the strut by 0·08 d, where d is the diameter of the strut. Determine the necessary diameter of the strut if the maximum compressive stress is not to exceed 8,000 lb. per sq. in. Young's Modulus equals 30×10^6 lb. per sq. in. If the compression limit of elasticity is 45,000 lb. per sq. in., what is the factor of safety of the strut ? (S. & A.)

(4) The pull on a tie rod 4 in. wide and $\frac{1}{2}$ in. thick is $\frac{1}{4}$ in. out of the axial line in the direction of the breadth. What is the maximum stress under a load of 8 tons ?

(5) A mild-steel strut rectangular in cross-section, the breadth being four times its thickness, is 9 ft. long and has pin ends. Determine the cross-section for 24 tons and a factor of safety of 5. Use Rankine's formula and take f equal to 67,000 lb. per sq. in. and the constant $\frac{1}{5{,}000}$. (B.Sc., Lond.)

(6) A hollow, circular mild-steel column is 30 ft. long and has rigidly fixed ends. It is to support a load of 50 tons. The external diameter is 6 in. and the factor of safety 4. Find the thickness of the metal by Rankine's and Johnson's formulas, taking the yield stress as 42,000 lb. per sq. in. (Young's Modulus $= 30 \times 10^6$ lb. per sq. in.)

(7) Four wrought-iron struts rigidly held at the ends, all of section 1 in. $\times$ 1 in. and of length 15·0, 30·0, 60·0, and 90·0 in. respectively, are found to buckle under loads of 15·9, 11·3, 7·7, and 4·35 tons. Test whether these satisfy Rankine's and Gordon's formulæ, and if so, find the average values of the constants involved. (B.Sc., Lond.)

(8) What will be the probable collapsing load of a mild-steel strut

which is freely hinged at the ends, 15 ft. long between the hinges, with a solid section 3 in. in diameter ? (Young's Modulus $= 30 \times 10^6$ lb. per sq. in.)

(9) Find the diameter of each of four mild-steel struts which may be substituted for the single strut in the preceding question.

(10) Find the safe load on a rolled steel tee strut 6 in. $\times$ 4 in. $\times$ $\frac{1}{2}$ in. $\times$ 10 ft. long, fixed at one end and free at the other. (Young's Modulus $=30 \times 10^6$ lb. per sq. in.)

(11) A reinforced-concrete pillar fixed at its lower end is 9 in. square and is reinforced with four steel bars $1\frac{1}{4}$ in. diameter. Each bar is placed near a corner of the pillar with its centre 1 in. from the nearest sides. Find the maximum stress in the concrete and the stress in the bars when the pillar supports a load of 10 tons (1) if the pillar is short and the load axial ; (2) if the pillar is short and the load acts 1 in. from the centre on a line parallel to a pair of sides ; (3) if the pillar is 30 ft. long and the load acts 1 in. from the centre ; (4) if the pillar is short and the load acts 6 in. from the centre.

CHAPTER VIII

TORSION—TORSION COMBINED WITH BENDING—HELICAL SPRINGS—THE VIBRATION AND WHIRLING OF SHAFTS

75. Simple Twist of Circular Shaft—In Fig. 120 is shown a view of a circular elastic shaft of radius R and length l fixed at one end and free at the other. If a torque T is applied to the free end, it will rotate, and it will be evident that, since T is constant along the length l, the amount of rotation will be proportional to l. When the torque is applied the point B will move to C and the end of the shaft will rotate through the angle θ. Thus

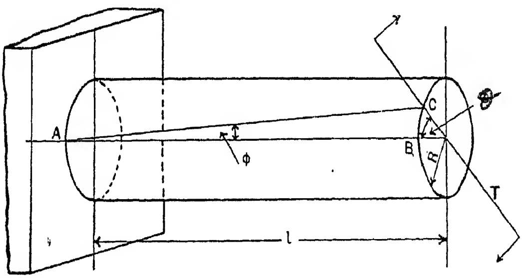


Fig. 120.

the straight line AB on the unstrained shaft will become a helical line AC, and if ϕ denote the angle BAC, then $\phi l = R\theta$ or $\phi = \frac{R\theta}{l}$.

The resisting moment of the shaft will involve pure shear strain on normal plane sections along the circumference of circles concentric with the axis. For equilibrium there must also be shear strain on axial planes along directions parallel to the axis of the shaft, and if a small square be drawn on the surface of the unstrained shaft with one side coincident with the line AB, then when the shaft is twisted the square will become a rhombus. The angles of the square will be changed by the amount ϕ, which is the shear strain on the surface of the shaft, and if S is the

corresponding shear stress, then $\frac{S}{\phi} = G$, where G is the modulus of rigidity. Substituting $\phi = \frac{R\theta}{l}$ then $S = \frac{R\theta G}{l}$.

In Fig. 121 is shown the end view of a shaft subjected to a torque T. Let the tangential stress on the ring of radius r' and small width $\delta r'$ be S', then the total stress on the ring is $2\pi r'\delta r'S'$ and the moment of this about the centre is $2\pi r'^2\delta r'S'$. The strain at any point in the shaft is proportional to the distance of the point from the centre, and since stress is proportional to strain, it follows that the stress is also proportional to the distance from the centre, hence $\frac{S}{R} = \frac{S'}{r'}$ where R is the radius of the shaft and S the stress on the surface. The torque T is accordingly given by

$$T = \frac{2\pi S}{R}\int_0^R r'^3\,dr'$$

$$= \frac{\pi S R^3}{2}$$

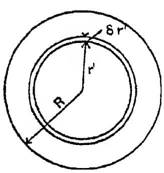


FIG. 121.

or, in terms of D, the diameter of the shaft

$$T = \frac{\pi S D^3}{16}$$

If the shaft is hollow with internal radius and diameter r and d respectively, then

$$T = \frac{2\pi S}{R}\int_r^R r'^3\,dr'$$

$$= \frac{\pi S(R^4 - r^4)}{2R}$$

$$= \frac{\pi S(D^4 - d^4)}{16D}$$

Substituting $S = \frac{D\theta G}{2l}$ in the above equations, then for a solid shaft

$$T = \frac{\pi G\theta D^4}{32l}$$

and for a hollow shaft

$$T = \frac{\pi G\theta(D^4 - d^4)}{32l}$$

These last two equations can also be written $T = \frac{GJ\theta}{l}$, where J

is the moment of inertia of a normal section of the shaft about the axis. If the shaft is not circular J is not the moment of inertia but has a smaller value.

Most problems in engineering relating to the torsion of shafts are concerned with the determination of the stress, diameter, or the twist in a given length when the shaft is transmitting power. Let the horse-power transmitted be denoted by HP and let N be the number of revolutions per minute, then $HP=\frac{2\pi NT}{33,000}$, where T is the torque in lb. ft. If ω be the angular velocity in radians per second, then $HP=\frac{T\omega}{550}$. The value of T can be obtained from these equations and used in the preceding formulæ.

For a solid shaft
$$T=\frac{\pi SD^3}{16}$$

hence
$$\frac{33,000\times 12\times HP}{2\pi N}=\frac{\pi SD^3}{16}$$

$$D^3=\frac{321,400\ HP}{N.S}$$

or
$$D=68\cdot5\sqrt[3]{\frac{HP}{N.S}}$$

The corresponding equation for a hollow shaft is

$$\frac{D^4-d^4}{D}=\frac{321,400\ HP}{N.S}$$

so that D and d can be calculated if their ratio is known.

The following values of the safe stress S may be taken:

For medium	carbon-steel	shafts	S = 8,000 lb. per sq. in.
,,	mild-steel	,,	S = 6,000 ,, ,, ,, ,,
,,	wrought-iron	,,	S = 5,000 ,, ,, ,, ,,

The twist in a length l is given by $\theta=\frac{lT}{GJ}$ where θ is in radius.

Hence
$$\theta=\frac{12\times 33,000\times HP\times l}{2\pi NGJ}$$

In order to convert θ from radius to degrees multiply it by $\frac{360}{2\pi}$

and $\theta^\circ=3\cdot41\times 10^6\frac{HP.l}{N.G.J.}$ where all the dimensions are in inches and HP is the horse-power. The modulus of rigidity for wrought iron and steel may be taken as 12×10^6 lb. per sq. in.

Example—A circular shaft transmits 100 H.P. at 100 R.P.M. and

the maximum turning moment is $\frac{3}{2}$ of the mean. Find the diameter if the greatest shear stress is 4 tons per sq. in. Also find the angle of twist in a 20 ft. length. Take $G = 6{,}000$ tons per sq. in.

$$D = 68{\cdot}5 \sqrt[3]{\frac{150}{100 \times 2{,}240 \times 4}}$$

$$= 3{\cdot}79 \text{ in.}$$

Also
$$\frac{\pi}{16} SD^3 = \frac{\pi G \theta D^4}{32l}$$

$$\theta^\circ = \frac{2lS}{GD} \times \frac{180}{\pi}$$

$$= \frac{40 \times 12 \times 4 \times 180}{6{,}000 \times 3{\cdot}79 \times \pi} = 4{\cdot}84 \text{ degrees.}$$

76. Twist beyond Elastic Limit—When a mild-steel shaft is twisted the stress while the shaft is elastic is proportional to the strain and thus the stress is greatest on the surface. As the twist is increased the material on the surface will yield, and if the twist is further increased there will be a concentric layer of more or less plastic material covering an elastic cylindrical core. A still further increase in the twist will cause the cylindrical boundary surface between the elastic core and the concentric plastic layer to decrease, until finally all the shaft is sensibly plastic.

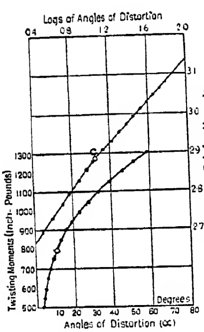


FIG. 122.

In Fig. 122 are shown two graphs which are typical of a torsion experiment on mild steel. The lower curve shows the relation between torques and resulting angles of distortion, and the upper curve was obtained by plotting the logarithms of these. It will be noted that the upper curve consists of two intersecting straight lines. The torque at the point of intersection *c* is the "change-point," or point where all the material of the shaft has just become sensibly plastic, so that there is no elastic core. That perfect plasticity is never attained is evidenced by the fact that if a shaft be twisted to just beyond the change-point, then, if the torque be kept constant, distortion will continue for some time, but will finally cease. These conditions hold until the torque is increased considerably beyond the change-point, when fracture will occur before the distortion has ceased. If the material

were perfectly plastic there would be no distortion until the torque reached a certain value, and for a slightly greater value distortion would commence and continue indefinitely or till fracture occurred. In all probability the material near the surface of the mild-steel shaft where the distortion is the greatest approaches the most nearly to the plastic condition. The material near the centre will accordingly be less plastic and there is probably always a fine thread of unstrained material about which the shaft has twisted.

In Plate II (*b*) is shown a portion of a mild-steel torsion test-piece which has been twisted to fracture. The spiral line was straight before the piece was twisted, the inclination to its original direction being about 60 deg. for mild steel, whilst it is usually a little less for wrought iron.

In the case of torque applied to a perfectly plastic shaft the shear stress S will be independent of the strain, and, referring to Fig. 122, the torque T will be

$$T = 2\pi S \int_0^R r^2 dr = \frac{2}{3}\pi R^3 = \frac{\pi S D^3}{12}$$

and the corresponding formula for a plastic shaft is

$$T = \frac{\pi S(D^3 - d^3)}{12},$$

where D and d are the external and internal diameters respectively. That the above relations sensibly hold for soft-steel and wrought-iron shafts when approaching the point of fracture has been shown in the results of published experiments. In some torsion experiments * on solid and hollow mild-steel test-pieces by W. C. Popplewell and Mr. E. G. Coker (now Professor Coker) the solid test-pieces had a diameter of 0·473 in. and the hollow ones of the same material had external and internal diameters of 0·500 in. and 0·313 in. respectively. The elastic formula gave an ultimate stress for the solid pieces of 29·09, and the plastic formula 21·82. Corresponding results for the hollow pieces were 27·51 and 23·15. The shear strength of the material from direct shear experiments was 22·45 tons per sq. in.

77. Direct Shear—An apparatus for finding the shear stress of flat bars of steel or iron is shown in Fig. 123. The bar is clamped to the block D through the plates PP. The portion of the bar which bridges the space S is securely held between the blocks EE, and the thrust T is applied as shown by the arrows. If a is the area of the bar, then the maximum single-shear stress is $\frac{T}{2a}$, where T is the thrust necessary to shear the specimen.

* "Min. Proc. Inst.C.E.," vol. cxxii.

In the actual apparatus the width of the space S is usually about 2 in. and the blocks EE should be a good sliding fit. The edges of the blocks which shear the specimen are hardened and ground to a well-defined corner. The ultimate shear stress is most accurately obtained from torsion tests on short hollow cylinders with thin walls, but the results given by the above method agree very closely with these.

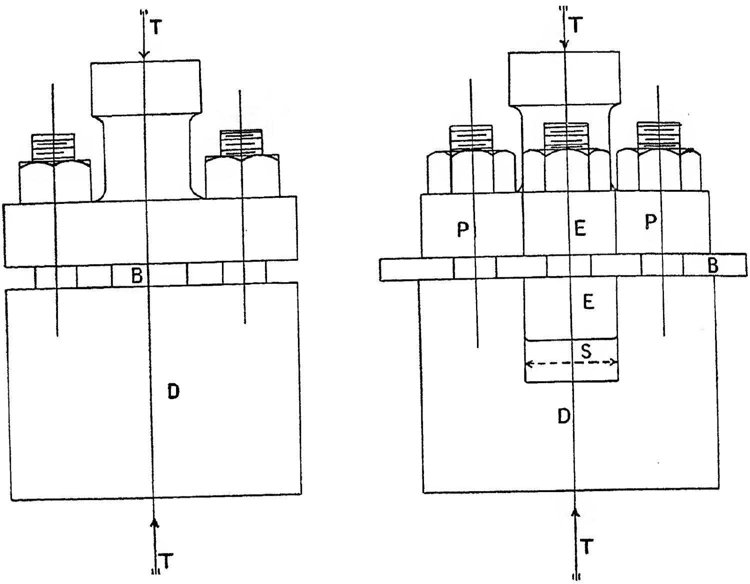


FIG. 123.

78. Torsion combined with Bending—This case resolves itself into finding the magnitude of the principal stresses arising from the shear stress due to torsion and the direct stress due to bending. In Fig. 124 is shown a view of a small triangular prism situated in a shaft which is subjected to torsion and to pure bending. The triangular base ABC of the prism coincides with the surface of the shaft where the direct tensile stress due to bending is greatest, that is, with the surface which is farthest from the neutral axes, the side AB being perpendicular to the axis of the shaft. The direct stress due to bending is denoted by p and the torsional stress by q, whilst r is one of the two

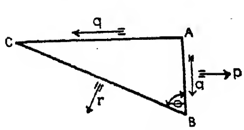


FIG. 124.

mutually perpendicular principal stresses and acts normal to the side BC. Considering the equilibrium of the prism and resolving horizontally

$$pAB = r\cos\theta.BC + qAC$$
$$(p-r)\cos\theta = q\sin\theta,$$

also resolving vertically

$$r\sin\theta.BC = -qAB,$$
$$r\sin\theta \quad = -q\cos\theta.$$

Eliminating $\sin\theta$ and $\cos\theta$, $(p-r)r=-q^2$ or $r^2-rp-q^2=0$.

Hence $$r=\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q^2}$$

The maximum shear stress S is one half the difference of the principal stresses (Art. 17) or

$$S=\sqrt{\frac{p^2}{4}+q^2}$$

The value of p is given by $p=\frac{MD}{2I}=\frac{32M}{\pi D^3}$, where D is the diameter of the shaft and M the bending moment.

Also $q=\frac{16T}{\pi D^3}$ where T is the torque. Hence r, the maximum principal stress, is given by

$$r=\frac{16M}{\pi D^3}+\sqrt{\left(\frac{16M}{\pi D^3}\right)^2+\left(\frac{16T}{\pi D^3}\right)^2}$$
$$=\frac{16}{\pi D^3}\{M+\sqrt{M^2+T^2}\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also S, the maximum shear stress, becomes

$$S=\frac{16}{\pi D^3}\sqrt{M^2+T^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The theory that a material fails when the maximum principal stress reaches a certain value was originally due to Lamé, but is sometimes called Rankine's theory. This theory is correct for most brittle materials, for which formula (1) should be used. In the case of ductile materials it is usual to accept Guest's theory, which asserts that failure begins when the maximum shear stress is reached, so that formula (2) should be used.

It may be noted that it is possible for direct strain to exist without corresponding stress, or direct stress without corresponding strain. In the case of a bar subjected to a simple tensile stress there will be lateral strain but no corresponding stress. Also, if a cube be subjected to normal stresses on two pairs of faces, the ratio of the stresses can be so adjusted that there will be

no strain in the direction of one series of stresses. There is accordingly another theory, which is due to St. Venant, according to which failure begins when the maximum principal strain reaches a certain value. Let e denote this strain, then $e=\frac{r_1}{\mathrm{E}}-\frac{r_2}{m\mathrm{E}}$, where r_1 and r_2 are the principal stresses and $\frac{1}{m}$ is Poisson's Ratio. Hence

$$e=\frac{1}{\mathrm{E}}\left\{\frac{p}{2}+\sqrt{\frac{p^2}{4}+q^2}-\frac{1}{m}\left(\frac{p}{2}-\sqrt{\frac{p^2}{4}+q^2}\right)\right.$$

and the equivalent stress is

$$\mathrm{E}e=\frac{p}{2}\left(1-\frac{1}{m}\right)+\sqrt{\frac{p^2}{4}+q^2}\left(1+\right.$$

If $\frac{1}{m}=\frac{1}{4}$ then $\mathrm{E}e=\frac{3}{8}p+\frac{5}{4}\sqrt{\frac{p^2}{4}+q^2}$. Thus r', the equivalent stress, is

$$r'=\frac{16}{\pi\mathrm{D}^3}\left\{\frac{3}{4}\mathrm{M}+\frac{5}{4}\sqrt{\mathrm{M}^2+\mathrm{T}^2}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

It will be noted that the stress given by equation (3) is greater than that given by equation (1). In practice it is usual to use equation (1) for all cases and to take the value of the factor of safety on the high side for ductile materials.

Equation (1) can be written

$$\mathrm{T}_e=\mathrm{M}+\sqrt{\mathrm{M}^2+\mathrm{T}^2}$$

or

$$\mathrm{M}_e=\frac{1}{2}\{\mathrm{M}+\sqrt{\mathrm{M}^2+\mathrm{T}^2}\}$$

where T_e and M_e are the equivalent torque and bending moment respectively. Either of these acting alone would induce direct stresses in the shaft equal to those induced by the combined action of M and T. Equations (2) and (3) can also be written in the above form.

In the case just considered of combined torsion and pure bending the maximum stresses will occur on the surface of the shaft. Due to the transverse loads there are, however, as a rule, additional shear stresses which are a maximum at the centre of the shaft, but, except in special cases of very short shafts, the maximum resultant stresses always occur on the surface.

Example—The crank-pin of an overhung crank is subjected to a thrust normal to the plane containing the axes of the crank-pin and crank-shaft of 33,100 lb. The throw is 15 in. and the

distance between the centre of the bearing and the centre of the crank-pin in the direction of their axes is 18 in. Taking a safe stress of 7,000 lb. per sq. in., find the diameter of the crank-shaft.

$$T_e = 33{,}100(18 + \sqrt{18^2 + 15^2})$$

$$= 33{,}100 \times 41{\cdot}4 = \frac{\pi}{16} \times 7{,}000\text{D}^3.$$

hence $$\text{D} \quad \frac{1{,}370{,}000}{1{,}375} = 10 \text{ in.}$$

79. The Torsion of Shafts of Non-circular Section—In Fig. 125 is shown a view of the normal cross-section of a non-circular shaft, and OP is any line drawn from the centre of the section to the boundary. Suppose that forces are applied to the shaft so that it is twisted about the point O and normal cross-sections remain plane. The shear stress at the point P will act in a direction normal to OP, and if OP does not meet the boundary normally the stress can be resolved into two components, PB and PC, which act tangentially and normally respectively to the boundary. In order that the component PC may exist, there must be a shear stress acting on the surface of the shaft in the direction of its length. In simple torsion this stress is not applied; the point P moves in the direction of the length of the shaft and the originally plane cross-section becomes warped. The problem of the torsion of non-circular shafts was first solved by St. Venant, and is given in Todhunter and Pearson's "History of the Theory of Elasticity."

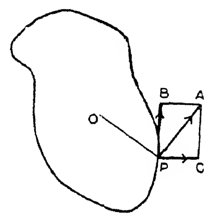


Fig. 125.

The relation between the torque T and the angle of twist θ in a length l is given by $T = \frac{G\theta}{l}J$ where J is the polar moment of inertia in the case of circular shafts. Expressions for J for other shapes of sections were obtained by St. Venant, and are as follows: For a square section $J = 0{\cdot}140a^4$ where a is the length of one side. In the case of an elliptic section of major and minor axis a and b respectively $J = \frac{\pi a^3 b^3}{16(a^2 + b^2)}$. For a rectangular section of breadth b and depth d the approximate value of J is

$$\frac{1}{3}bd^3\left\{1 - 0{\cdot}630\frac{d}{b}\left(1 - \frac{1}{12}\left(\frac{d}{b}\right)^4\right)\right\}$$

which is correct for all values of b and d with an error of less than 4 per cent.

If $d < \frac{1}{3}b$ the value of J becomes

$$J = \frac{1}{3} bd^3 \left\{ 1 - 0{\cdot}630 \frac{d}{b} \right\}$$

with an error of less than 1 per cent.

For an equilateral triangle of side a,

$$J = 0{\cdot}0216a^4.$$

In the case of any symmetrical shape of cross-section where the ratio of any two dimensions through the centre is not great, the approximate value of $J = \frac{A^4}{40I}$, where A is the area of the cross-section and I is the polar moment of inertia.

The relation between the torque T and maximum shear stress S, is for a square shaft $T = 0{\cdot}208Sa^3$, where a is the side of the square; for an elliptic shaft $T = \frac{\pi}{16} Sab^2$, where a and b are the major and minor axes, and for a shaft whose cross-section is an equilateral triangle $T = 0{\cdot}050Sa^3$, where a is the length of one side. For a shaft of rectangular cross-section of breadth b and depth d, $T = \frac{b^2d^2}{3b + 1{\cdot}8d}S$, with an error of less than 4 per cent. It may be noted that the maximum stress in the above sections occurs at points on the boundary which are nearest to the centre.

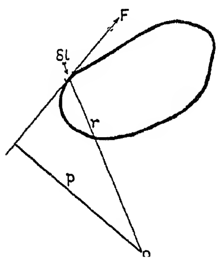


FIG. 125A.

Torsion of Tubes—In papers recently published * simple formulæ have been derived connecting the torque and twist and the torque and shear stress for both open and closed tubes. The formulæ are accurate enough for most practical purposes where the tubes are thin, as, for instance, the tubular members used in aeroplane construction. They can also be applied to I, channel sections, etc. A method will be given of deriving the formulæ for a closed tube. In Fig. 125A the closed curve represents the normal cross-section of a thin-walled tube which is twisted about an axis parallel to its length, whose end view is the point 0. The tangential shearing force F acts in the plane of the section on an element δl of the perimeter l of the section and p is the perpendicular from the point 0 on the line of action of F. If T is the torque, then, taking moments about 0,

* Cyril Batho, "Engineering," Oct. 15, 1915, and Nov. 24, 1916. John Prescott, "Phil. Mag.," Nov., 1921.

$T = \Sigma pF\delta l$. The area of the triangle of base δl and apex at 0 is $\frac{p\delta l}{2}$, hence the quantity. $\Sigma p\delta l = 2A$ where A is the area of the section. The result of this summation is independent of the position of the point 0,* and since the torque in the tube is also independent of the position of 0, it appears that F is constant round the perimeter. Hence $T = 2AF$. If the thickness t is variable and S denote the stress, then $F = St$, so that the stress is greatest where the thickness is least. The area A is the area enclosed by the median line which passes midway between the inner and outer boundaries of the section, or alternately the arithmetic mean of the areas enclosed by the inner and outer boundaries.

The shear strain in the walls of the tube is τp where τ is the twist per unit length. $\frac{S}{\tau p} = G$, where G is the modulus of rigidity, and therefore $\Sigma S\delta l = G\tau\Sigma p\delta l$, thus $S = \frac{2G\tau A}{l}$ Substituting this value of S in the above equation, $T = \frac{4G\tau A^2 t}{l}$, where t is the mean thickness. It is usually accurate enough to use the mean thickness in this equation. A more correct equation is $T = \frac{4G\tau A^2}{\int \frac{dl}{t}}$.

The last two equations can also be written $T = G\tau J$—where J—which depends only on the dimensions of the section—is given by either

$$\frac{4A^2 t}{l} \text{ or } \frac{4A^2}{\int \frac{dl}{t}}.$$

For open tubes the reader is referred to the papers above quoted for the methods of deriving the corresponding equations. In the case of an open tube $J = \frac{t^3 l}{3}$ where t is the mean thickness and $S = \frac{Tt}{J}$ where t is the maximum thickness. It will thus be noted that for an open tube the stress is greatest where the thickness is greatest, whilst for a closed tube the reverse is the case. The more accurate value of J when the thickness is somewhat variable is $J = \frac{1}{3}\int$

* "Calculus," H. Lamb, Art. 101.

Let b be the breadth of the flanges and d the depth of the web for an I beam or channel, and let t_1 and t_2 be the corresponding thicknesses, then $J=\left\{\frac{2t_1^3b}{3}+\frac{t_2^3d}{3}\right\}$. Since for an open tube such as an I or channel section the stress is greatest where the thickness is greatest, the maximum stress will usually occur in the vicinity of the mid-breadth of the flanges. It is, however, important that the section contain no sharp corners and that if the radius of curvature at a corner be small, the thickness of metal in the vicinity of the corner be slightly greater than that of the rest of the tube. It may be noted that the stress at a corner of a thin tube is independent of whether the corner projects inwards or outwards. This is contrary to the case of a solid section, where the stress at a corner which projects inwards is usually much greater than at one which projects outwards.

For an open tube the relation between the torque and the twist is obtained by substituting the values of J given above in the equation $T = G\tau J$. In the case of an I or channel section, for instance, the value of J is as given in the last equation. The above equations will apply to most commercial tubular sections and also to I sections, channels and tees with sufficient accuracy for most practical purposes. It is, however, advisable, especially when determining the stress, to use the actual thicknesses of the tubes etc., rather than the nominal thicknesses as given in a catalogue.

The shear stress and torsional rigidity of a shaft of any shape of cross-section can be determined from measurements of the displacement of a thin membrane such as a soap film stretched across a hole—of the same shape as the cross-section of the shaft—in a flat plate and displaced slightly from the plane of the plate by a uniform difference in pressure on the two sides. The contour lines of the film will be identical with the lines of shearing stress in the cross-section of the prism, and if γ denote the slope of the film at any point, t the surface tension, and p the small difference in pressure, then the shear stress S at the point is given by $S=\frac{4t}{p}G\frac{\theta}{l}\gamma$ and $T=\frac{8t}{p}G\frac{\theta}{l}v$, where v is the volume displaced by the loaded film.

The above analogy was first pointed out by L. Prandt (1903),* and was elaborated and applied by Griffiths and Taylor,† who devised an apparatus for measuring the displacement and inclination of the film.

* Love's "Elasticity," Art. 224.

† "Proc. Inst. Mech. Eng.," Dec., 1917.

80. Helical Springs—In Fig. 126 is shown a helical spring made from wire of circular cross-section and subjected to an axial pull P. Neglecting the obliquity of the coils, the forces on any normal cross-section of the wire are (1) a torsional couple PR and (2) a direct shearing force P across the section. The comparative effect of the direct force on the extension is small and the wire is sensibly subjected to pure torsion, the effect being the same as if it were in the form of a shaft and subjected to a torque PR. If x denote the extension of the spring, then $\frac{x}{R}=\theta$ where θ is the total twist in the wire. The expression for θ is $\theta=\frac{lT}{GJ}$ (Art. 75), where l is the length of the wire, T the torque, J the polar moment of inertia in the case of circular wire, and G the modulus of rigidity. Denoting the number of coils in the spring by n, then $l=2\pi Rn$ and

$$\frac{x}{R}=\frac{2\pi RnT}{GJ} \quad \text{or} \quad x=\frac{2\pi PR^3n}{GJ}$$

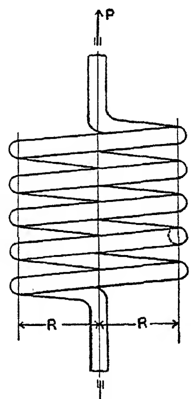


Fig. 126.

Let the lower end of the spring be subjected to a couple M acting in a plane normal to the axis of the helix and in a direction to increase the number of coils. Neglecting the inclination of the coils, the only force acting on the wire will be a bending moment of amount M. If R and R′ denote the radius of curvature of the coils before and after the couple is applied, and n and n' their number respectively, then if the radius of curvature of the coils is several times as great as the diameter of the wire, $\frac{1}{R'}-\frac{1}{R}=\frac{M}{EI}$, where I is the moment of inertia of a section of the wire about a diameter. But $2\pi R'n'=2\pi Rn=l$ where l is the length of the wire. Hence $\frac{1}{R'}=\frac{2\pi n'}{l}$ and $\frac{1}{R}=\frac{2\pi n}{l}$, so that $2\pi(n'-n)=\frac{Ml}{EI}=\phi$ where ϕ is the angle turned through by the couple.

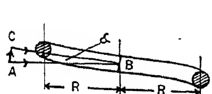


Fig. 127.

In Fig. 127 is shown half a coil of a helical spring of which the helix is inclined at angles α with planes normal to the axis. Normal cross-sections of the wire will be subjected to a torsional couple PR, but the axis of the couple will make an angle α with

normals to the cross-sections. The couple can therefore be resolved into two components, CB and AC, with axes normal and tangential respectively to the cross-section. The normal component is pure torsion of amount PR cos α. The extension of the spring due to this couple is $\frac{PR^2 l \cos \alpha}{GJ}$ in a direction making an angle α with the axis of the helix. The component x_1 of this in the direction of the axis is therefore

$$x_1 = -\frac{PR^2 l \cos^2 \alpha}{GJ}$$

The tangential component is PR sin α and acts in a direction to uncoil the spring. The angle ϕ turned through by the end of the spring due to this couple is $\phi = \frac{PRl \sin \alpha}{EI}$, the angle being in a plane whose normal makes an angle α with the axis of the helix. The component of ϕ in a vertical direction is $\frac{PRl \sin^2 \alpha}{EI}$ and the corresponding extension $x_2 = \frac{PR^2 l \sin^2 \alpha}{EI}$ The total extension of the spring is therefore

$$x = PR^2 l \left\{ \frac{\cos^2 \alpha}{GJ} + \frac{\sin^2 \alpha}{EI} \right.$$

where l, the length of the wire, is given by $l = 2\pi Rn \sec \alpha$, n being the number of coils.

In the case of circular wire J = 2I, and taking $\frac{E}{G} = \frac{5}{2}$, the extension becomes

$$x = \frac{PR^2 l}{EI} \left\{ \frac{5}{4} \cos^2 \alpha + \sin^2 \alpha \right.$$

or

$$x = \frac{PR^2 l}{GJ} \left\{ \cos^2 \alpha + \frac{4}{5} \sin^2 \alpha \right.$$

In Fig. 128 is shown a curve indicating the effect of the inclination of the coils on the extension. The extension corresponding with $\alpha = 0$ is taken as unity, and the curve is plotted for values of α from 0 to 45 deg. The percentage reduction in the extension when α is taken into account is given by

$$1 - \left(\cos^2 \alpha + \frac{4}{5} \sin^2 \alpha \right) \Big\} \times 100 = 20 \sin^2 \alpha.$$

When an open-coiled helical spring is extended there is a rotation of the end of the spring about the axis of the helix. The total angle θ turned through by the wire when it is twisted

by a torsional couple T induced by an axial pull P on the spring is $\theta=\frac{lT}{GT}$, and since $T=PR\cos\alpha$ then $\theta=\frac{PRl\cos\alpha}{GJ}$ The plane of this angle makes an angle α with the axis of the helix, and hence the rotation in a plane normal to the axis is

$$\frac{PRl\sin\alpha\cos\alpha}{GJ}$$

The rotation of the spring due to the couple PR sin α is

$$\frac{PRl\sin\alpha}{EI}$$

in a plane whose normal makes an angle α with the axis, and the resolution of this in a plane normal to the axis is

$$\frac{PRl\sin\alpha\cos\alpha}{EI}$$

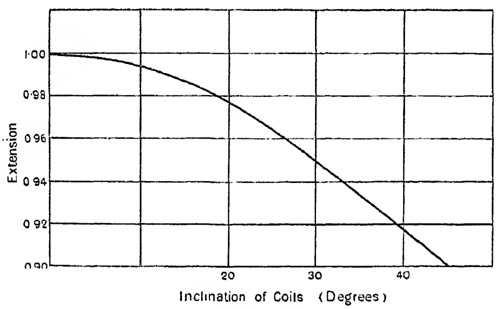


FIG. 128.

so that ϕ, the total angle, is given by

$$\phi=PRl\sin\alpha\cos\alpha\left(\frac{1}{GJ}-\frac{1}{EI}\right)$$

or

$$\phi=\frac{PRl\sin 2\alpha}{2}\left(\frac{1}{GJ}-\frac{1}{EI}\right)$$

which is a maximum when $\alpha=45$ deg. It may be noted that the effect of the torsional couple PR cos α is to increase the number of coils, and the bending couple PR sin α decreases them. In the case of circular wire and taking $\frac{E}{G}=\frac{5}{2}$, $\phi=\frac{PRl\sin 2\alpha}{10GJ}$, where l, the length of the spring, is given by $l=2\pi Rn\sec\alpha$.

If the wire is not of circular cross-section the values of J given in Art. 79 should be used in all the above formulæ. The moment of inertia I is about the neutral axis in a normal section about

which bending is taking place. It is possible by making a spring from wire of rectangular or other elongated shape of cross-section to cause the rotation of the end to be considerable and either positive or negative. Profs. Ayrton and Perry * devised springs on this principle for the measurement of small changes in length, the rotation of a pointer fixed to the end of the spring being used to measure the amount of the change.

81. Strain Energy stored in Springs—When an open-coiled spring is extended or compressed or subjected to a torsional couple about its axis the wire is subjected to both torsion and bending stresses. Let T_1 denote the torsional couple about the axis of the wire and M_1 the bending moment on the wire. The energy stored due to T_1 is $w_1 = \frac{1}{2}T_1\theta$ where θ is the total angle turned through by a length l of the wire about its axis, the value of θ being given by $\theta = \frac{lT_1}{GJ}$. Hence $w_1 = \frac{lT_1^2}{2GJ}$. Also the energy stored by the bending moment is $w_2 = \frac{1}{2}M_1\theta_1$ where $\theta_1 = \frac{Ml}{EI}$. The angle θ_1 is in a plane whose normal makes an angle α with the axis of the helix, θ_1 being the angle subtended at the axis of the helix by the displacement of the end of a length l of the wire. Hence $w_2 = \frac{lM_1^2}{2EI}$ and the total strain energy w stored in the spring is $w = \frac{l}{2}\left(\frac{T_1^2}{GJ} + \frac{M_1^2}{EI}\right)$. If the spring is extended an amount x by a pull P then $w = \frac{Px}{2}$ and the combination of these two equations yields the same expression for x as was previously obtained by a different method.

Let the spring be subjected to a couple M which acts in a plane normal to the axis of the spring and in a direction to wind it up, that is, to increase the number of coils. On every normal section there will be a couple acting whose axis is parallel to the axis of the spring and makes an angle α with the sections where α is the inclination of the coils. The couple can be resolved into normal and tangential components T_1 and M_1 respectively, where $T_1 = M \sin\alpha$ and $M_1 = M\cos\alpha$. The work done by the couples is $\frac{1}{2}M\phi$ where ϕ is the angle turned through and therefore

$$\frac{1}{2}M\phi = \frac{M^2l}{2}\left(\frac{\sin^2\alpha}{GJ} + \frac{\cos^2\alpha}{EI}\right)$$

* "Proc. Roy. Soc.," Vol. 36, 1884.

and
$$\phi = Ml\left(\frac{\sin^2 \alpha}{GJ} + \frac{\cos^2 \alpha}{EI}\right)$$

where l, the length of the wire, is given by $l = 2\pi Rn \sec \alpha$. The value of ϕ in degrees is

$$\phi^\circ = 360\ MRn \sec \alpha\left(\frac{\sin^2 \alpha}{GJ} + \frac{\cos^2 \alpha}{EI}\right.$$

For wire of circular cross-section and taking $\frac{E}{G} = \frac{5}{3}$

$$\phi = \frac{Ml}{GJ}\left(\sin^2 \alpha + \frac{4}{5}\cos^2 \alpha\right.$$

or
$$\phi = \frac{Ml}{EI}\left(\frac{5}{4}\sin^2 \alpha + \cos^2 \alpha\right.$$

The relation between ϕ and α is shown in Fig. 129. The value

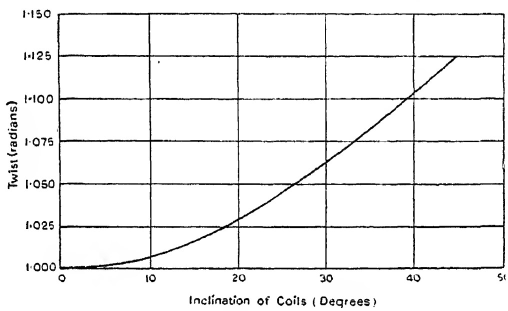


FIG. 129.

of ϕ is taken as one radian when $\alpha = 0$ and the curve gives values of ϕ from $\alpha = 0$ to $\alpha = 45$ deg. The percentage increase in the rotation when α is taken into account is given by

$$\left.\frac{5}{4}\sin^2 \alpha + \cos^2 \alpha - 1\right\} \times 100 = 25 \sin^2 \alpha.$$

In addition to the rotation of the spring about the axis, there will also be a change in length. This can be found by resolving the rotation as before, and the change x is given by

$$x = MlR \sin \alpha \cos \alpha\left(\frac{1}{NJ} - \frac{1}{EI}\right)$$

which is a maximum when $\alpha = 45$ deg. For circular wire and taking $\frac{E}{G}$, then

$$x = \frac{MlR \sin \alpha \cos \alpha}{4EI} \quad \frac{MlR \sin \alpha \cos \alpha}{5GJ}$$

For circular wire J is the polar moment of inertia and is equal to $\frac{\pi}{32}d^4$ where d is the diameter of the wire. If the wire is not circular J has the values given in Art. 79. Also for circular wire $\mathrm{I}=\frac{\pi}{64}d^4$ and for other shapes of cross-section I is about the axis in a normal cross-section passing through its centre of gravity and perpendicular to the normal to the axis of the spring from the centre of gravity of the section.

It should be noted that the values of α and R—the inclination and radius of the coils—have been treated as constants, but the errors so introduced are small and the formulæ are nearly exact.

82. Stresses in the Material—The force on any normal section of a helical spring can be resolved into a torsional couple T and a bending moment M. Also if the spring is subjected to a longitudinal load P there will be a direct shearing force of amount P across the section. The maximum stress in the material is found as for a shaft subjected to combined twisting and bending (Art. 78), the effect of the direct shearing force being as a rule negligible. Thus if the inclination of the coils to planes normal to the axis is α, the torque and bending moment produced by the load P will be PR $\cos\alpha$ and PR $\sin\alpha$ respectively. Hence for circular wire the maximum shear stress S is given by

$$\mathrm{S}=\frac{16}{\pi d^3}\sqrt{\mathrm{M}^2+\mathrm{T}^2}=\frac{16}{\pi d^3}\sqrt{(\mathrm{PR}\sin\alpha)^2+(\mathrm{PR}\cos\alpha)^2}$$

$$=\frac{16\mathrm{PR}}{\pi d^3}$$

so that the shear stress is independent of the inclination of the coils.

In the case of square wire the maximum shear stress is

$$s=\quad p^2+q^2,$$

where
$$p=\frac{\mathrm{M}y}{\mathrm{I}}=\frac{6\mathrm{M}}{a^3}$$

and
$$q=\frac{-}{0{\cdot}208a^3}$$

where a is the length of a side of the square. Hence

$$\mathrm{S}=\sqrt{\frac{9\mathrm{M}}{a}+\quad\frac{23{\cdot}1\mathrm{T}^2}{}}$$

$$= \frac{3}{a^3}\sqrt{M^2 + 2{\cdot}57T^2}$$

$$= \frac{3PR}{a^3}\sqrt{\sin^2 \alpha + 2{\cdot}57 \cos^2 \alpha}$$

$$= \frac{3PR}{a^3}\sqrt{1 + 1{\cdot}57 \cos^2 \alpha},$$

which decreases as α increases.

The stress in wire of rectangular or elliptical cross-section is obtained in a similar manner, the value of q being found from the formulæ given in Art. 79. It should be noted that q is a maximum in the middle of the longer sides in the case of rectangular wire and at the ends of the minor axis of elliptical wire. When combining p and q, care should be taken to notice whether they both have their maximum values at the same point on the boundary of the section. In the case, for instance, of wire of rectangular cross-section with the shorter sides in the direction of the axis of the spring, the value of p will be maximum on the shorter sides whilst q will be a maximum in the middle of the long sides. Hence it will usually be necessary to evaluate the maximum induced stress at both places in order to determine the greatest stress in the material.

It is important that the proof resilience of the material from which springs are made should be large. Considering the wire of the spring to be arranged in the form of a shaft of length l, the proof resilience is $\frac{1}{2}T\theta$ where T is the torque on the wire and θ the angle turned through by the end of the shaft at the elastic limit. For hollow circular wire of outside and inside diameter of D and d respectively

$$T = \frac{\pi}{16} S \frac{D^4 - d^4}{D}$$

and
$$\theta = \frac{lT32}{G\pi(D^4 - d^4)}$$

hence
$$\frac{1}{2}T\theta = \frac{D^2 + d^2}{D^2} \times \frac{S^2}{4G} \times \text{volume}.$$

When the wire is solid $d = 0$ and the resilience per unit volume is $\frac{S^2}{4G}$. If the wire is hollow the resilience is greater than this and approaches to its maximum value of $\frac{S^2}{2G}$ as d approaches to D. In the case of wire of square section the resilience per unit volume by a similar method reduces to $0{\cdot}154\frac{S^2}{G}$. It may be

noted that the torsional resilience of a hollow circular shaft or wire is higher than that of any other shape of hollow section. Also the resilience of a solid circular section is less than that of a hollow circular section but greater than that of any other shape of solid section. The value of the elastic limit shear stress for spring steel wire $\frac{1}{8}$ in. diameter or less is about 80 tons per sq. in. and diminishes as the diameter increases, and is about 60 tons per sq. in. for wire of $\frac{1}{2}$ in. diameter to 1 in. diameter. The safe working stress is about $\frac{1}{2}$ to $\frac{1}{3}$ of these values, the lower stress being used for tension springs which are liable to be overstretched. The value of G can be taken as 12×10^6 lb. per sq. in.

Example 1—A helical spring 4·8 in. outside diameter is made from steel wire 0·96 in. diameter and is subjected to a compressive load of 3 tons. Neglecting the inclination of the coils and taking $G=12\times 10^6$ lb. per sq. in., find (1) the decrease in length per coil, (2) the max. shear stress in the wire, and (3) the resilience per unit volume.

The mean diameter of the spring is $4{\cdot}8-0{\cdot}96=3{\cdot}84$ in. and the torque on the wire is $\dfrac{3\times 3{\cdot}84}{2}=5{\cdot}76$ ton in.

If x be the shortening per coil, then $\dfrac{x}{\mathrm{R}}=\dfrac{l\mathrm{T}}{\mathrm{GJ}}$ where R is the radius and l the length of one coil. Hence

$$x=\frac{1{\cdot}92\times 3{\cdot}84\times \pi\times 5{\cdot}76\times 2{,}240\times 32}{12\times 10^6\times \pi\times (0{\cdot}96)^4}=0{\cdot}30 \text{ in.}$$

The maximum shear stress is obtained from the formula

$$\mathrm{T}=\frac{\pi}{16}\mathrm{S}d^3$$

where d is the diameter of the wire. Thus

$$\mathrm{S}=\frac{16\mathrm{T}}{\pi d^3}=\frac{16\times 5{\cdot}76}{\pi\times(0{\cdot}96)^3}=33{\cdot}1 \text{ tons per sq. in.}$$

The resilience per cubic in. is given by

$$\frac{\mathrm{S}^2}{4\mathrm{G}}=\frac{(33{\cdot}1\times 2{,}240)^2}{4\times 12\times 10^6}=115 \text{ in. lb.}$$

Example 2—A spring of square steel wire is to shorten 0·4 in. under a pressure of 1,800 lb. If the mean radius of the coils is 2 in. and the safe shear stress 25 tons per sq. in., find the dimensions of the cross-section and length of the wire, neglecting the inclination of the coils. Take $G=12\times 10^6$ lb. per sq. in.

The torque on the wire is $\mathrm{T}=2\times 1{,}800=3{,}600$ lb. in. The length a of a side of the square is obtained from

$$\mathrm{T}=0{\cdot}208\, a^3\mathrm{S}$$

or $$a=\sqrt[3]{\frac{3{,}600}{0{\cdot}208\times 25\times 2{,}240}}=0{\cdot}675 \text{ in.}$$

The length l of the wire is derived from $\theta=\dfrac{7{\cdot}11Tl}{Ga^4}$,

where $$\theta=\frac{0{\cdot}4}{2}=0{\cdot}2 \text{ rad.}$$

Hence $$l=\frac{0{\cdot}2\times 12\times 10^6\times(0{\cdot}675)^4}{3{,}600\times 7{\cdot}11}=195 \text{ in.}$$

Example 3—An Ayrton-Perry spring has a mean radius of 0·1 in. and is made from strip steel 0·45 in. wide and 0·0015 in. thick and the inclination of the coils is 45 deg. Taking $E=32\times 10^6$ lb. per sq. in. and $G=14\times 10^6$ lb. per sq. in., find expressions for the axial extension and twist when the spring is subjected to a longitudinal pull P.

Let x be the extension and ϕ the axial twist, then

$$x=PlR^2\left(\frac{\cos^2\alpha}{GJ}+\frac{\sin^2\alpha}{EI}\right)$$

and $$\phi=PlR\sin\alpha\cos\alpha\left(\frac{1}{GJ}-\frac{1}{EI}\right)$$

where l is the length of the strip and R the mean radius of the coils.

The moment of inertia

$$I=\frac{0{\cdot}45\times(0{\cdot}0015)^3}{12}=1{\cdot}26\times 10^{-10}$$

and $$J=\frac{0{\cdot}45\times(0{\cdot}0015)^3}{3}\left(1-\frac{0{\cdot}630\times 0{\cdot}0015}{0{\cdot}45}\right)=5{\cdot}08\times 10^{-10}$$

Therefore $$x=\frac{Pl\times 0{\cdot}01}{2}\left(\frac{1}{14\times 5{\cdot}08\times 10^{-4}}+\frac{1}{32\times 1{\cdot}26\times 10^{-4}}\right)$$
$$=Pl\times 0{\cdot}005(142+248)$$
$$=1{\cdot}95Pl$$

Also $$\phi=\frac{Pl\times 0{\cdot}1}{2}(142-248)$$
$$=-5{\cdot}30\,Pl.$$

Hence if the spring is extended 0·1 in. the rotation of the end in degrees is $\dfrac{5{\cdot}30\times 180}{1{\cdot}95\times 10\times\pi}=15{\cdot}5$ deg.

83. Simple Elastic Vibrations—A typical case is that of a weight suspended from the end of a spring or elastic rod. Suppose the weight be raised until there is just no tension in the spring and then allowed to fall. When the weight has fallen

through a distance x the tension in the spring will be Kx where K is a constant. Hence the total force on the weight is $\mathrm{W} - \mathrm{K}x$ where W is the weight, and therefore

$$\frac{\mathrm{W}}{g}\frac{d^2x}{dt^2} = \mathrm{W} - \mathrm{K}x$$

or

$$\frac{d^2x}{dt^2} + \frac{g\mathrm{K}}{\mathrm{W}}x = g.$$

The solution of this equation is $x = a\cos(pt + e) + \frac{g}{p^2}$, where a is the amplitude of the vibration, e is a constant, and $p = \sqrt{\frac{g\mathrm{K}}{\mathrm{W}}}$. It is evident that a particular value of x will recur whenever pt increases by 2π. The interval between each recurrence is the periodic time or period of the vibration and is given by

$$t = \frac{2\pi}{p} = 2\pi\sqrt{\frac{\mathrm{W}}{g\mathrm{K}}}$$

The constant K is the stiffness of the spring or the force necessary to produce unit extension. The extension produced by the weight W when it is at rest is $\frac{\mathrm{W}}{\mathrm{K}} = c$, and therefore the equation can be written $t = 2\pi\sqrt{\frac{c}{g}}$. Hence the period can be determined if the quantity c corresponding with any particular weight is known. The number of vibrations per unit time, or frequency n, is expressed by $n = \frac{1}{t}$.

The expression for the frequency can also be obtained from a consideration of the kinetic and potential energy of the weight and the strain energy stored in the spring. Since no external work is done by the system, the total energy at any time must be constant. Reckoning the potential energy of the weight as zero when it is at its lowest point, then if the weight vibrates through a distance of $2c$, the total energy of the system when the weight is at its highest point is the potential energy of the weight and equals $2\mathrm{W}c$, the kinetic energy of the weight and the strain energy stored in the spring being zero. When the weight is passing through its static position the kinetic energy is $\frac{1}{2}\frac{\mathrm{W}}{g}(2\pi nc)^2$, the potential energy is Wc, and the strain energy stored in the spring is $\frac{1}{2}\mathrm{W}c$. Hence

$$\frac{1}{2}\frac{\mathrm{W}}{g}(2\pi nc)^2 + \mathrm{W}c + \frac{1}{2}\mathrm{W}c = 2\mathrm{W}c$$

or
$$n=\frac{1}{2\pi}\sqrt{\frac{g}{c}}.$$

It may be noted that the frequency is independent of the amplitude.

The last formula applies to the majority of cases of a vibrating weight where the effect of the period of the structure or member supporting it can be neglected. In the case of a shaft fixed at the ends and supporting a weight in the middle the period is given by $n=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$, where $\delta=\frac{1}{192}\cdot\frac{Wl^3}{EI}$. Similar cases can be treated in the same way.

If the weight is not in the middle then in addition to the transverse vibration there will also be a rotary vibration of the weight about an axis perpendicular to the plane of vibration.

In Fig. 130 is shown a cantilever supporting a heavy weight W of radius of gyration k about an axis perpendicular to the plane of vibration, and c is the static deflection, whilst θ is the

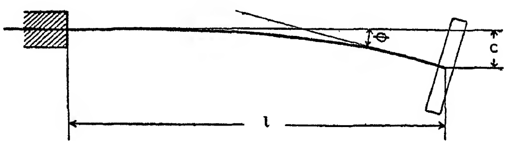


FIG. 130.

static inclination of the end. The frequency can be found by the same method as that used for the spring. Thus

$$\frac{1}{2}\frac{W}{g}(2\pi nc)^2+\frac{1}{2}\frac{W}{g}k^2(2\pi n\theta)^2=\frac{1}{2}Wc$$

or
$$n=\frac{1}{2\pi}\sqrt{\frac{cg}{c^2+k^2\theta^2}}$$

In order to find θ consider any section distant x from the fixed end. The bending moment at the section is $W(l-x)$, and therefore

$$EI\frac{d^2y}{dx^2}=W(l-x)$$

$$EI\frac{dy}{dx}=Wlx-\frac{Wx^2}{2}$$

The constant of integration vanishes, hence $\theta=\frac{Wl^2}{2EI}$, and since

$c = \frac{Wl^3}{3EI}$ then $\theta = \frac{3c}{2l}$. Thus the equation for n becomes

$$n = \frac{1}{2\pi}\sqrt{\frac{}{1+\frac{9}{4}\left(\frac{k}{l}\right)^2}}$$

This equation is deduced on the assumption that the static shape of the loaded shaft is the same as the dynamic. The corresponding equation for whirling can be deduced in a similar manner (Art. 85) and is

$$n = \frac{1}{2\pi}\quad\frac{g}{1-\frac{9}{4}\left(\frac{k}{l}\right.}$$

Suppose the radius of the weight is 6 in. and $l = 4$ in. Then the ratio $\left(\frac{k}{l}\right) = \frac{81}{64}$ and therefore the denominator in the last expression is negative and thus the expression fails. The correct expression is

$$(2\pi n)^2 = \frac{2EIg}{Wlk^2}\left\{\frac{3k^2}{l^2} - 1 + \sqrt{1 - 3\frac{k^2}{l^2} + 9\frac{k^4}{l^4}}\right\}*$$

If $+k^2$ be substituted for $-k^2$ the expression will apply to vibrations. Expanding the terms under the root sign up to $\frac{k^4}{l^4}$ the expression becomes

$$(2\pi n)^2 = \frac{2EIg}{Wlk^2}\left\{3\frac{k^2}{l^2} - 1 + 1 - \frac{3}{2}\frac{k^2}{l^2} + \frac{9}{2}\frac{k^4}{l^4} - \frac{9}{8}\frac{k^4}{l^4}\right.$$

$$= \frac{2EIg}{Wlk^2}\left\{\frac{3}{2}\frac{k^2}{l^2} + \frac{27}{8}\frac{k^4}{l^4}\right\}$$

$$= \frac{3EIg}{Wl^3}\left\{1 + \frac{9}{4}\frac{k^2}{l^2}\right.$$

$$\therefore \quad n = \frac{}{2\pi}\quad\frac{g}{c\left\{1-\frac{9}{4}\left(\frac{k}{l}\right)^2\right.}\quad \text{as before.}$$

It therefore appears that when considering rotary inertia the assumption that the static deflection is the same as the dynamic will involve appreciable errors unless the ratio $\left(\frac{k}{l}\right)^2$ is small.

In the case of a cantilever or beam supporting a number of weights W_1, W_2 . . . . . with corresponding deflections and

* "The Steam Turbine," Stodola.

inclinations of $c_1, c_2 \ldots\ldots$ and $\theta_1, \theta_2 \ldots\ldots$ respectively, the energy equation is

$$\frac{2\pi^2 n^2}{2g}\{\Sigma Wc^2 + \Sigma Wk^2\theta^2\} = \frac{1}{2}\Sigma Wc$$

and

$$n = \frac{1}{2\pi}\sqrt{\frac{g\Sigma Wc}{\Sigma Wc^2 + \Sigma Wk^2\theta^2}}$$

The effect of the weight of the beam is neglected.

In the case of a weight vibrating at the end of a spring the fraction of the weight of the spring to be added to the vibrating weight in order to determine the correct frequency can be found by considering the kinetic energy of the system. Let the weight per unit length of the spring be w, then the weight of a length δx is $w\delta x$. Also the velocity of a point distant x from the fixed end is $\frac{x}{l}v$ where v is the velocity of the vibrating weight and l is the length of the spring. Hence the total kinetic energy is given by

$$\text{KE} = \frac{1}{2}\,\frac{W}{g}v^2 + \frac{1}{2}\int_0^l \left(\frac{xv}{l}\right)^2 \frac{w}{g}dx$$

$$= \frac{1}{2g}\left(W + \frac{1}{3}wl\right)v^2$$

The inertia of the spring can therefore be allowed for by increasing the suspended weight by one-third the weight of the spring. As an approximation, one-half of the weight of the spring is sometimes added to the suspended weight. The higher fraction approximately allows for the effect of the end of the spring, which is bent over to support the weight and vibrates through the same amplitude.

A method similar to the above can be adopted for finding the effect of the weight of a cantilever vibrating with a weight at its free end. Let c be the static deflection of the end of the lever and W the suspended weight. The bending moment at any point distant x from the fixed end is $W(l - x)$. Hence if y be the deflection distant x from the fixed end

$$EI\frac{d^2y}{dx^2} = W(l - x)$$

and

$$EIy = W\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$$

The total energy of the vibrating system at any time is constant and is equal to either the max. kinetic energy or the max. strain energy. The max. kinetic energy is

$$\frac{1}{2}\frac{W}{g}(2\pi nc)^2 + \frac{1}{2}\,\frac{w}{g}(2\pi n)^2\int_0^l y^2dx,$$

where w is the weight per unit length of the lever and l the total length. The max. strain energy is $\frac{1}{2}\mathrm{W}c$ where $c=\frac{\mathrm{W}l^3}{3\mathrm{EI}}$.

Hence the energy equation becomes

$$\frac{1}{2}\frac{\mathrm{W}}{g}(2\pi nc)^2+\frac{1}{2}\frac{w}{g}(2\pi n)^2\int_0^l y^2dx=\frac{1}{2}\mathrm{W}c\,dx.$$

On substituting the values for y and c given above the solution of the equation comes out to be $n=\frac{1}{2\pi}\sqrt{\frac{3g\mathrm{EI}}{l^3(\mathrm{W}+\frac{33}{140}wl)}}$. Therefore the effect of the vibrating shaft is equivalent to $\frac{33}{140}$ of its weight assumed concentrated at the end. If the weight W is zero

$$n=\frac{0{\cdot}568}{l^2}\sqrt{\frac{g\mathrm{EI}}{w}}$$

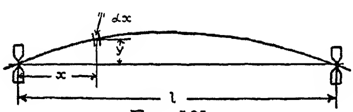


FIG. 131.

It should be noted in the treatment of the last case that the shape of the rod when vibrating has been assumed to be similar to that produced by a static load at the end. That such assumptions are justified when the rotary inertias of the weights in the plane of vibration are small was shown by Lord Rayleigh, and the error so introduced is very small, as is indicated by comparing the last result with the more exact one below.

84. Lateral Vibrations of Elastic Rods—The general case will be considered by reference to Fig. 131, which represents a uniform rod whose ends are free to rotate about and slide through the supports. The maximum force on an elemental length δx of the rod is $\frac{w\delta x}{g}(2\pi n)^2 y$, where w is the weight per unit length, y the amplitude of the element, and n the frequency. The intensity of loading per unit length is $\mathrm{EI}\frac{d^4y}{dx^4}$ (Art. 54), hence

$$\mathrm{EI}\frac{d^4y}{dx^4}=\frac{w(2\pi n)^2y}{g\mathrm{EI}} \quad\text{or}\quad \frac{d^4y}{dx^4}-m^4y=0, \text{ where } m^4=\frac{w(2\pi n)^2}{g\mathrm{EI}}$$

The solution of this equation is

$$y=\mathrm{A}\cos mx+\mathrm{B}\sin mx+\mathrm{C}\cosh mx+\mathrm{D}\sinh mx \qquad (1)$$

where A, B, C, and D are constants.

For the case shown in the figure $y=0$ and $\frac{d^2y}{dx^2}=0$ when $x=0$ and when $x=l$. With these four relations the values of the constants can be determined. Differentiating equation (1) twice gives

$$\frac{1}{m^2}\frac{d^2y}{dx^2}=-\mathrm{A}\cos mx-\mathrm{B}\sin mx+\mathrm{C}\cosh mx+\mathrm{D}\sinh mx \quad . \quad . \quad (2)$$

Putting $y=0$ and $x=0$ in equation (1) $\mathrm{A}+\mathrm{C}=0$ also, since $y=0$, when $\frac{d^2y}{dx^2}=0$, equation (2) gives $\mathrm{A}-\mathrm{C}=0$, hence $\mathrm{A}=\mathrm{C}=0$ and the equation becomes

$$y=\mathrm{B}\sin ml+\mathrm{D}\sinh ml \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting the relations $y=0$ and $\frac{d^2y}{dx^2}=0$ when $x=l$, equation (3) gives

$$0=\mathrm{B}\sin ml+\mathrm{D}\sinh ml$$
$$0=-\mathrm{B}\sin ml+\mathrm{D}\sinh ml.$$

Hence $\mathrm{B}\sin ml=\mathrm{D}\sinh ml=0$, and since ml is not zero, $\sinh ml$ has a value and therefore $\mathrm{D}=0$. Also B has a value or y would vanish for all values of x, and hence $y=\mathrm{B}\sin mx$. Thus the vibrating rod takes the form of a sine curve. When $x=l$ then since $y=o$ and B has a value $ml=\pi$, 2π, 3π, etc. The lowest frequency is determined from $ml=\pi$, hence

$$m^4=\frac{\pi^4}{l^4}=\frac{w(2\pi n)^2}{g\mathrm{EI}}$$

and

$$n=\frac{\pi}{2l^2}\sqrt{\frac{g\mathrm{EI}}{w}}=\frac{1{\cdot}571}{l^2}\sqrt{\frac{g\mathrm{EI}}{w}}.$$

The next highest frequency is $n=6{\cdot}283\sqrt{\frac{g\mathrm{EI}}{w}}$ and there is a node in the middle of the span.

Other cases of vibrating rods supported in various ways at their ends can be similarly treated. The general formula is of the form $n=\frac{\mathrm{K}}{\ }\sqrt{\frac{g\mathrm{EI}}{w}}$, where $\mathrm{K}=\frac{m^2}{2\pi}$. In the case of a rod fixed at both ends a consideration of the end conditions with the general equation for y yields $\cos ml \coth ml=1$, the solutions of which are $ml=3{\cdot}011\pi\ \frac{5\pi}{2}, \frac{7\pi}{2}$ correct to

four figures. Hence the value of K for the first and second frequencies is 3·57 and 9·82 respectively. For a cantilever $\cos ml \cosh ml = -1$ and $ml = \frac{1 \cdot 194\pi}{2}, \frac{2 \cdot 988\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \ldots\ldots$ and the value of K for the first and second frequencies is 0·561 and 3·52 respectively. If the rod is fixed at one end and free to turn about and slide through the other, $\cot ml = \coth ml$, and the first two values of K are 2·46 and 7·95.

The above cases of vibrating rods have an application in the blades of steam turbines, and failures have been caused by the blades being set vibrating owing to their natural periods approaching to some multiple of the period of rotation of the rotor. If the blades are parallel the period can be calculated from the above formulæ, the degree of accuracy depending mainly upon the accuracy with which the moment of inertia I is determined. A more satisfactory method is to clamp the blade to a heavy weight or vice and obtain the pitch of the note given out when the blade is struck. The pitch can be obtained by the use of a monochord or wire stretched along a board and tuning forks of known frequencies. If the frequency is too low to yield a note, a shorter length of the blade should be set vibrating by altering the position of the clamp, and the frequency so obtained, multiplied by the square of the ratio of the vibrating length to the total length of the blade, will give the required frequency. In the case of taper blades it is always better to determine the frequency experimentally. If the frequency is too low for a note to be heard a short brass pointer should be brazed to the end of the blade. The blade is then set vibrating and the pointer made to trace out a time displacement curve by bringing it into contact with indicator paper placed on a drum revolving at a known speed. The result will be a wavy curve, the height of the waves gradually diminishing as the vibrations die down, and the period is obtained from the length to scale of a complete wave.

85. The Whirling of Shafts. Critical Speeds—The determination of whirling and critical speeds is analogous to the determination of natural periods of vibration. In the case, for instance, of a straight parallel revolving shaft, the critical speeds are the same as the frequencies of the natural transverse vibrations, the end conditions being the same in both cases. In the vicinity of these speeds the centre line of the shaft, if slightly disturbed, will become curved and the shaft will whirl. The whirling is most pronounced when the speed synchronizes with a natural frequency, and if the shaft is run for long at such speed it may become permanently bent or broken. If the speed is increased beyond the first whirling speed, the shaft

becomes stable until the second critical speed is reached. Thus as the speed is increased a number of critical speeds are passed through in the vicinity of which the shaft is unstable and whirling is probable, whilst between these speeds the shaft is stable.

A similar relation holds in the case of loaded shafts, neglecting the rotary inertia of the weights. If the weight of the shaft is neglected, there is only one period of natural vibration or critical speed. Suppose the shaft supports a number of weights W_1, W_2, W_3 . . . . . with corresponding static deflections c_1, c_2, c_3 . . . . . The maximum kinetic energy is $\frac{(2\pi n)^2}{2g}\Sigma Wc^2$ and the maximum strain energy is $\frac{1}{2}\Sigma Wc$. Since these are equal $(2\pi n)^2=\frac{g\Sigma Wc}{\Sigma Wc^2}$, from which the frequency can be obtained. The values of c can be determined from a drawing of the static shape of the shaft obtained as described in Art. 55.

In the case of weights vibrating on a shaft, and taking account of the rotary motion of the weights, the expression for n has been shown to be (Art. 83) $(2\pi n)^2=\frac{g\Sigma Wc}{\Sigma Wc^2+\Sigma Wk^2\theta^2}$, where W, c, and θ are the weights and corresponding static deflections and inclinations and k is the radius of gyration of the weights about axes perpendicular to the plane of vibration. It will be noted that the effect of the rotary vibration of a weight is to decrease the frequency, the decrease being caused on account of the inertia couple $\frac{W}{g}k^2(2\pi n)^2\theta$. In the case of whirling this is absent, and there is a centrifugal couple of the same amount acting in the opposite direction. Hence the energy equation for weights W_1,W_2 . . . . . and corresponding static deflections and inclinations of c_1, c_2 . . . . . and θ_1, θ_2 . . . . . respectively is

$$\frac{(2\pi n)^2}{2g}\{\Sigma Wc^2-\Sigma Wk^2\theta^2\}=\frac{1}{2}\Sigma Wc$$

and

$$(2\pi n)^2=\frac{g\Sigma Wc}{\Sigma Wc^2-\Sigma Wk^2\theta^2}$$

the effect being an increase in the frequency. The values of c and θ can be obtained from a drawing of the static shape of the bent shaft. In order to find the period including the effect of the weight of the shaft the static curve should be drawn, taking account of the weight of the shaft in addition to the concentrated loads, the shaft being divided into segments for the purpose. The weight and corresponding deflection of each segment should then be included in the above summation.

If there is an end load on the shaft in the direction of its axis,

the treatment is exactly the same, the static shape of the loaded shaft being obtained as described in Art. 73.

86. Alternative Graphical Method for Finding Whirling Speed—The following method, due to Stodola,* is often convenient where the whirling speed is approximately known and the effect of the frequency of the shaft is neglected. Assume a curve of static deflection and let the deflections of the weights $W_1, W_2, \ldots$ be $y_1, y_2 \ldots$ respectively. Also assume a corresponding angular velocity ω_a and calculate the quantities $\frac{W_1}{g}\omega_a^2 y_1$, $\frac{W_2}{g}\omega_a^2 y_2$, etc., and draw another deflection curve with these quantities acting as lateral forces on the shaft. If the new curve is reasonably similar in shape to the assumed curve the critical velocity ω_c is given by $\omega_c = \omega_a$ [illegible], where y_c and y_a are the deflections at corresponding points on the new and assumed curve respectively. The validity of the above method depends upon the fact that when the shaft is about to whirl the equilibrium is neutral.

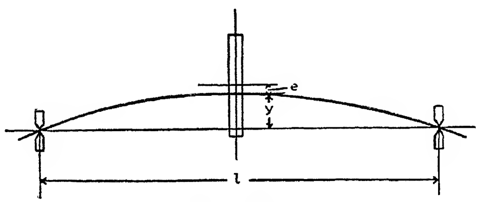


FIG. 132.

87. Approximate Formula for Several Loads—In the case of a shaft carrying several loads the whirling speed or frequency is approximately given by the empirical formula $\frac{1}{n^2} = \frac{1}{n^2_1} + \frac{1}{n^2_2} + \frac{1}{n^2_3} + \ldots$ where n_1 is the frequency of the unloaded shaft and n_2, n_3, etc., are the frequencies for each weight taken separately, neglecting the weight of the shaft, n being the frequency of the whole system. This formula, which was derived by Dunkerley, can be used as a rough alternative to, or as a check on, the graphical methods given, and is usually correct to within a few per cent.

88. Rotor of Laval Turbine—For the purpose of considering whirling the Laval Rotor can be assumed to consist of a heavy disc mounted on a flexible shaft. In Fig. 132 is shown such a disc mounted on a shaft of length l, and y is the eccentricity of

* "The Steam Turbine," Stodola.

the shaft when it is whirling, whilst e is the distance of the centre of gravity of the disc from the axis of the shaft. Let K be the force necessary to produce unit deflection in the middle of the span and W the weight of the disc. Then $\frac{W}{g}\omega^2(y+e) = Ky$, where ω is the angular velocity in radians per sec. Hence $y = \frac{W\omega^2 e}{Kg - W\omega^2}$, and if $Kg = W\omega^2$ then y is infinite and the critical velocity is accordingly given by $\omega_c = \frac{Kg}{W}$ or $\omega_c = \frac{g}{\ }$, where c is the static deflection produced by the weight. It may be noted that the critical speed is independent of the eccentricity of the disc.

If the speed is increased beyond the critical, $y = -\frac{W\omega^2 e}{W\omega^2 - Kg}$, which means that the sign of e changes and the centre of the disc is now diametrically opposite to its previous position, the conditions being as shown in Fig. 133. Hence, changing the sign

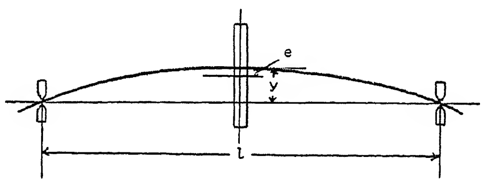


FIG. 133.

of e, the expression for y becomes $y = \frac{c}{1 - \frac{Kg}{W\omega^2}} = \frac{e}{1 - }$. Thus as the speed is increased beyond the critical the centre of gravity of the disc approaches more and more nearly to the axis of rotation. In the case of the De Laval rotor the value of $\frac{\omega_c}{\omega}$ is about one-seventh.

It is important that masses which revolve at high speeds should be well balanced, as, for instance, in the case of rotors of electrical generators and steam turbines. The rotor can be tested for any unbalanced force by placing its ends on level knife-edges at the same level and noting whether the equilibrium is neutral. There may also be an unbalanced couple which cannot be detected by such methods. Its presence may be indicated by revolving the rotor in bearings fixed to springs and noting whether any vibration is set up.

For further information on whirling the reader is referred to papers by Prof. Dunkerley,* Dr. Chree,† Dr. Morley,‡ Mr. W. Kerr,§ and Mr. H. A. Webb,‖ the latter of whom has given a new graphical method.

Example 1—A steel helical spring has ten coils of 3 in. mean diameter, the wire being $\frac{1}{4}$ in. diameter. Neglecting the inclination of the coils and the weight of the spring, find the frequency if a weight of 10 lb. is suspended from the end. Take $G = 12 \times 10^6$ lb. per sq. in.

The extension of the spring due to the weight is given by $x = \frac{l\mathrm{T}}{\mathrm{GJ}}$, where x is the extension.

The torque T on the wire $= 10 \times 1{\cdot}5 = 15$ lb. in. The length l of the wire is $10 \times \pi \times 3 = 30\,\pi$ in. The polar moment of inertia

$$\mathrm{J} = \frac{\pi}{32}d^4 = \frac{\pi}{32 \times 256}$$

Hence $$x = \frac{30\pi \times 15 \times 32 \times 256}{12 \times 10^6 \times \pi} = 0{\cdot}307 \text{ in.}$$

and the frequency is given by $n = \frac{60}{2\pi} \quad \frac{32{\cdot}2 \times 12}{0{\cdot}307} = 338$ complete vibrations per minute.

Example 2—A steel shaft $\frac{1}{2}$ in. diameter revolves in a long bearing at its upper end with its axis vertical. A disc 12 in. diameter and weighing 100 lb. is fixed to the lower end of the shaft, the length of the projecting shaft to the centre of gravity of the disc being 12 in. Neglecting the weight of the shaft, find the frequency and whirling speed (1) neglecting the rotary motion of the weight and (2) taking account of it. Take $E = 30 \times 10^6$ lb. per sq. in.

If the rotary motion is neglected the frequency is the same as the whirling speed and is given by $n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ where δ is the static deflection due to the weight if the axis of the shaft were horizontal.

$$\delta = \frac{100 \times (12)^3 \times 64 \times 16}{3 \times 30 \times 10^6 \times \pi} = 0{\cdot}625 \text{ in.}$$

* "Phil. Trans. Roy. Soc." (1894), Vol. 185A.
† "Phil. Mag.," May, 1904.
‡ "Engineering," July 30 and August 13, 1909.
§ "Engineering," Vol. 101 (1916).
‖ "Engineering," November 2, 9, and 16 (1917).

Hence for case (1) $n=\frac{60}{2\pi}\sqrt{\frac{32 \cdot 2\times 12}{0 \cdot 625}}=238$ per minute.

For case (2) the frequency and whirling speed is given by

$$n=\frac{1}{2\pi}\qquad\times\left\{1\pm\frac{9}{4}\left(\frac{k}{l}\right)\right.$$

where the positive and negative signs refer respectively to vibration and whirling and k is the radius of gyration of the disc about a diameter.

$$\frac{9}{4}\left(\frac{k}{l}\right)^2=\frac{9\times 36}{4\times 4\times 144}=0 \cdot 141.$$

Hence the frequency of the transverse vibrations is $238\times 0 \cdot 936$ $=222$ per minute, and the whirling speed is $238\times 1 \cdot 08=257$ revs. per minute.

89. Torsional Vibrations—A simple case is that of the unifilar suspension, which consists of a wire or shaft supporting a heavy weight which is capable of torsional vibration about the axis of the wire. Let the torque necessary to twist the wire through one radian be denoted by K. When the weight is vibrating and has turned through an angle θ from its central position the accelerating force is $-\mathrm{K}\theta$, and therefore, neglecting the effect of the weight of the shaft,

$$\mathrm{I}\frac{d^2\theta}{dt^2}=-\mathrm{K}\theta \text{ or } \frac{d^2\theta}{dt^2}+\frac{\mathrm{K}}{\mathrm{I}}\theta=0$$

where I is the mass polar moment of inertia of the weight. The solution of the equation is $\theta=\phi\sin(pt+\alpha)$, where ϕ is the amplitude of the vibrations, $p=\sqrt{\frac{\mathrm{K}}{\mathrm{I}}}$ and α is a constant. Hence n the frequency is given by

$$n=\frac{1}{2\pi}\sqrt{\frac{\mathrm{K}}{\mathrm{I}}}$$

The expression for K is obtained from the relation $\mathrm{T}=\frac{\mathrm{GJ}\theta}{l}$

When $\theta=1$ then $\mathrm{T}=\mathrm{K}$ or $\mathrm{K}=\frac{\mathrm{GJ}}{l}$.

If the shaft is of variable diameter with values of $\mathrm{J}=\mathrm{J}_1$, J_2, J_3 . . . . . etc., with corresponding lengths of l_1, l_2, l_3 . . . . . etc., then

$$\frac{\theta}{\mathrm{T}}=\frac{1}{\mathrm{G}}\left(\frac{l_1}{\mathrm{J}_1}+\frac{l_2}{\mathrm{J}_2}+\frac{l_3}{\mathrm{J}_3}+\cdots\cdots\right)=\frac{1}{\mathrm{G}}\frac{l}{\mathrm{J}}$$

where J and l refer to an equivalent parallel shaft. When $\theta = 1$ radian $T = K$, whose value as given by the last equation can be used in the equation for the frequency.

In the case of a circular shaft or wire, J is the polar moment of inertia, but for other shapes of cross-section it has the values given in Art. 79. For a circular shaft consisting of several lengths, each with a different diameter,

$$\frac{1}{K} = \frac{32}{\pi G}\Sigma\left(\frac{l}{d^4}\right)$$

and

$$n = \frac{1}{2\pi}\sqrt{\frac{\pi G}{32 I \Sigma\left(\frac{l}{d^4}\right)}}$$

$$\frac{1}{20}\sqrt{\frac{G}{I\Sigma\left(\frac{l}{d^4}\right)}}$$

If a parallel shaft has a weight at either end which is caused to vibrate about the axis, there will be a node, or section of the shaft which is at rest somewhere between the weights. Let the moment of inertia of the weights be I_1 and I_2 and the corresponding distances from the weights to the node l_1 and l_2 respectively. Then $n = \frac{1}{2\pi}\sqrt{\frac{K_1}{I_1}} = \frac{1}{2\pi}\sqrt{\frac{K_2}{I_2}}$. Hence $\frac{K_1}{I_1} = \frac{K_2}{I_2}$. For a parallel shaft $K_1 = \frac{GJ}{l_1}$ and $K_2 = \frac{GJ}{l_2}$, and therefore $\frac{l_1}{l_2} = \frac{I_2}{I_1}$, from which the values of l_1 and l_2, that is, the position of the node, can be obtained. The frequency is then given by one of the above formulæ for n.

Let K be the torque necessary to produce unit twist of the whole shaft. Then

$$K = \frac{GJ}{l} \text{ and } K_1 = \frac{Kl}{l_1} = \frac{K(l_1 + l_2)}{l_1} = \frac{K(I_1 + I_2)}{I_2}$$

Hence

$$n = \frac{1}{2\pi}\sqrt{\frac{K_1}{I_1}} = \frac{1}{2\pi}\sqrt{\frac{K(I_1 + I_2)}{I_1 I_2}} = \frac{1}{2\pi}\sqrt{K\left(\frac{1}{I_1} + \frac{1}{I_2}\right)}$$

It may be noted that the last equation can be written in the form $n^2 = n^2{}_1 + n^2{}_2$, where $n_1 = \frac{1}{2\pi}\sqrt{\frac{K}{I_1}}$ and $n_2 = \frac{1}{2\pi}\sqrt{\frac{K}{I_2}}$, n_1 and n_2 being the frequencies if the shaft were fixed at one end with the weights of moments of inertia I_1 and I_2 respectively vibrating at the other.

In order to allow for the effect of the moment of inertia of the shaft the fraction of the equivalent moment to be assumed vibrating at the end is obtained by considering the kinetic energy of the system. Let ω be the angular velocity of the weight of mass moment of inertia I. Then for a circular shaft the kinetic energy is given by $KE=\frac{1}{2}I\omega^2+\int_0^l\left(\frac{x}{l}\right)^2\omega^2 I_1 dx$, where I_1 is the mass polar moment of inertia per unit length of the shaft. Hence $KE=\frac{1}{2}\left(I+\frac{1}{3}I_1\right)\omega^2$, so that the equivalent moment of inertia is one-third that of the whole shaft.

Torsional vibrations are liable to occur in cases where rotating masses are subjected to periodic torques. A familiar case is that of a gas engine with two flywheels. Torsional vibrations have also been troublesome in the case of a reciprocating engine driving generators whose rotors were mounted on an extension of the crank-shaft.* In such cases there is also a danger of torsional vibrations if the speed is $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. of, or 2, 3, 4, etc. times the natural period of the system.

90. Longitudinal Vibration of Loaded Rods—Cases of longitudinal vibrations of loaded rods are similar to those for a loaded spring which was considered in Art. 83. Neglecting the effect of the inertia of the rod, the frequency is given by $n=\frac{1}{2\pi}\sqrt{\frac{g}{c}}$, where c is the static extension produced by the weight. The equation can also be written $n=\frac{1}{2\pi}\sqrt{\frac{gK}{W}}$ where K is the stiffness or force necessary to produce unit extension. For a parallel rod of area A the extension x produced by a force F is given by $F=\frac{EAx}{l}$ where A is the area of the rod and l the length. When $x=1$ then $F=K$, hence $K=\frac{EA}{l}$

If the rod is of variable cross-section with areas A_1, A_2, A etc., and corresponding lengths l_1, l_2, l_3, etc., then

$$\frac{x}{F}=\left(\frac{l_1}{EA_1}+\frac{l_2}{EA_2}+\frac{l_3}{EA_3}+\ldots\right)=\frac{1}{K}$$

or $K=\dfrac{E}{\Sigma\frac{l}{A}}$. Hence $n=\dfrac{1}{2\pi}\sqrt{\dfrac{Eg}{W\Sigma\frac{l}{A}}}$

91. Longitudinal and Torsional Vibration of Unloaded

* Frith and Lamb, "Journal of Inst. of Elect. Eng.," Vol. xxxi.

Rods—In the case of longitudinal vibration where L is the length of a wave the frequence is given by $n=\frac{1}{L}\sqrt{\frac{E}{\varrho}}$,* where ϱ is the density of the material of the rod. The equation for torsional vibrations is similar, and $n=\frac{1}{L}\sqrt{\frac{G}{\varrho}}$. If w be the weight of a unit length of the rod, then $\frac{w}{g}=A\varrho$ and the equations become $n=\frac{1}{L}\sqrt{\frac{EgA}{w}}$ and $n=\frac{1}{L}\sqrt{\frac{GgA}{w}}$ respectively. For a rod fixed at one end and free at the other there is a node at the fixed end and an antinode at the free end and therefore $L=4l$ where l is the length of the rod. If the rod is fixed in the middle or at each end, $L=2l$. It may be noted that for an unloaded rod fixed at one end and free at the other the amplitude at a section distant x from the fixed end is for both longitudinal and torsional vibrations $\sin\left(\frac{x}{l}\frac{\pi}{2}\right)$ times that at the free end. In the case of a loaded rod the corresponding fraction is $\frac{x}{l}$. The velocity of sound in the rod is given by nL.

Example 1—A gas engine flywheel weighs 10 tons and has a radius of gyration of 4·5 ft. It is keyed to a crank-shaft 10 in. diameter and the distance from the centre of gravity of each flywheel to the mid-length of the crank-pin is 20 in. Taking $G=12\times 10^6$ lb. per sq. in., find the frequency of natural torsional vibration.

The frequency n is given by $n=\frac{1}{2\pi}\sqrt{\frac{K}{I}}$ where

$$K=\frac{GJ}{l}=\frac{12\times 10^6\times\pi\times 10^4}{20\times 32}$$

and
$$I=\frac{\pi}{2}\times(4{\cdot}5\times 12)^2\times\frac{10\times 2{,}240}{32{\cdot}2\times 12}$$

Hence
$$\frac{K}{I}=\frac{6\times 10^6\times 32{\cdot}2\times 12}{2{,}240\times(4{\cdot}5\times 1{\cdot}2)^2\times 4}=8{,}820.$$

and $n=15$ per sec. or 900 vibrations per minute.

Example 2—A piece of spruce with its length in the direction of the grain is 2 ft. 6 in. long and $\frac{1}{4}$ in. square section and weighs 0·0305 lb. Six in. of one end is clamped in a vice and the projecting 2 ft. is set vibrating longitudinally by rubbing with a

* "The Dynamical Theory of Sound," Lamb.

resined cloth. The frequency of the note emitted is found by means of a siren to be 2,290 per second. Find the value of Young's Modulus. The relation between the frequency n and E is

$$n = \frac{1}{4l}\sqrt{\frac{EgA}{w}}$$

hence

$$E = \frac{16l^2n^2w}{gA}$$

$$= \frac{16 \times (24 \times 2{,}290)^2 \times 0{\cdot}0305 \times 16}{32 \times 2 \times 30 \times 12}$$

$$= 2{\cdot}03 \times 10^6 \text{ lb. per sq. in.}$$

The value of E determined by vibration experiments of high frequency as in the last example is slightly higher than that obtained by static methods. The difference is owing to the fact that in the case of vibration the entropy is sensibly constant, the temperature being variable, whilst for static methods the reverse is the case. For structural materials the difference is less than 1 per cent., which is not of practical importance.

EXAMPLES. VIII

(1) If a round bar 1 in. in diameter and 40 in. between supports deflects 0·0936 in. under a load of 100 lb. in the middle and twists through an angle of 0·037 radian when subjected to a twisting moment of 1,000 lb. in. throughout its length of 40 in., find Young's Modulus and the moduli of rigidity and volume. (Victoria.)

(2) Find the diameter of a wrought-iron shaft to transmit 90 H.P. at 130 revs. per minute. If there is a bending moment acting equal to the twisting moment, what ought to be the diameter ? Safe stress = 5,000 lb. per sq. in. (S. & A.)

(3) A steel rod $\frac{3}{8}$ in. diameter is clamped in a vice so that 24 in. projects outwards and the projecting length is rubbed longitudinally and emits a note. If Young's Modulus is 30×10^6 lb. per sq. in. and the weight of a cubic in. of material is 0·28 lb., find the lowest frequency of the note. If the rod is fixed at both ends with a length of 24 in. between the fixings, find the frequency.

(4) A cast-iron disc 12 in. diameter and 1 in. thick is supported with its diameters horizontal by three parallel flexible strings 10 ft. long. The strings are fixed to three equally spaced points on the disc, each being 5 in. from the centre. If the weight of a cubic in. of the material of the disc is 0·26 lb., find the frequency of the torsional vibrations.

(5) A gas engine has two flywheels on its shaft ; each flywheel weighs half a ton and has a radius of gyration of 27 in. If the two flywheels are 24 in. apart centre to centre, and if the diameter of the crank-shaft is $3\frac{1}{2}$ in., determine the natural frequency of torsional oscillations. The crank is attached to the shaft midway between

the two flywheels. (Modulus of rigidity $= 12 \times 10^6$ lb. per sq. in.) (S. & A.)

(6) An open coiled helical spring is made from steel wire $\frac{3}{8}$ in. diameter. The coils make an angle of 23 deg. with planes perpendicular to the axis of the helix. There are fifteen coils and their mean diameter is 8 in. If the spring carries an axial load of 36 lb., find the maximum intensity of shear stress in the steel, the axial elongation of the spring, and the angular twist of the free end.

(7) A hollow steel shaft has to transmit 500 H.P. at a speed of 65 revolutions per minute. The maximum twisting moment in each revolution exceeds the mean by 32 per cent. Determine the necessary dimensions of the cross-section of the shaft if the shearing stress is not to exceed 5 tons per sq. in. and if the internal diameter is to be $\frac{7}{10}$ of the external diameter. (S. & A.)

(8) A propeller shaft has to be designed to support (1) a maximum twisting moment of 500,000 lb. ft., (2) a simultaneous bending moment of 90,000 lb. ft., and (3) an axial thrust of 24 tons. The shaft is made 15 in. in external diameter and 8 in. in internal diameter. Determine the maximum intensity of the compressive principal stress.

(9) A long steel rod is firmly fixed at one end in a vertical position and carries a weight of 50 lb. at the free end. A steady force of 10 lb. applied horizontally at the free end is found to produce a displacement of 0·1 ft. The free end is pulled aside a distance of 0·3 ft. and let go. Find the time of vibration and the position after two seconds.

(10) A shaft transmits 100 horse-power at 200 revs. per minute. What diameter is required if the twist is not to exceed 1 degree in 10 ft.? (Modulus of rigidity $=$ 5,500 tons per sq. in.)

(11) A vertical shaft 2 in. in diameter, of length 5 ft., has a wheel of 1,610 lb. keyed to its middle. The distance of the centre of gravity of the wheel from the axis of rotation being called x, let x be 0·001 ft. when the wheel is not rotating. What is x when the speed is a radians per second? What is the critical value of a? When the shaft is run at 10 per cent., 400 per cent., and 900 per cent. greater speeds than the critical, what is x? Take Young's Modulus as 3×10^7 lb. per sq. in. Neglect the stiffness of the fastenings of the shaft. (S. & A.)

(12) A steel shaft 1 in. diameter revolves in a long bearing at its upper end with its axis vertical. A disc 18 in. diameter and weighing 200 lb. is fixed at the lower end of the shaft, the length of the projecting shaft to the centre of gravity of the disc being 20 in. Find the approximate frequency and whirling speed, (1) neglecting the rotary motion of the weight, (2) taking account of it. Take Young's Modulus as 30×10^6 lb. per sq. in.

CHAPTER IX

THIN AND THICK CYLINDERS—ROTATING DISCS

92. Thin Cylinders—The simplest case of stress in the material of a cylinder is that of a thin cylinder—that is, one whose wall thickness is small compared with the mean radius—when subjected to internal pressure of a fluid. Such cases occur in practice in the shells of steam boilers, thin steam pipes, compressed air tanks, and many other similar instances.

In Fig. 134 is shown a transverse and longitudinal section of

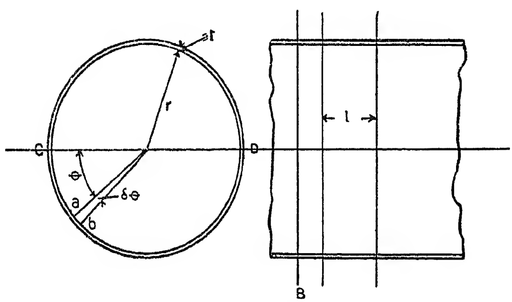


FIG. 134.

a cylinder of internal radius r and small thickness t. Let the internal pressure be p units per unit of area. This pressure will act in a radial direction at all points on the internal surface of the cylinder, and the total load on the area abl will be $pabl$ where l is the length of the cylinder under consideration. Resolving this normal to CD the load becomes $pabl \sin\theta$, and since $ab = r\delta\theta$, then the total force tending to fracture the material across the longitudinal section CD is $2prl\int_0^{\frac{\pi}{2}} \sin\theta d\theta = 2prl$. This force is balanced by the tension in the material at C and D, and if f_1 is the circumferential or hoop stress assumed uniform across the thickness, this tension is $2f_1t$. Thus $pr = f_1t$ or $f_1 = \frac{pr}{t}$.

The total force tending to fracture the material across the transverse section AB is $\pi r^2 p$, which is resisted by the longi-

tudinal force in the material. Since t is small compared with r, this force is sensibly given by $2\pi r f_2 t$ where f_2 is the longitudinal stress, and therefore $pr = 2f_2 t$ or $f_2 = \frac{pr}{2t}$. Thus $f_1 = 2f_2$ or the hoop stress is twice the longitudinal stress, and hence the longitudinal seams in thin cylinders—notably boilers—should be twice as strong as the circumferential seams.

In the case of boilers the seams consist of rivetted joints which are always weaker than the solid plate. Considering the reduction of hoop strength caused by the longitudinal joints, it will be evident that a gain can be obtained by using spiral joints. If the spiral is inclined at an angle θ with the transverse section, the stress normal to the spiral is $p_d = f_1 \sin^2 \theta + f_2 \cos^2 \theta$. Also the shear stress along the spiral is $p_s = \frac{f_1 - f_2}{2} \sin 2\theta$ and the resultant stress is $q = \sqrt{p^2_d + p^2_s}$ (Arts. 17 and 18), which acts obliquely to the spiral. Economy by the use of spiral joints is scarcely probable, as the cost of the greater work in forming such joints would outweigh any possible ultimate saving.

In the case of a thin spherical shell the conditions across any diametrical plane are the same as those across the transverse section of a thin cylinder. With the above notation

$$2\pi r f t = \pi r^2 p \text{ or } f = \frac{pr}{2t}$$

where f is the hoop stress in the material.

93. Strain in Thin Cylinders and Sphere—Considering first the case of the cylinder, let e_1 and e_2 be the hoop and longitudinal strains respectively. Then $e_1 = \frac{f^1}{E} - \frac{f_2}{mE} = \frac{f_1}{E}\left(1 - \frac{1}{2m}\right)$, where E is Young's Modulus and $\frac{1}{m}$ is Poisson's Ratio. Hence if $\frac{1}{m} = \frac{1}{4}$ then $e_1 E = \frac{7}{8} f_1$. Similarly $e_2 = \frac{f_2}{E}\left(1 - \frac{2}{m}\right)$ and $e_2 E = \frac{1}{2} f_2 = \frac{1}{4} f_1$. The ratio of the hoop to the longitudinal strain is accordingly $\frac{e_1}{e_2} = \frac{7}{2}$, whereas the ratio of the corresponding stresses is $\frac{f_1}{f_2} = 2$.

The initial internal volume of the cylinder is $\pi r^2 l$, where r is the initial radius and l the length under consideration. The volume when the pressure is acting is $\pi r^2 (1 + e_1)^2 l (1 + e_2)$, and since e_1

and e_2 are very small, this is sensibly equal to $\pi r^2 l(1 + 2e_1 + e_2)$. Hence the increase in volume is given by

$$\delta v = \pi r^2 l\left\{\frac{2f_1}{E}\left(1 - \frac{1}{2m}\right) + \frac{f_2}{E}\left(1 - \frac{2}{m}\right.\right.$$

$$\frac{\pi r^2 l f_1}{E} \qquad \frac{2}{m}$$

$$= \frac{\pi r^3 l p}{Et}\left\{\frac{5}{2} - \frac{2}{m}\right.$$

where p is the internal pressure and t the thickness of the material. If $\frac{1}{m} = \frac{1}{4}$ then $\delta v = \frac{2\pi r^3 l p}{Et}$.

In the case of the sphere let e be the hoop strain. Then $= \frac{f}{E}\left(1 - \frac{1}{m}\right)$, where f is the hoop stress. The volume of the sphere when the pressure is acting is $\frac{4}{3}\pi r^3(1 + e)^3$, and since e is small the increase in volume δv caused by the pressure p is given by

$$\delta v = 4\pi r^3 e = \frac{4\pi r^3 f}{E}\left(1 - \frac{1}{m}\right.$$

$$\frac{2\pi r^4 p}{Et}\left(1 - \frac{1}{m}\right) \text{ and if } \frac{1}{m} = \frac{1}{4}$$

then $$\delta v = \frac{3\pi r^4 p}{4Et}$$

94. Stresses in Thick Cylinders—The theory given below is due to Lamé.* In Fig. 135 is shown the end view of a small prism of unit length supposed to be situated in the walls of a thick cylinder which is subjected to both internal and external pressure. The prism is held in equilibrium by the hoop stress q and the radial stresses p and $p + \delta p$. Compressive and tensile stresses will be considered as positive and negative respectively. Resolving the forces on the prism radially outwards

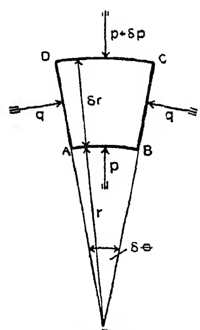


FIG. 135.

$$pAB - (p + \delta p)CD + qAD \sin\frac{\delta\theta}{2} + qBC \sin\frac{\delta\theta}{2} = 0$$

or $$pr\delta\theta - (p + \delta p)(r + \delta r)\delta\theta + 2q\delta r\frac{\delta\theta}{2} = 0.$$

* "Leçons sur la théorie . . . de l' élasticité." Paris, 1852.

Hence $-r\delta p - p\delta r + q\delta r = 0$, neglecting the squares and products of the small quantities δp and δr. Therefore

$$r\frac{dp}{dr} = q - p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let the longitudinal stress in the walls be f and the corresponding strain e_3. Then at any radius r, $e_3 = \frac{f}{E} - \frac{1}{mE}\{p+q\}$. It is assumed that the longitudinal stress is uniformly distributed over the section or that f is constant and independent of r. Also that plane transverse sections remain plane when the pressure is acting, so that e_3 is constant and independent of r. Hence

$$p + q = 2a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $2a$ is a constant. Subtracting equation (1) from (2)

$$r\frac{dp}{dr} + 2p = 2a$$

and multiplying both sides by r

$$r^2\frac{dp}{dr} + 2pr = 2ar,$$

hence

$$\frac{d}{dr}(pr^2) = \frac{d}{dr}(ar^2 + b),$$

therefore

$$pr^2 = ar^2 + b \text{ or } p = a + \frac{b}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also from (2) and (3)

$$q = a - \frac{b}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It may be noted that the values of p and q are independent of the longitudinal stress f.

The values of the constants a and b are usually best obtained from the conditions of the problem to be solved, but general formulæ for p and q can be deduced. Let the internal and external pressures and radii be p_0, r_0 and p_1, r_1 respectively. Then, substituting in equation (3)

$$p_0 = a + \frac{b}{r_0^2}$$

$$p_1 = a + \frac{b}{r_1^2},$$

and the solution of these equations gives

$$a = -\frac{p_1 r_1^2 - p_0 r_0^2}{r_1^2 - r_0^2}, \; b = \frac{(p_0 - p_1) r_1^2 r_0^2}{r_1^2 - r_0^2}$$

The general formula is obtained by substituting these values in equations (3) and (4). In many practical cases, as, for instance,

hydraulic mains, p_1 is zero and the constants a and b reduce respectively to

$$a = \frac{p_0 r_0^2}{r_1^2 - r_0^2} \qquad b = \frac{p_0 r_1^2 r_0^2}{r_1^2 - r_0^2}$$

Hence
$$p = \frac{p_0 r_0^2}{r_1^2 - r_0^2} + \frac{p_0 r_1^2 r_0^2}{(r_1^2 - r_0^2) r^2}$$

or
$$p = \frac{p_0 r_0^2}{r_1^2 - r_0^2} \left\{ \left(\frac{r_1}{r} \right)^2 - 1 \right\} \tag{5}$$

Similarly
$$q = - \frac{p_0 r_0^2}{r_1^2 - r_0^2} \left\{ \left(\frac{r_1}{r} \right)^2 + 1 \right\} \tag{6}$$

From a consideration of the above equations it will appear that the radial stress p is always positive or compressive, whilst the hoop stress q is always negative or tensile. Also that both stresses have their greatest numerical values when $r = r_0$, that is, on the inside surface of the cylinder, and their smallest numerical values on the outside surface, the value of p on the outside surface being zero. Further, on adding equations (5) and (6)

$$p + q = - \frac{2 p_0 r_0^2}{r_1^2 - r_0^2} = \text{constant.}$$

The equation for q can be written in the form

$$q = - \frac{p_0 r_0^2}{(r_1 - r_0)(r_1 + r_0)} \left\{ \left(\frac{r_1}{r} \right)^2 + 1 \right\}$$

In the case of a thin cylinder of mean radius r and thickness t,

$$q = - \frac{p_0 r_0^2}{2tr} \left\{ \left(\frac{r_1}{r} \right)^2 + 1 \right\}$$

and since r_0, r_1, and r are sensibly equal, then $q = - \frac{p_0 r}{t}$, which is the same relations as was previously otherwise deduced. Similarly the equation for p becomes

$$p = \frac{p_0 r}{2t} (1 - 1) = 0.$$

The relations between p, q, and r in the case of a thick cylinder will be considered in connexion with the following example.

Example—A thick cylinder with internal and external radii of 2 in. and 4 in. respectively is subjected to an internal pressure of 2 tons per sq. in. Draw curves showing the relation between the radial and hoop stresses and the radius.

Substituting the above values of r and p in the equation

$$p = a + \frac{b}{r^2}$$

it becomes $\qquad 2=a+\frac{b}{4}$ and $0=a+\frac{b}{16}$

whence $\qquad a=-\frac{2}{3} \quad b=\frac{32}{3}$

and therefore $\qquad p=-\frac{2}{3}+\frac{32}{3r^2}$

and $\qquad q=-\frac{2}{3}-\frac{32}{3r^2}$

When $r=2$, $p=2$ and $q=-3{\cdot}33$ tons per sq. in.
When $r=3$, $p=0{\cdot}52$ and $q=-1{\cdot}85$ tons per sq. in.
When $r=4$, $p=0$ and $q=-1{\cdot}33$ tons per sq. in.

The curves are shown drawn in Fig. 136, the signs being reckoned positive upwards and negative downwards.

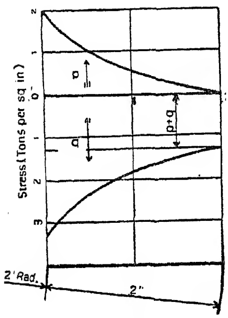


FIG. 136.

The variation of hoop stress represented by the Lamé equation has been confirmed experimentally by Prof. Goodman.* A thick cast-iron cylinder was used with small holes drilled into the walls from the outside surface to various depths. An internal pressure was applied without end load and the movement at the bottom of the holes was measured with a delicate extensometer. From the amount of the movement the stresses were calculated and curves constructed which closely agreed with Lamé's theory.

In the above example the longitudinal stress in the material is given by $-\pi(16-4)f=\pi\times 4\times 2$.

Hence $f=-\frac{2}{3}$ tons per sq. in., which is smaller than the values of p and q on the inside surface.

From equations (5) and (6) the values of p and q when $r=r_0$, that is, on the inside surface, are given by $p=p_0$ and $q_0=-\frac{p_0(r_1+r_0)}{r_1^2-r_0^2}$ Writing $\frac{r_1}{r_0}=\text{K}$, the expression for q becomes $q_0=-p_0\frac{\text{K}^2+1}{\text{K}_2-1}$, and since K is always greater than 1, the intensity of the stress q_0 is always greater than p_0. The longitudinal stress f is given by

$$f=-\frac{\pi r_0^2 p_0}{\pi(r_1^2-r_0^2)}=-\frac{p_0}{\text{K}^2-1}$$

* "Mechanics Applied to Engineering," Goodman.

and therefore $\frac{q_0}{f} = K^2 + 1$ or q_0 is always greater than f and becomes sensibly equal to $2f$ when K is nearly unity, as was previously found for thin cylinders. Hence of the three principal stresses q_0 the hoop stress is always the greatest and if f_t be the tensile stress in the material, then $f_t = p_0\frac{K^2+1}{K^2-1}$ or $p_0 = f_t\frac{K^2-1}{K^2+1}$, so that p_0 is always less than f_t.

The maximum shear stress is one-half the difference between the algebraically greatest and least principal stresses. Denoting the shear stress in the material by f_s,

$$f_s = \frac{1}{2}(p_0 - q_0) = \frac{1}{2}\left(p_0 + p_0\frac{K^2+1}{K^2-1}\right) = p_0\frac{K^2}{K^2-1} \text{ or } p_0 = f_s\cdot\frac{K^2-1}{K^2}$$

Hence $p_0 = f_s\frac{K^2-1}{K^2}$, so that p_0 is always less than f_s.

Drs. Cook and Robertson* found experimentally that for mild steel $p_0 = 0{\cdot}6\ f_t\frac{K^2-1}{K^2}$, which gives $f_s = 0{\cdot}6\ f_t$.

95. Manner of Failure of Thick Cylinders—It may be stated briefly that whereas thin cylinders usually fail by sudden or rapid bursting, the manner in which a thick cylinder becomes destroyed depends largely on the character of the material. When the material is brittle, like cast-iron and some of the harder steels, failure begins to show when the tensile principal stress reaches a certain maximum value equal to the tensile strength of the material. The stresses cause a crack to appear which extends at once to the outer surface. The crack lies in an axial direction and generally extends over a part only of the complete length. An example of this type of failure is that of the cast-iron cylinder described in Chap. XII, tested by one of the authors (W. C. Popplewell), and it is clearly evident that the failure was governed by the maximum tensile stress.

In the case of ductile materials, including all steels except the very hardest, failure begins when the maximum shear stress of the material is reached. A certain amount of bulging takes place and bursting may subsequently occur, but shear failure is the first manifestation.

In some experiments on thick cast-iron and mild-steel cylinders, conducted by Doctors Cook and Robertson,† it was found that the maximum direct stress theory held for cast iron, whilst the maximum shear stress theory was well supported by the experiments on mild steel. In a more recent paper by Mr. W. W.

* 'Engineering," December 15, 1911.
† 'Engineering," December 15, 1911.

Gaylord, read before the American Society of Mechanical Engineers, dealing with steel pipes and fittings for water pressures up to 5,000 lb. per sq. in., the author also concluded from observed failures that for such materials as ductile steel the best results are obtained by designing according to the maximum shear stress theory. Hence for mild steel the formula to use is $p_0 = f_s\frac{K^2 - 1}{K^2}$, and since $\frac{f_s}{f_t} = 0{\cdot}6$ approximately, this becomes $p_0 = 0{\cdot}6 f_t\frac{K^2 - 1}{K^2}$, where f_t and f_s are the ultimate tensile and shear stresses respectively. For brittle materials the formula is $p_0 = f_t\frac{K^2 - 1}{K^2 + 1}$. If in either of these formulæ f_t, p_0 and the factor of safety are given, K can be found, so that if the value either of r_1 or r_0 is fixed, the other value can be obtained. In most cases it is sufficient to use the formula $p_0 = f_t\frac{K^2 - 1}{K^2 + 1}$ for both brittle and ductile materials. Since $0{\cdot}6\, f_t\frac{K^2 - 1}{K_2}$ is greater or less than $f_t\frac{K^2 - 1}{K^2 + 1}$ according as K^2 is less or greater than 1·5, the value of f_t should be taken on the low side for ductile materials if K^2 is greater than 1·5.

It may be noted that if K is very great in the formula $p_0 = f_t\frac{K^2 - 1}{K^2 + 1}$ it sensibly becomes $p_0 = f_t$, which means that an infinitely thick wall will not prevent the cylinder from failing. An instance of this can be seen at the ends of holes containing the explosive during blasting in hard rock. Here p_0 is extremely high and must be greater than f_t, and it is seen that cracks are started by the explosion which have time to run radially some way into the body of the rock before the force of the explosion has spent itself.

96. Strain in Thick Cylinders—If the three principal stresses at any point in the wall of a thick cylinder and Young's Modulus and Poisson's Ratio are known, then the strain at the point in any one of the three corresponding directions can be found. Let p, q and f be the stresses in the radial circumferential and longitudinal directions respectively, and e_1, e_2, and e_3 the corresponding strains reckoned positive when compressive. Then

$$e_1 = \frac{p}{E} - \frac{q}{mE} - \frac{f}{mE} = \frac{1}{E}\left\{p - \frac{1}{m}(q + f)\right.$$

$$=-\frac{q}{E}+\frac{p}{mE}-\frac{f}{mE}=\frac{1}{E}\left\{-q+\frac{1}{m}(p-f)\right\}$$

$$e_3=-\frac{f}{E}+\frac{p}{mE}-\frac{q}{mE}=\frac{1}{E}\left\{-f+\frac{1}{m}(p-q)\right\}$$

where E and $\frac{1}{m}$ are Young's Modulus and Poisson's Ratio respectively. If the strains are required it is usually best to find the numerical values of p, q, and f from the formulæ given and insert them in these equations. In some cases the longitudinal stress is zero and the strains then become

$$e_1=\frac{p_0}{E}\left\{1-\frac{K^2+1}{m(K^2-1)}\right\}$$

$$e_2=-\frac{p_0}{E}\left\{\frac{K^2+1}{K^2-1}-\frac{1}{m}\right\}$$

$$e_3=-\frac{2p_0}{mE(K^2-1)}$$

where p_0 is the internal pressure and K is the ratio of the external to the internal radius.

97. Built-up Cylinders—In the case of thick solid cylinders subjected to internal pressure the greatest induced stress is the hoop stress on the inside surface, which diminishes rapidly towards the outside. Hence if a cylinder be built up of two tubes, the outer being shrunk on the inner, the hoop stress on the inside surface of the inner tube will be of the opposite sign to the stress induced when the pressure is acting, and therefore the range of stress for the built-up tube will be the greater.

Consider the case of a solid cylinder of 2 in. internal and 4 in. external radius and that of a cylinder of the same weight but built up of a cylinder of 3 in. and 4 in. internal and external radii respectively shrunk on one of 2 in. and 3 in. internal and external radii. Using the maximum principal stress theory and taking the safe tensile stress as 15 tons per sq. in., it is required to find the maximum internal pressure that can be applied in the two cases. For the solid cylinder the formulæ

$$p=a+\frac{b}{r^2}\text{ and }q=a-\frac{b}{r^2}$$

yield $\quad 0=a+\frac{b}{16}$ and $-15=a-\frac{b}{4}$ respectively.

Hence $a=-3$ and $b=48$ and therefore

$$p_0=a+\frac{b}{4}=-3+\frac{48}{4}=9\text{ tons per sq. in.}$$

In the case of the built-up cylinder let the radial and hoop stresses on the inside surfaces of the inner and outer cylinders be p_0, q_0 and p_1, q_1 respectively. Also let the corresponding stresses on the outside surfaces of the two cylinders be p_1', p_2 and q_1', q_2 respectively. This notation is shown in Fig. 137. Considering first the outer cylinder, $p_2=0$ and $q_1=-15$, and therefore $0=a+\frac{b}{16}$ and $-15=a+\frac{b}{9}$, whence $a=-\frac{27}{5}$ and $b=\frac{432}{5}$, therefore $p_1=a+\frac{b}{9}=-\frac{27}{5}+\frac{435}{45}=4\cdot2$ tons per sq. in. This pressure acts as an external radial pressure on the inner cylinder, and therefore $4\cdot2=a+\frac{b}{9}$ and $-15=a-\frac{b}{4}$, whence

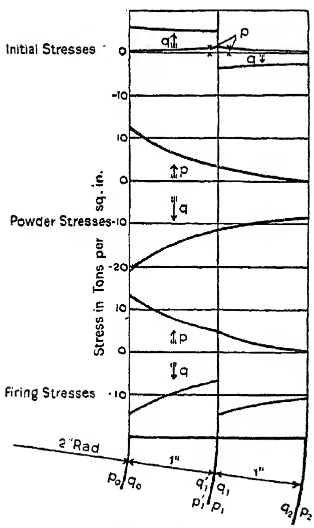


FIG. 137.

$a=0\cdot70$ and $b=53\cdot15$. Hence $p_0=a+\frac{b}{4}=-0\cdot70+13\cdot29=12\cdot6$ tons per sq. in., so that the allowable interval pressure in the case of the built-up cylinder is 40 per cent. higher than for the solid one. The other stresses on the surfaces of the cylinders can be found by the aid of the above equations. For instance, in the case of the inner cylinder $p_0+q_0=p_1'+q_1'$, and therefore

$$q_1'=12\cdot6-15-4\cdot2=-6\cdot6$$
tons per sq. in.

Also for the outer cylinder $p_1+q_1=p_2+q_2$, hence

$$q_2=4\cdot2-15-0=-10\cdot8 \text{ tons}$$
per sq. in.

In the case of built-up cylinders the diameters of the tubes from which they are built have to be calculated, so that when the tubes are shrunk on one another the stresses induced are as prearranged. In gun work the stresses in the complete cylinder before the pressure is applied are called "Initial Stresses." Also the stresses induced when the pressure is applied are called "Firing Stresses," whilst those that would be induced by the pressure if the cylinder were solid are "Powder Stresses." The initial stresses from which the allowances to be made for shrinkage are obtained are given by the relation :

Initial Stresses + Powder Stresses = Firing Stresses, which is true for both hoop and radial stresses.

In order to find the powder stresses for the last example, consider the cylinder as solid and let the hoop and radial stresses at the inside, middle, and outside of the walls be Q_0, Q_1, Q_2 and P_0, P_1, P_2 respectively. Then

$$P_0 = a + \frac{b}{4} = 12{\cdot}6, \quad P_2 = a + \frac{b}{16} = 0,$$

whence $$a = -4{\cdot}2 \text{ and } b = 67{\cdot}2$$

and $$P_1 = -4{\cdot}2 + \frac{67{\cdot}2}{9} = 3{\cdot}27$$

$$Q_0 = -4{\cdot}2 - \frac{67{\cdot}2}{4} = -21{\cdot}0$$

$$Q_1 = -4{\cdot}2 - \frac{67{\cdot}2}{9} = -11{\cdot}67$$

$$Q_2 = -4{\cdot}2 - \frac{67{\cdot}2}{16} = -8{\cdot}4.$$

All the stresses are tabulated below and curves showing the relation between the stresses and the radius are given in Fig. 137.

Firing Stresses.	Powder Stresses.	Initial Stresses.
$q_0 = -15$	$Q_0 = -21$	$q_0 = 6{\cdot}0$
$q_1' = -6{\cdot}6$		$q_1' = 5{\cdot}07$
$q_1 = -15$	$Q_1 = -11{\cdot}67$	$q_1 = -3{\cdot}33$
$q_2 = -10{\cdot}8$	$Q_2 = -8{\cdot}4$	$q_2 = -2{\cdot}4$
$p_0 = 12{\cdot}6$	$P_0 = 12{\cdot}6$	$p_0 = 0$
$p_1' = 4{\cdot}2$		$p_1' = 0{\cdot}93$
$p_1 = 4{\cdot}2$	$P_1 = 3{\cdot}27$	$p_1 = 0{\cdot}93$
$p_2 = 0$	$P_2 = 0$	$p_2 = 0$

The initial tensile hoop strain at the inside surface of the outer cylinder is $-\frac{q_1}{E} + \frac{p_1}{mE}$, and taking $E = 13{,}000$ tons per sq. in. and $\frac{1}{m} = \frac{1}{4}$ this becomes

$$\frac{3{\cdot}33}{13{,}000} + \frac{0{\cdot}53}{4 \times 13{,}000} = 0{\cdot}000266.$$

Hence the increase in the diameter is

$$\frac{\pi \times 6 \times 0{\cdot}000266}{\pi} = 0{\cdot}001596 \text{ in.}$$

The initial compressive hoop strain at outside surface of inner tube is

$$\frac{q_1'}{E} - \frac{p_1'}{mE} = \frac{5{\cdot}07}{13{,}000} - \frac{0{\cdot}53}{4 \times 13{,}000} = 0{\cdot}00038$$

and therefore the decrease in diameter is $6 \times 0{\cdot}00038 \times 0{\cdot}00228$ in. Thus the total difference in diameter before the outer tube is shrunk on should be $0{\cdot}00228 + 0{\cdot}00160 = 0{\cdot}00388$ in. Hence if the outer cylinder is bored to 6 in. diameter, the inner one must be turned to 6·0039 in. diameter. Cases of cylinders built up of three or more tubes can be similarly treated.

98. Wire-wound Cylinders—From the above example it may be concluded that the greater the number of tubes used to build up a cylinder of a given weight the more nearly the average hoop stress throughout the walls will approach to that of the inside surface, and consequently the more efficient the design. An important practical difficulty in the manufacture of such cylinders is the expense involved in turning and boring the surfaces and the difficulty in making them exactly to size. The more usual method is to use a solid inner cylinder and wind round it square or rectangular wire of high tensile strength. This method is used notably for guns, and when the winding is completed an outer cylinder with no initial stress is slipped over it.

The treatment of the case is very similar to that of a gun built up of shrunk tubes, and in order to outline it briefly it will be assumed that the dimensions are known and also the initial and firing hoop stresses on the inside surface. Let the stresses be $+26$ and -18 tons per sq. in. respectively and the internal and external radii of the finished gun and of the coil be r_0, r_3, and r_1 and r_2 respectively. The powder hoop stress is given by $-18-26=-44$ tons per sq. in., and the other powder stresses can be obtained by finding the values of the constants a and b from the equations $-44 = a - \frac{b}{r_0^2}$, $0 = a + \frac{b}{r_3^2}$. With the aid of these constants the values of the powder stresses at any radius can be calculated.

The firing stresses in the inner tube can be found in a similar manner, the Lamé equations becoming $-18 = a - \frac{b}{r_0^2}$ and $p_0 = a + \frac{b}{r_0^2}$, where p_0 is the firing radial stress at the inside surface and has the same value as the radial powder stress there. When the values of these constants are known the radial firing pressure p_1 at the outside surface of the inner tube, which is the same as that at the inside surface of the coil, can be calculated.

Also the radial firing pressure at the outside surface of the coil is known, for it is the same as the radial powder stress at the inside surface of the outer tube. Hence if t, the firing tension in the wire, is to be constant, then

$$t(r_2 - r_1) = p(r_1 - r_2)$$

or
$$t = \frac{p_1 r_1 - p_2 r_2}{r_2 - r_1}.$$

The value of t may be as high as 50 tons per sq. in.

By the above procedure the powder and firing stresses can be found, from which the initial stresses are given by the relation

Initial Stresses + Powder Stresses = Firing Stresses.

Consider next the coil and let θ be the winding-on tension of the wire at radius r which lies between r_1 and r_2. When the coil is completed there is a radial pressure at the radius r, which induces a compressive hoop stress in the wire, and the algebraic sum of this induced stress and the winding-on tension is the initial hoop stress in the wire. Let q_r be the induced compressive stress caused by the winding-on from r to r_2 and let ϕ be the initial hoop stress or tension, then $\theta - q_r = \phi$. The value of q_r is obtained by considering the gun as homogeneous from r_0 to r, with zero pressure inside and subjected to an external pressure p, where p is the initial radial pressure at radius r. Hence

$$p = a + \frac{b}{r^2} \qquad 0 = a + \frac{b}{r_0^2}$$

whence
$$a = \frac{pr^2}{r^2 - r_0^2} \text{ and } b = \frac{pr_0^2}{r_0^2 - r^2}$$

Therefore
$$q_r = a - \frac{b}{r^2} = p\frac{r^2 + r_0^2}{r^2 - r_0^2}$$

and
$$\theta = \phi + p\frac{r^2 + r_0^2}{r^2 - r_0^2}$$

and since ϕ and p are known for all values of r, a curve connecting θ and r can be drawn, from which the winding-on tension at any radius can be obtained.

It will be noted that the winding-on tension is greater than the initial, which is owing to the fact that, as the winding is continued beyond a particular radius, the tension at the radius decreases owing to the induced compressive hoop stresses.

99. Stress in Thin Revolving Ring—The method of determining the hoop stress in a thin revolving ring is analogous to that for a thin cylinder subjected to an internal pressure. Let the thickness and mean radius of the ring be t and r respectively and w the weight of a cubic inch of the material. When the ring

is revolving the outward radial pressure on a sq. in. is $\frac{w}{g}\omega^2 rt$, where ω is the angular velocity. This acts like the internal pressure in a thin cylinder, and therefore $\frac{w}{g}\omega^2 rtr = ft$, where f is the hoop stress in the ring. Hence $\omega^2 r^2 = \frac{gf}{w} = v^2$ where v is the peripheral speed, or $f = \frac{wv^2}{g}$. If v is the velocity in ft. per second, the equation becomes

$$f = \frac{w \times v^2 \times 12 \times 12}{32{\cdot}2 \times 12} = 0{\cdot}372\, wv^2.$$

It will be noted that f is independent of the radius r. If v is 88 ft. per second, that is, 60 miles per hour, and w for cast iron is 0·26 lb. per cu. in., then $f = 0{\cdot}372 \times 0{\cdot}26 \times (88)^2 = 750$ lb. per sq. in.

100. Stresses in Thin Revolving Disc—When a thin disc is revolving about an axis through its centre and perpendicular to its plane, the stress in the direction of the thickness of the disc can be assumed zero. Let u be the radial displacement at radius r. The radial strain is $\frac{du}{dr}$ and the hoop strain is $\frac{u}{r}$, hence

$$\frac{du}{dr} = \frac{1}{E}\left\{p - \frac{q}{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{u}{r} = \frac{1}{E}\left\{q - \frac{p}{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where p and q are the radial and hoop stresses at radius r, whilst E is Young's Modulus and $\frac{1}{m}$ is Poisson's Ratio. Multiplying equation (2) by r and differentiating with respect to r, it becomes

$$\frac{du}{dr} = \frac{1}{E}\frac{d}{dr}\left\{qr - \frac{pr}{m}\right.$$

and, combining this with equation (1),

$$\frac{d}{dr}\left(qr - \frac{pr}{m}\right) = \left(p - \frac{q}{m}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Another equation is required, which is obtained by considering the equilibrium of a small prism situated in the disc, as was done for thick cylinders. Referring to Fig. 135, but reckoning compressive forces as negative, the resolution outwards of the forces on the prism should equal its mass times its outward acceleration.

Hence, referring to Art. 94 and omitting a term for the thickness of the disc which would cancel out,

$$-pr\delta\theta+(p+\delta p)(r+\delta r)\delta\theta-2q\delta r\frac{\delta\theta}{2}=-\frac{w}{g}\left(r+\frac{\delta r}{2}\right)\delta r\delta\theta\omega^2\left(r+\frac{\delta r}{2}\right)$$

or

$$-r\delta p-p\delta r+q\delta r=\frac{w}{g}\omega^2r^2\delta r,$$

neglecting the squares and products of the small quantities δp and δr. Thus

$$q=p+r\frac{dp}{dr}+\frac{w}{g}\omega^2r^2 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting for q in equation (3) it becomes

$$r^2\frac{d^2p}{dr^2}+3r\frac{dp}{dr}\left(3+\frac{1}{m}\right)\frac{w}{g}\omega^2r^2=0.$$

The solution of this equation is

$$p=A+\frac{B}{r^2}-\frac{1}{8}\left(3+\frac{1}{m}\right)\frac{w}{g}\omega^2r^2$$

and from equation (4)

$$q=A-\frac{B}{r^2}-\frac{1}{8}\left(3+\frac{1}{m}\right)\frac{w}{g}\omega^2r^2,$$

where A and B are constants. If B has a value, p and q approach infinity when r is very small. Since the stresses are finite for all values of r, then $B=0$, also $A=\frac{1}{8}\left(3+\frac{1}{m}\right)\frac{w}{g}\omega^2r_1^2$ where r_1 is the outside radius of the disc. Therefore

$$p=\frac{w\omega^2}{8g}\left(3+\frac{1}{m}\right)(r_1^2-r^2)$$

and

$$q=\frac{w\omega^2}{8g}\left\{\left(3+\frac{1}{m}\right)r_1^2-\left(1+\frac{3}{m}\right)r^2\right\}$$

The stresses have their greatest value at the centre of the disc, where $r=0$, and become

$$p=q=\frac{wv^2}{8g}\left(3+\frac{1}{m}\right)$$

which is $\frac{3m+1}{8m}$, or if $\frac{1}{m}=\frac{1}{4}$, it is $\frac{13}{32}$ times the hoop stress in a thin ring with the same peripheral speed.

If the disc has a concentric hole through it of radius r_0 then

$$A=\frac{w\omega^2}{8g}\left(3+\frac{1}{m}\right)(r_1^2+r_0^2) \text{ and } B=-\frac{w\omega^2}{8g}\left(3+\frac{1}{m}\right)r_1^2r_0^2$$

and therefore

$$p=\frac{w\omega^2}{8g}\left(3+\frac{1}{m}\right)\left(r_1^2+r_0^2-\frac{r_1^2r_0^2}{r^2}-r^2\right)$$

$$q=\frac{w\omega^2}{8g}\left(3+\frac{1}{m}\right)\left(r_1^2+r_0^2+\frac{r_1^2r_0^2}{r^2}\right)-\left(1+\frac{3}{m}\right)r^2$$

It is evident that q has its greatest value when $r=r_0$, that is, at the circumference of the hole, and becomes

$$q=\frac{w\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)r_1^2+\left(1-\frac{1}{m}\right)r_0^2\right\}$$

If the hole is very small $q=\frac{wv^2}{4g}\left(3+\frac{1}{m}\right)$, where v is the peripheral speed. It will be noted that this stress is just twice that for the solid disc.

The value of r for the maximum value of p can be found by differentiating the equation for p with respect to r and equating to zero. This gives $r=\sqrt{r_1r_0}$ and the maximum value of p becomes $p=\frac{w\omega^2}{8g}\left(3+\frac{1}{m}\right)(r_1-r_0)^2$. The manner in which the stresses p and q vary with the radius for both the solid disc and the one with a concentric hole in it is shown in Fig. 138. The disc is of steel and 2 ft. diameter, the diameter of the hole being 8 in. The weight of a cubic in. of steel is taken as 0·28 lb., the speed as 7,000 r.p.m., and Poisson's Ratio as $\frac{1}{4}$.

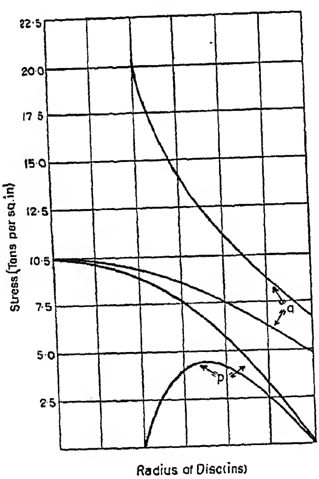


FIG. 138.

101. Rotating Disc of Uniform Strength—In the case of a rotating disc of uniform strength the stresses p and q will each be constant and independent of r. Let this constant stress be f and the thickness of the disc at the radii r and $r+\delta r$ be t and $t+\delta t$ respectively. Then, considering a small prism situated in a disc, the equation of equilibrium is obtained in a similar manner to that for a disc of uniform small thickness and is

$$frt\delta\theta+f(r+\delta r)(t+\delta t)\delta\theta+2f\delta r\frac{\delta\theta}{2}\left(t+\frac{\delta t}{2}\right)$$

$$w\omega^2/_{r+\frac{\delta r}{2}}\ {}^2/_{t+\frac{\delta t}{2}}$$

which reduces to $\frac{dt}{dr}+\frac{w\omega^2}{gf}rt=0$. The solution of this equation is $t=Ae^{-\frac{w\omega^2r^2}{2gf}}$. At the centre of the disc $r=0$ and $t=t_0$, and therefore $t=t_0e^{-\frac{w\omega^2r^2}{2gf}}$. The shape of half a disc of uniform strength is shown in Fig. 139. The diameter of the disc is 2 ft. 0 in. and the thickness at the axis is 8 in. The uniform stress is taken as 10 tons per sq. in., the speed as 14,000 revs. per minute, and the weight of a cubic inch of material as 0·28 lb. per cubic in.

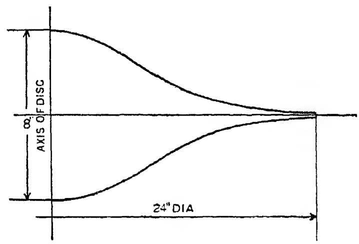


FIG. 139.

EXAMPLES. IX

(1) If the tensile stress due to centrifugal force on the rim of a cast-iron pulley is not to exceed 1,500 lb. per sq. in., find the maximum peripheral velocity which is permissible. Weight of cast iron,=0·26 lb. per cubic in.

(2) A hydraulic cylinder is 8 in. in internal diameter, the metal is 3 in. thick, and the water pressure is 2,000 lb. per sq. in. Calculate the intensities of circumferential and radial stresses at radii of 4 in., 5 in., 6 in., and 7 in., and plot diagrams to show the variation of stress. (S. & A.)

(3) A boiler 6 ft. in diameter is built of plates $\frac{3}{4}$ in. thick. Find the maximum strain in the plates when the internal pressure is 120 lb. per sq. in. and calculate the stress in simple tension which would produce a strain of equal magnitude. (Young's Modulus$=30\times 10^6$ lb. per sq. in. and Poisson's Ratio$=\frac{1}{4}$.)

(4) Find the stress in the rims of the wheels of a railway train when the train is running at 60 miles per hour. Treat the rims as thin rings and take the weight of a cubic in. of the material as 0·28 lb.

(5) A compound cylinder is formed by shrinking a tube of 7 in. external diameter on to another tube of $5\frac{1}{2}$ in. external diameter and 4 in. internal diameter. If the radial pressure at the common surface is 2 tons per sq. in., find the circumferential stress at the common inside and outside surfaces. Also find the difference in diameter to be allowed for in shrinking on, taking E equal to 30×10^6 lb. per sq. in. and Poisson's Ratio$=\frac{1}{4}$.

(6) A steel disc is to be 2 ft. 6 in. diameter and is to run at 7,500 r.p.m. If the thickness at the periphery is 1 in., find the thickness at the axis so that the direct stress in the disc shall be everywhere 5 tons per sq. in. Weight of a cubic in. of steel equals 0·28 lb.

(7) A thin steel disc 2 ft. diameter revolves at 5,000 revs. per minute. Find the maximum stress induced (1) when the disc is solid, (2) when it has a concentric hole 12 in. diameter through it, (3) if the diameter of the concentric hole approaches the diameter of the disc so that the disc can be considered as a thin ring. Weight of a cubic in. of steel equals 0·28 lb. and Poisson's Ratio $= \frac{1}{4}$.

(8) Find the thickness of a copper pipe to carry steam at a pressure of 250 lb. per sq. in. The mean diameter is 7 in. and the stress in the metal is not to exceed 2,000 lb. per sq. in.

(9) At a certain cross-section of a crank-shaft 6 in. diameter there is a bending moment of 50 ton in. and a twisting moment of 120 ton in. Find the position, direction, and magnitude of the maximum tensile and shearing stresses.

CHAPTER X

PLANE SPRINGS—RESILIENCE—STRESSES IN CURVED BARS— FLAT PLATES—STABILITY OF EARTH AND DAMS

102. Plane Spiral Springs—In Fig. 140 is shown a plane spiral spring such as is used in watches and clocks. One end of the spring is fixed to an arbor, A, which can turn freely in bearings, the other end being fixed to a point B. Suppose there are initially no forces acting, and then let a torque T be applied to the arbor, which turns it through an angle ϕ. The torque will cause a force to act at the point B sensibly in the direction from C to B, which must be equal and opposite to the resultant of the forces between the arbor and its bearings. Denoting these forces by F and the distance between their lines of action by h, then $T = Fh$.

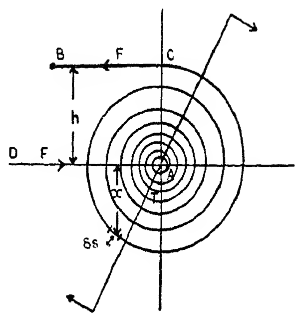


Fig. 140.

Let the bending moment on a short length δs of the spring be M. Then, assuming that the portion of the spring from δs to B is removed, it will appear that $M = T + Fx$. If the change in the angle between the tangents to the ends of δs be $\delta\phi$, then $\frac{\delta\phi}{\delta s} = \frac{M}{EI}$ and $EI\Sigma\delta\phi = T\Sigma\delta s + F\Sigma x\delta s$. The quantity $\Sigma x\delta s$ is the moment of the spring about the line DE, which can usually be assumed to pass through the centre of gravity of the spring, so that the quantity is zero and therefore $\phi = \frac{Tl}{EI}$, where l is the length of the spring and ϕ the angle of winding-up in radians. It will be noted that ϕ is proportional to T.

The energy stored in the spring is given by $\frac{1}{2}T\phi = \frac{T^2 l}{2EI}$.

The change in curvature at any point distant x from DE is $\frac{\phi}{s} = \frac{1}{R} = \frac{M}{EI}$. Near the arbor x approaches zero and therefore

sensibly $\frac{1}{R}=\frac{T}{EI}$. Also, near the outside at C, $x=h$ and $\frac{1}{R}=\frac{T+Fh}{EI}=\frac{2T}{EI}$. Let t be the thickness of the strip from which the spring is made, then $f=\frac{Et}{2R}=\frac{Tt}{I}$, where f is the stress at the point C. If the breadth of the strip is b, this becomes $f=\frac{Tt}{2I}=\frac{6T}{bt^2}$. It is assumed that the thickness t at any point is not great compared with the initial and final radii of curvature at the point.

The energy stored in the spring by the application of a torque T has been shown to be $\frac{T^2l}{2EI}$ and the maximum stress f to be $\frac{Tt}{I}$. Hence the energy stored in terms of f is

$$\frac{I^2f^2l}{2EIt^2}=\frac{f^2}{24E}\times btl=\frac{1}{24}\frac{f^2}{E}\times \text{volume of spring.}$$

This is the resilience, which is $\frac{1}{24}\frac{f^2}{E}$ per unit volume of the material, and is only $\frac{1}{4}$ of that for a closely coiled helical spring subjected to an axial twist.

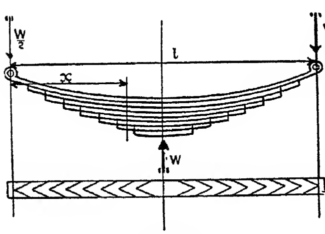


FIG. 141.

103. Built-up Plate Springs—A view of an ordinary laminated spring such as is used for supporting railway and other vehicles is shown in Fig. 141, the ends being hinged and the load applied at the middle. In its unloaded state such a spring consists of a number of narrow steel plates or strips which all have the same curvature before they are assembled.

Let the initial curvature be denoted by R and the bending moment and corresponding maximum stress at a section distant x from one end and between the end and the centre be M and f respectively. Then $f=\frac{Mt}{2n_1I}$, where I is the moment of inertia of a plate, t the thickness of the plates, all assumed the same, and n_1 the number of plates at the section. The moment of inertia $I=\frac{bt^3}{12}$ where b is the breadth of the plates, and therefore

$f = \frac{6M}{n_1 bt^2}$. The stress in all the plates at the section is evidently the same, and if the stress at every section is the same, then $\frac{M}{n_1}$ is constant, and since $M = \frac{W}{2}x$, then x should be proportional to n_1. Thus the number of plates should increase uniformly from each end to the middle. The bending moment in the middle is $\frac{Wl}{4}$, and if n be the total number of plates, then $f = \frac{3Wl}{2nbt^2}$ where W is the central load and l the length of the spring, which is assumed to be the same before and after it is loaded. In an actual spring an equivalent to the uniform increase in the number of plates is attained by tapering each end of the plates. This is shown in Fig. 141.

Suppose that when the central load W is applied the central deflection is δ. The bending moment at any section of any plate is constant and the relation for the deflection may be obtained by considering the longest plate and assuming it to be initially straight and then bent to an arc of a circle of radius R and central deflection δ. Then $\delta(2R - \delta) = \frac{l^2}{4}$, where l is the length of the plate. Hence, neglecting δ^2,

$$\frac{l^2}{8R}$$

and since

$$\frac{1}{R} = \frac{M}{EI}$$

$$\frac{Ml^2}{8EI} = \frac{3Wl^3}{8nEbt^3}$$

The energy stored in the spring, or resilience, is given by

$$\frac{1}{2}W\delta = \frac{3W^2l^3}{16nEbt^3}$$

Substituting

$$f^2 = \frac{9W^2l^2}{4n^2b^2t^4}$$

then

$$\frac{1}{2}W\delta = \frac{f lbtn}{12E} = \frac{f^2}{6E} \times \text{volume of spring.}$$

In Fig. 142 are shown two load-deflection curves of the front axle spring of a Ford car, and it will be noted that when the plates are ungreased the lag in the deflection is much the greater. The spring was suspended at the ends in the same manner as in

the actual car, which probably accounts for the lines not being straighter. The dimensions of the spring were: span, 31 in.; number of plates in middle, 7; breadth and thickness of plates, 1·5 in. and 0·25 in. respectively. Hence the central deflection corresponding with a load of $\frac{1}{2}$ ton is given by

$$\delta = \frac{3Wl^3}{8nEbt^3} = 2{\cdot}54 \text{ in.}$$

whilst from the graphs it is about 2·9 in.

104. Resilience—Cases of resilience, or the energy stored in an elastic body when it is strained, have been considered in

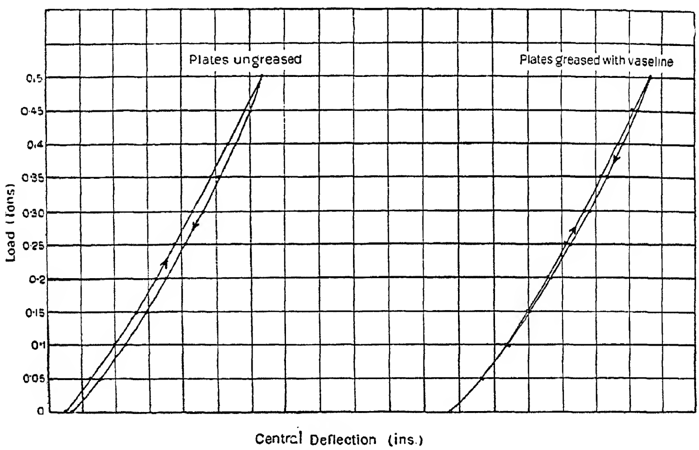


FIG. 142.—Deflection Curves of Front Axle Spring of a Ford Car.

various previous articles. If a force or couple F produces a deflection or twist ϕ the resilience is $\frac{1}{2}F\phi$. This can usually be written in terms of the maximum stress in the material and the modulus of elasticity, and is then of the form $B\frac{f^2}{M} \times$ volume of body, where B is a constant and M is the modulus. For instance, in the case of a short vertical pillar or bar firmly fixed at its lower end and supporting a weight W at its upper end, the strain energy stored is $\frac{1}{2}Wx$, where x is the shortening caused by the weight. Let A and l be the area and length of the bar respectively.

Then $\frac{W}{A}=f$ where f is the stress in the material. Also $\frac{fl}{x}=E$ where E is Young's Modulus and therefore

$$\frac{1}{2}Wx=\frac{f^2Al}{2E}=\frac{f^2}{2E}\times\text{ volume of bar,}$$

or $\frac{f^2}{2E}$ per unit volume of the material. The strain energy stored when the maximum stress is equal to the elastic limit of the material is called the proof resilience.

Let the area of the bar at the upper end be A_1 and at the fixed end A_2. Also let A_1 be constant for a length l_1 from the upper end and A_2 be constant for a length l_2 from the fixed end where $l_1+l_2=l$, the total length of the bar. If f be the stress in the portion A_1, then $\frac{A_1}{A_2}f$ is the stress in the portion A_2, so that the total energy stored in the bar is $\frac{f^2}{2E}\left\{A_1l_1+\frac{A_1^2}{A_2}l_2\right\}$ Let $A_2=KA_1$ where K is greater than unity. Then the resilience becomes $\frac{f^2A_1}{2E}\left\{l_1+\frac{l_2}{K}\right\}$. If the resilience of the two portions is to be the same, $l_1=\frac{l_2}{K}$, and since K is greater than unity, the length of the smaller area should be less than that of the larger.

If $l_1=l_2=\frac{l}{2}$ the resilience becomes $\frac{f^2A_1l}{4E}\left(1+\frac{1}{K}\right)$ which is always less than $\frac{f^2A_1l}{2E}$, which is the resilience of a bar of uniform area A_1. Also the ratio of the resilience of the portion of smaller area A_1 to that of the larger A_2 is evidently K, the ratio of the areas.

In the case of the parallel bar of area A and length l, let the load be placed so as just to touch the top of the bar and be then released. The bar will then act like a spring, and if x be the static shortening, then the shortening when the weight is so applied will be $2x$. Hence the stress will be $2f$ where f is the static stress and the strain energy stored when the weight is at its lowest point is $\frac{2f^2}{E}\times$ volume of bar.

If the weight be dropped from a height h and there is no loss of energy at impact, the energy lost by the weight will be stored as strain energy in the bar. Let the maximum shortening of the bar be x and assume that the weight moves with the top of the bar. Then when the weight is at its lowest point, and neglecting the effect of the weight of the bar itself, $W(h+x)=\frac{fAx}{2}$, where A is the area of the bar and f the

maximum induced stress. Also $x = \frac{fl}{E}$ where l is the length of the bar, and therefore

$$\frac{f^2Al}{2E} - \frac{fWl}{E} - Wh = 0$$

or
$$f = \frac{W}{A}\left(1 + \sqrt{1 + \frac{2AEh}{Wl}}\right)$$

Since there is always some loss of energy at impact the induced stress will be less than that given by this equation. In some cases the loss of energy may be so great as to cause deformation of the bodies. When a wooden post is driven into the ground with a light hammer a large proportion of the available energy at impact is spent in deforming the top of the post. If a heavier hammer is used the proportion of dissipated energy is smaller, so that a larger proportion is available for driving the post into the ground. Another instance is that of the Izod impact machine, where a swinging pendulum knocks off the end of a notched bar. That some energy is lost is evidenced by the fact that a groove usually appears in the bar where it has been struck by the knife-edge of the pendulum.

105. The Bending of Curved Bars—In the case of the bending of an initially straight bar or beam the neutral axes pass through the centroids of the normal sections. For this case the relations $\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$ have been established in chapter IV, where M is the bending moment, I the moment of inertia of a section about the neutral axis, f the stress in fibres distant y from the neutral plane, E Young's Modulus, and R the radius of curvature of the neutral line of the bent beam. If the beam is initially curved the neutral axes do not pass through the centroids of the sections, but if the initial curvature is fairly great compared with the dimensions of the section, then sensibly $\frac{M}{I} = \frac{f}{y} = E\left\{\frac{1}{R'} - \frac{1}{R}\right\}$, where R′ and R are the final and initial radii of curvatures respectively. In cases where the initial radius of curvature is comparable with the dimensions of the section, as, for instance, in crane hooks and chain links, the last formulæ do not apply. The neutral plane is at an appreciable distance from the centroids of the sections, which necessitates a modification of the formulæ.

In Fig. 143 A, B, C, D is the side view of a portion of an initially curved elastic bar where R is the radius of curvature of the plane which passes through the centroids of the sections and is normal to the plane of bending, the line EF representing the edge view

of the plane. When the bending moment is applied the portion becomes as shown at A′, B′, C′, D′, where the radius of curvature of the plane through the centroids is denoted by R′. Consider a fibre of material of initial length LM and distant x from the centroid plane. After bending, the length of the fibre will change to L′M′ and its distance from the centroid plane to x'. If f_x denote the stress in the fibre, then

$$\frac{L'M' - LM}{LM} = \frac{f_x}{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also, considering a fibre in the centroid plane,

$$\frac{E'F' - EF}{EF} = \frac{f_c}{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where f_c is the stress in the fibre.

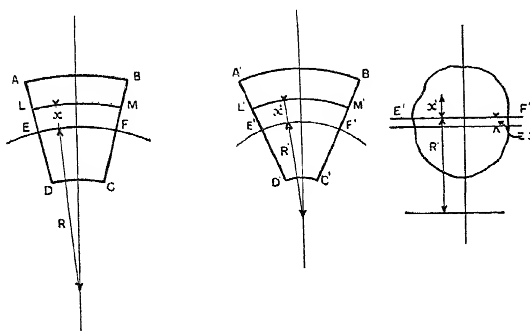


FIG. 143.

Next considering the symmetry of the figures and assuming, as in the theory of straight beams, that plane normal cross-sections remain plane—

$$\frac{LM}{EF} = \frac{x + R}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{L'M'}{E'F'} = \frac{x' + R'}{R'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From equations (1) to (4) relations for the stress in any fibre and the bending moment can be obtained. Equations (1) and (2) can be written

$$\frac{L'M'}{LM} = 1 + \frac{f_x}{E}$$

$$\frac{E'F'}{EF} = 1 + \frac{f_c}{E}$$

Dividing these equations,

$$\frac{L'M' \times EF}{LM \times E'F'} = \left(1 + \frac{f_x}{E}\right)\left(1 + \frac{f_c}{E}\right)^{-1}$$

Substituting from equations (3) and (4)

$\frac{R(x' + R')}{R'(x + R)} = 1 + \frac{f_x}{E} - \frac{f_c}{E}$ since $\frac{f_x}{E}$ and $\frac{f_c}{E}$ are small quantities. Hence

$$f_x = f_c + E\left\{\frac{\frac{x'}{R'} + 1}{\frac{x}{R} + 1}\right\} - 1$$

$$= f_c + E\left\{\frac{\frac{x'}{R'} - \frac{x}{R}}{\frac{x}{R} + 1}\right\}$$

In the theory of the bending of straight beams it is assumed that $x' = x$, and making this assumption,

$$f_x = f_c + E\left(\frac{1}{R'} - \frac{1}{R}\right)\frac{xR}{x + R} \quad . \quad . \quad . \quad . \quad (5)$$

This assumption introduces a very small error which appreciably simplifies subsequent expressions. The question has been fully considered by Prof. Morley.*

The bending moment M on the portion of the beam is given by $M = \Sigma f_x x \delta a$, where δa is the area of the fibre distant x from the centroid plane. Hence

$$M = \Sigma f_c x \delta a + E\Sigma\left(\frac{1}{R'} - \frac{1}{R}\right)\frac{x^2 R}{x + R}\,\delta a$$

but

$$\Sigma f_c x \delta a = f_c \Sigma x \delta a = 0,$$

and therefore

$$M = E\left(\frac{1}{R'} - \frac{1}{R}\right)\Sigma\frac{x^2 R}{x + R}\delta a.$$

Writing $\Sigma\frac{x^2 R}{x + R}\delta a = Ah^2$ where A is the area of the normal section and h^2 a constant for the section, this becomes

$$\frac{M}{Ah^2} = E\left(\frac{1}{R'} - \frac{1}{R}\right).$$

The corresponding formula for a beam with large initial curvature

* "Engineering," September 11 and 25, 1914.

R is $\frac{M}{Ak^2}=E\left(\frac{1}{R'}-\frac{1}{R}\right)$, where Ak^2 is the moment of inertia of a normal section about the neutral axis. In order to find the stress in any fibre, consider equation (5), which is

$$f_x=f_c+E\left(\frac{1}{R'}-\frac{1}{R}\right)\frac{xR}{x+R}$$

Since the total longitudinal force on the portion of the beam is zero, then

$$\Sigma f_x\delta a=0$$

and $$o=f_c\Sigma\delta a+E\left(\frac{1}{R'}-\frac{1}{R}\right)\Sigma\frac{xR}{x+R}\delta a.$$

Hence $$f_c=-\frac{E}{A}\left(\frac{1}{R'}-\frac{1}{R}\right)\Sigma\frac{xR}{x+R}\delta a$$

but $$\frac{xR}{x+R}\delta a=x\delta a-\frac{x^2}{x+R}\delta a$$

hence $$\Sigma\frac{xR}{x+R}\delta a=\Sigma x\delta a-\Sigma\frac{x^2}{x+R}\delta a$$

$$=0-\Sigma\frac{x^2}{x+R}\delta a.$$

Since $$\Sigma\frac{x^2R}{x+R}\delta a=Ah^2,$$

then $$\Sigma\frac{x^2}{x+R}\delta a=\frac{Ah^2}{R}=-\Sigma\frac{xR}{x+R}\delta a$$

and $$f_c=\frac{Eh^2}{R}\left(\frac{1}{R'}-\frac{1}{R}\right)$$

and the stress in any fibre distant x from the centroid plane is given by

$$f_x=\frac{Eh^2}{R}\left(\frac{1}{R'}-\frac{1}{R}\right)+E\left(\frac{1}{R'}-\frac{1}{R}\right)\frac{xR}{x+R}$$

$$=M\left\{\frac{1}{RA}+\frac{x}{Ah^2}\left(\frac{R}{x+R}\right)\right\}$$

where A is the area of a normal section. At the neutral plane $f_x=0$, and therefore

$$\frac{x+R}{x}=-\frac{R^2}{h^2}$$

$$x\left(1+\frac{R^2}{h^2}\right)=-R$$

and $$x_0=-\frac{R}{1+\frac{R^2}{h^2}}$$

Thus the neutral plane is nearer the centre of curvature than the centroid plane by the amount x_0. If the stress in an extreme fibre distant y from the neutral plane is denoted by f, then

$$f = M\left\{\frac{1}{R.A} + \frac{y}{Ah^2}\left(\frac{R}{y+R}\right)\right.$$

$$= \frac{M}{RA}\left\{1 + \frac{R^2 y}{h^2(y+R)}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where y is positive or negative according as it is measured away from or towards the centre of curvature.

If R is very great this reduces to $\dfrac{f}{y} = \dfrac{M}{Ah^2}$ where

$$Ah^2 = \Sigma\frac{x^2 R}{x+R}\delta a = \Sigma\frac{x^2}{\frac{x}{R}+1}\delta a = \Sigma x^2 \delta a = Ak^2,$$

so that the formula becomes $\dfrac{f}{y} = \dfrac{M}{I}$, as for a straight beam.

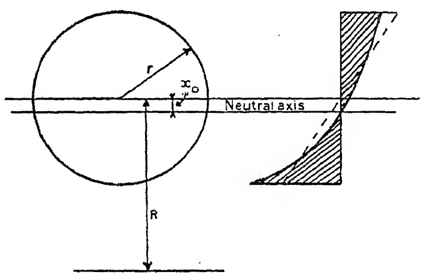


Fig. 144.

In Fig. 144 is shown a curve representing the stress variation across a circular bar which is initially bent to an arc of a circle of radius $2r$, where r is the radius of the section of the bar. The dotted line shows the stress variation if the bar were initially straight.

The quantity h^2 which is involved in the above expressions can be deduced for simple symmetrical sections by integration over the section. In the case of a rectangle, and referring to Fig. 145 :

$$\frac{Ah^2}{R^2} = -\Sigma\frac{x}{x+R}\delta a = -b\int_{-\frac{d}{2}}^{+\frac{d}{2}}\frac{x}{x+R}dx$$

$$= -b\int_{-\frac{d}{2}}^{+\frac{d}{2}}\left(1 - \frac{R}{x+R}\right)dx$$

$$= -b\left(d - R\log_e\frac{2R+d}{2R-d}\right)$$

Hence $$Ah^2 = R^2 b\left(R \log_e \frac{2R+d}{2R-d} - d\right)$$

or expanding,

$$h^2 = R^2\left\{\frac{1}{12}\left(\frac{d}{R}\right)^2 + \frac{1}{80}\left(\frac{d}{R}\right) + \frac{1}{448}\left(\frac{d}{R}\right) + \qquad .\right\}$$

In Fig. 146 is shown a circular section of radius r.

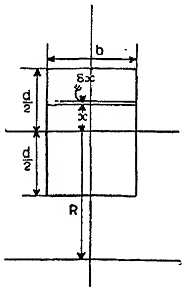


FIG. 145.

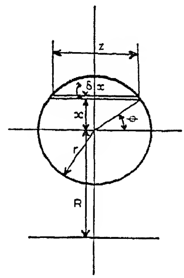


FIG. 146.

$$\frac{Ah^2}{R^2} = -\int_{-r}^{+r} \frac{zx}{x+R} dx$$

$$= R\int_{-r}^{+r} \frac{z}{x+R} dx - \int_{-r}^{+r} zdx$$

$$= zr^2 R\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\cos^2\theta}{r\sin\theta + R} d\theta - \pi r^2$$

On effecting the integration this becomes

$$2\pi R^2 - 2\pi R\sqrt{R^2 - r^2} - \pi r^2$$

$$\therefore\ h^2 = R^2\left\{2\left(\frac{R}{r}\right)^2 - \frac{2R}{r}\sqrt{\left(\frac{R}{r}\right)^2 - 1} - 1\right\}$$

There is a simple graphical method for determining the quantity h^2. In Fig. 147 is shown a section of a curved bar of radius of curvature R and radius of section r. The line EF passes through the centroid of the section and is perpendicular to the plane of bending. From any point p in EF draw the straight line op and produce to meet the circle in P. Next draw pP′ perpendicular to EF to cut the horizontal through P in P′. On repeating this construction for a number of points, such as p, a corresponding number of points P′ will be obtained. When these points P′ are joined by a smooth curve the figure denoted by a' results. Let A denote the area of the original figure and

A′ the area of the figure a'. Then $h^2 = \dfrac{R^2(A' - A)}{A}$; the area A′ − A is shown shaded in Fig. 147. To prove this last expression, consider the similar triangles OPQ and pPP′. Then

$$\frac{PP'}{PQ} = \frac{QC}{QO} = \frac{x}{x + R}$$

and

$$\frac{Ah^2}{R^2} = \Sigma\frac{xz}{x + R}\delta x = -\Sigma\frac{PP'}{PQ} \times PQ\,\delta x$$

$$= -\Sigma PP'\delta x$$

$$= -(\Sigma PQ\,\delta x - \Sigma P'Q\,\delta x)$$

$$= A' - A,$$

hence

$$h^2 = \frac{R^2(A' - A)}{A}$$

Other shapes of sections can be treated in a similar manner.

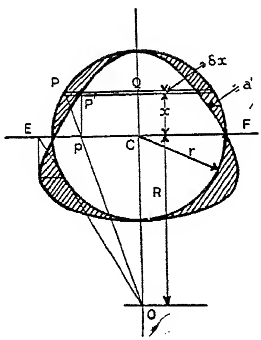


FIG. 147.

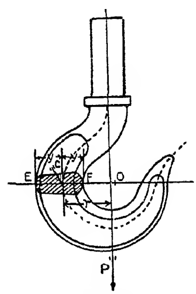


FIG. 148.

106. Crane Hooks—A familiar application of the bending of curved bars occurs in crane hooks, one of which is shown in Fig. 148. Here P is the load which acts at a distance r from c, which is on the dotted centroid line of the sections. The bending moment on the section EF is sensibly Pr, and, in addition, there will be a direct tensile stress $= \dfrac{P}{A}$, where A is the area of the section. A rough method is to neglect the initial curvature of the hook. Then the stress is given by $f = -\dfrac{P}{A} + \dfrac{My}{I}$, where y is positive or negative according as it is measured away from or towards the centre of curvature O, and tensile stresses are negative. The formula can be written in the form $f = \dfrac{P}{A}\left\{\dfrac{ry}{k^2} - 1\right\}$, where k^2 is

the radius of gyration of the section EF about the axis through c and perpendicular to the plane of bending. This formula will give the stress too small on the inside of the section and too great on the outside, the error usually being appreciable.

Recognizing the initial curvature of the hook, the more correct formula is

$$f=\frac{M}{RA}\left\{1+\frac{R^2y}{h^2(y+R)}\right\}-\frac{P}{A}$$

$$=\frac{P}{A}\left\{\frac{r}{R}+\frac{Rry}{h^2(y+R)}-1\right.$$

where tensile stresses are negative and the sign of y is as denoted above. The value of h^2 can be obtained by the graphical method given in the previous article.

107. Stresses in Rings—In Fig. 149 is shown a ring of initial radius R which is subjected to a pull P along a diameter. The dotted line indicates the positions of the centroids of the cross-sections. Consider the portion ECDF. This is evidently held in equilibrium by the forces $\frac{P}{2}$ and by

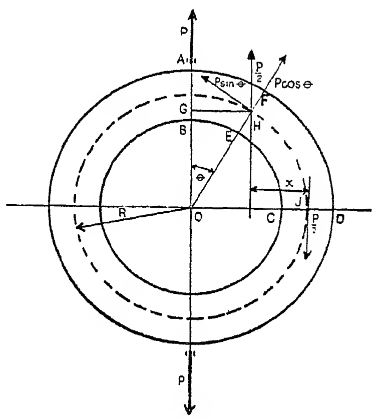


FIG. 149.

couples M and M_1 at H and J respectively. Taking moments about H, then

$$M=M_1+\frac{PR}{2}x=M_1+\frac{PR}{2}(1-\sin\theta),$$

where M and M_1 are reckoned positive when they tend to increase the curvature of the ring. But

$$M=Ah^2E\left(\frac{1}{R'}-\frac{1}{R}\right)$$

and therefore

$$\int_0^{\frac{\pi}{2}}Md\theta=Ah^2E\left\{\int_{\theta=0}^{\theta=\frac{\pi}{2}}\frac{ds'}{R'}-\int_{\theta=0}^{\theta=\frac{\pi}{2}}\frac{ds'}{R}\right.$$

Neglect the effect of the change in the shape of the ring on the bending moment, which introduces a negligible error. Hence

writing $\frac{ds'}{R} = \frac{ds'}{R'}$ where s' is the length of the centroid line

$$\int_0^{\frac{\pi}{2}} M d\theta = \frac{Ah^2E}{R}\left\{\frac{\pi}{2} - \frac{\pi}{2}\right\} = 0$$

and $$\int_0^{\frac{\pi}{2}} \left\{M_1 + \frac{PR}{2}\left(1 - \sin\theta\right)\right\} d\theta = \frac{M_1\pi}{2} + \frac{PR\pi}{2} - \frac{PR}{2} = 0,$$

whence $$M_1 = \frac{PR}{\pi}\left(1 - \frac{\pi}{2}\right).$$

There is in addition a bending moment caused by the stress which acts normal to normal cross-sections and is uniformly distributed over them. This stress is of amount $\frac{P\sin\theta}{2A}$, which increases an original length δs of the centroid line to

$$\left(1 + \frac{P\sin\theta}{2AE}\right)$$

Hence the total angle between the ends of the centroid line over one quarter of the circumference after the load is applied is

$$\int_0^{\frac{\pi}{2}} \left(1 + \frac{P\sin\theta}{2AE}\right)\delta\theta$$

$$= \frac{\pi}{2} + \frac{P}{2AE}$$

Therefore the change in direction is $\frac{Pk^2}{2EI}$ where $Ak^2 = I =$ the moment of inertia of a normal section about a line perpendicular to the plane of the ring and passing through the centre of gravity. Denoting this change by ϕ, then if m is the corresponding change in the bending moment

$$\phi = \int_0^{\frac{\pi R}{2}} \frac{m}{EI} ds = \frac{m\pi R}{2EI}$$

hence $$\frac{Pk^2}{2EI} = \frac{m\pi R}{2EI} \text{ or } m = \frac{Pk^2}{\pi R}$$

This bending moment tends to decrease the curvature of the ring, and therefore

$$M_1 = \frac{PR}{\pi}\left(1 - \frac{\pi}{2} - \frac{k^2}{R^2}\right)$$

and $$M = M_1 + \frac{PR}{2}\left(1 - \sin\theta\right) = PR\left(\frac{1}{\pi} - \frac{\sin\theta}{2} - \frac{k^2}{\pi R^2}\right)$$

When $\theta = 0$, $M = M_0 = \frac{PR}{\pi}\left(1 - \frac{k^2}{R^2}\right)$. Also when $M = 0$

$$\frac{1}{\pi} - \frac{\sin\theta}{2} - \frac{k^2}{\pi R^2} = 0 \text{ or } \sin\theta = \frac{2}{\pi}\left(1 - \frac{k^2}{R^2}\right)$$

If $\frac{k}{R}$ is small this becomes $\sin\theta = 0{\cdot}636$ and $\theta = 39{\cdot}5$ deg.

Curves showing the variation of the bending moment over half a ring of circular section of radius r where $R = 2r$ are shown in Fig. 150.

The resultant stress on a section such as EF (Fig, 149) at any point is made up of the sum of three stresses: (1) A direct stress $f_d = \frac{P\sin\theta}{2A}$ where A is the area of the section; (2) a stress

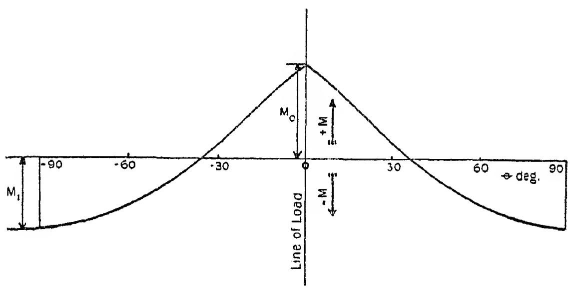


FIG. 150.

caused by bending $= f_b$; and (3) a shear stress f_s of average value $\frac{P\cos\theta}{2A}$. The maximum values of f_d and f_b occur at E and F on the inside and outside of the ring, whilst the value of f_s is zero at these points. The maximum average value of f_s occurs when $\theta = 0$ and 180 deg. and is equal to $\pm\frac{P}{2A}$ and in the case of a ring of rectangular cross-section the maximum value of f_s across the section AB, determined as in Art. 57, occurs along a line passing through the centroid of the section and perpendicular to the plane of the ring and $f_s = \frac{3P}{4A}$. For a ring of circular cross-section the maximum value of $f_s = \frac{2P}{3A}$. These values are approximate only and, the assumptions upon which they are based apply with less error to rectangular sections than circular.

The principal stresses induced by the maximum value of f_s are

less than the maximum value of f_b even when OH is nearly equal to OE. Hence the shear stress can be neglected and the maximum value of f_r obtained from the relation $f_r = f_b + f_d$.

$$= \frac{P\left(\frac{1}{\pi} - \frac{\sin\theta}{2} - \frac{k^2}{\pi R^2}\right)}{A}\left\{1 + \frac{R^2 y}{h^2(y+R)}\right\}$$

$$f_d = \frac{P\sin\theta}{2A}, \text{ and therefore}$$

$$f_r = \frac{P}{A}\left[\left(\frac{1}{\pi} - \frac{\sin\theta}{2} - \frac{k^2}{\pi R^2}\right)\left\{1 + \frac{R^2 y}{h^2(y+R)}\right\} + \frac{\sin\theta}{2}\right]$$

where y is positive or negative according as it is measured away from or towards 0, and tensile stresses are positive. When $\theta = 0$

$$f_r = \frac{P}{A\pi}\left[\left(1 - \frac{k^2}{R^2}\right)\left\{1 + \frac{R^2 y}{h^2(y+R)}\right\}\right]$$

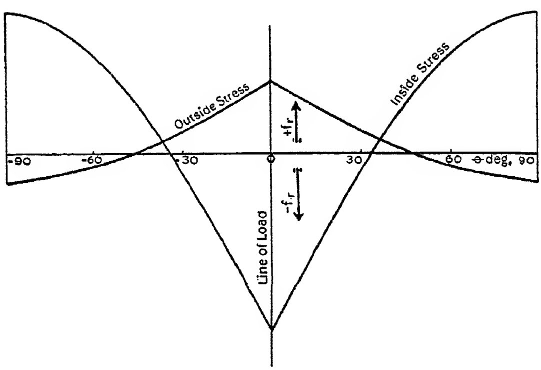


FIG. 151.

When $\theta = \frac{\pi}{2}$

$$f_r = \frac{P}{A}\left[\left(\frac{1}{\pi} - \frac{1}{2} - \frac{k^2}{\pi R^2}\right)\left\{1 + \frac{R^2 y}{h^2(y+R)}\right\} + \frac{1}{2}\right]$$

where A is the area of a normal section of the ring and

$$h^2 = \frac{R^2(A' - A)}{A}$$

the value of A′ being obtained as in Art. 105. Curves showing the variation of f_r for half a ring of circular cross-section of radius r, the mean radius of the ring, being $R = 2r$, are shown in Fig. 151.

If the cross dimensions of the ring are small compared with the mean radius R, then $A' - A$ is sensibly zero, and, as shown

in Art. 105, $h^2 = k^2$. In this case the general expression for f_r becomes

$$f_r = \frac{P}{A}\left[\left(\frac{1}{\pi} - \frac{\sin\theta}{2}\right)\left\{1 + \frac{R^2 y}{k^2(y+R)}\right\} + \frac{\sin\theta}{2}\right]$$

where the term $\frac{k^2}{\pi R^2}$ is omitted since $\frac{k}{R}$ is small.

108. Diametrical Strains—Consider an element of length δs situated at H on the centroid of the ring shown in Fig. 149. When the forces are acting let the inclination of this length to its original direction be $\delta\phi$. Then the movement of the end G of the line HG will be $y\delta\phi$ and the total decrease in the diameter in the line of the load is given by $2\int_0^{\frac{\pi}{2}} y\,d\phi$. Since $\frac{d\phi}{ds} = \frac{M}{EI}$, this decrease is equal to $2\int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{M}{EI}\,y\,ds$ where M is positive when it tends to increase the curvature of the ring. Inserting the value for

$$M = PR\left(\frac{1}{\pi} - \frac{\sin\theta}{2} - \frac{k^2}{\pi R^2}\right)$$

this becomes

$$2\int_0^{\frac{\pi}{2}} \frac{PR}{EI}\left(\frac{1}{\pi} - \frac{\sin\theta}{2} - \frac{k^2}{\pi R^2}\right) R\sin\theta R\,d\theta$$

$$= -\frac{2PR^3}{EI}\left(\frac{\pi}{3} - \frac{1}{\pi} + \frac{k^2}{\pi R^2}\right)$$

By a similar procedure the decrease in the diameter perpendicular to the line of load is

$$\frac{2PR^3}{EI}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\pi} - \frac{\sin\theta}{} - \frac{k^2}{\pi R^2}\right)\cos\theta\,d\theta$$

$$= \frac{2PR^3}{EI}\left(\frac{1}{\pi} - \frac{1}{4} - \frac{k^2}{\pi R^2}\right)$$

In addition to the diametrical change caused by the bending moment, there is also a small change caused by the normal stress on the normal sections. The normal stress $= \frac{P}{2A}\sin\theta$, and if δ is the original length of an element of the centroid line of the ring at G, this is increased by the normal stress to $\frac{P}{2AE}\sin\theta\delta s$.

Hence the total increase in length of the centroid line of the ring is $\frac{2PR}{AE}\int^{\frac{\pi}{2}}\sin\theta\,d\theta = \frac{2PR}{AE}$ where $\frac{ds}{d\theta} = R$. Therefore the in-

crease in length of a diameter is $\frac{2PR}{\pi AE}$. Thus the total increase in a diameter in the direction of the load is

$$\frac{2PR^3}{EI}\left(\frac{\pi}{8}-\frac{1}{\pi}+\frac{k^2}{\pi R^2}\right)+\frac{2PR}{\pi AE}$$

and the total decrease in the direction perpendicular to this is

$$\frac{2PR^3}{EI}\left(\frac{1}{\pi}-\frac{1}{4}-\frac{k^2}{\pi R^2}\right)-\frac{2PR}{\pi AE}$$

If $\frac{k}{R}$ is small these reduce to $\frac{0{\cdot}149PR^3}{EI}$ and $\frac{0{\cdot}137PR^3}{EI}$ respectively.

The above formulæ can be used for determining the value of Young's Modulus, as, for instance, in the case of the material of a thin tube where rings can be obtained but not pieces suitable for an extensometer. With such rings the loads, either compressive or tensile, can be applied in a small testing machine, and if $\frac{k}{R}$ is small the corresponding increases or diminutions of diameter are relatively great. The authors have found the method quite satisfactory in the case of rings cut from high tensile steel and phosphor bronze tubes.

109. Flat Plates—These occur frequently in engineering practice, as, for instance, in the steam and exhaust passages of turbines, in cylinder covers, boiler ends, and the sides and bottoms of tanks. In what follows the theory * will be given for plane circular plates and approximate theories applied to rectangular and elliptical plates. It will be assumed in all cases that there is no stress in the neutral surface of the plates, which is only justified if the deflections are small compared with the thickness. In cases where the deflections are not small compared with the thicknesses the reader is referred to papers by Dr. J. Prescott † and P. T. Steinthal. ‡ Some empirical formulæ on the strength of ribbed plates have been given by C. C. Pounder in the "Mechanical World," Vol. 62, 1917, but much experimental work yet remains to be done.

In Fig. 152 is shown a section through a circular plate loaded symmetrically about the central axis OZ. The line ABC represents the edge of the curved neutral plane of the plate. Let the co-ordinates of the point P in the figure be x, z. Also let R be the radius of curvature of the neutral surface at P in the plane

* "Theorie der Elasticität und Festigkeit," F. Grashof (1878), and "The Theory of Elasticity," Love.

† "Phil. Mag.," Jan., 1922. ‡ "Engineering," May, 1911.

OZX and let R_1 be the radius of curvature in the plane perpendicular to OZX and parallel to OZ. Then, if z is measured positive downwards

$$\frac{1}{R}=-\frac{\frac{d^2z}{dx^2}}{\left\{1+\left(\frac{dz}{dx}\right)^2\right\}^{\frac{3}{2}}}=-\frac{d^2z}{dx^2} \quad . \quad . \quad . \quad (1)$$

since $\frac{dz}{dx}$ is small, and

$$\frac{1}{R_1}=-\frac{1}{x}\frac{dz}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

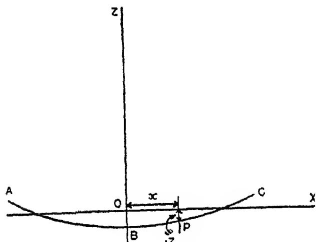


Fig. 152.

Let y be the sensibly vertical distance of a point in the plate on either side of P. Then, reasoning in the same way as in the ordinary theory of beams,

$$e_1=\frac{y}{R} \quad (3)$$

$$e_2=\frac{y}{R_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where e_1 and e_2 are the strains at the point in directions along OX and perpendicular to the plane OZX respectively. If f_1 and f_2 denote the stresses in the directions of e_1 and e_2 respectively, then

$$e_1=\frac{f_1}{E}-\frac{f_2}{mE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$e_2=\frac{f_2}{E}-\frac{f_1}{mE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and from equations (1) to (6) is obtained

$$f_1=\frac{mEy}{m^2-1}\left(m\frac{d^2z}{dx^2}+\frac{1}{x}\frac{dz}{dx}\right) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$f_2=\frac{mEy}{m^2-1}\left(\frac{d^2z}{dx^2}+\frac{m}{x}\frac{dz}{dx}\right) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

These equations give the relations between the stresses and the co-ordinates of the point, but before they can be solved it is necessary to find a relation between z and x. This relation can be found by considering the equilibrium of the semicircular ring of material of inside radius x, width δx, and thickness δz, shown in Fig. 153. The stresses in the direction of, and perpendicular to, the radius x are denoted by f_1 and f_2 respectively. Resolving all

the forces in the direction of OX, the force exerted on the ring by the outer portion of the plate is $\{2xf_1 + 2\delta(xf_1)\}\delta z$. The force exerted by the material on the inside of the ring is $-2xf_1\delta z$, and by the other half of the ring it is $-2f_2\delta x\delta z$. Also if s be the shear stress on the upper surface of the ring and $s+\delta s$ the shear stress on the lower surface, there will be a force in the direction OX of amount $\{(s+\delta s)-s\}2x\delta s = 2x\delta s\delta x$. Since the ring is in equilibrium the algebraic sum of all the above forces is zero, and

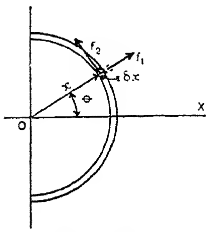


FIG. 153.

$$\frac{ds}{dz} = \frac{f_2}{x} - \frac{1}{x}\frac{d(xf_1)}{dx} = \frac{ds}{dy}$$

Substituting the value of f_1 and f_2 from equations 7 and 8,

$$\frac{ds}{dy} = \frac{m^2 Ey}{m^2-1}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} \quad \frac{1}{x^2}\frac{dz}{dx}\right)$$

The quantity in brackets is independent of y, hence, integrating,

$$s = \frac{m^2 Ey^2}{2(m^2-1)}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx}\right) + A$$

where A is a constant. Let t be the thickness of the plate. Then, since $s = 0$ when $y = \pm\frac{t}{2}$,

$$A = -\frac{m^2Et^2}{8(m^2-1)}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx}\right)$$

and
$$s = -\frac{m^2E(t^2-4y^2)}{8(m^2-1)}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx}\right) \quad . \quad . \quad (9)$$

Suppose the plate supports a central load P and also a uniformly distributed load of w units per unit of area. The total shearing force on a cylindrical surface of the plate of radius x from OZ is $\pi x^2 w + P$, and therefore

$$2\pi x\int_{-\frac{t}{2}}^{+\frac{t}{2}} s\,dz = 2\pi x\int_{+\frac{t}{2}}^{+\frac{t}{2}} s\,dy = \pi x^2 w + P$$

and from this equation and equation (9) is obtained

$$\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx} = -\frac{6(m^2-1)}{m^2Et^3}\left(wx + \frac{P}{\pi x}\right). \quad . \quad (10)$$

Hence from equations (9) and (10) the shear stress is

$$s = \frac{3(t^2-4y^2)}{4t^3}\left(wx + \frac{P}{\pi x}\right) \quad , \quad , \quad , \quad , \quad (11)$$

It will be noted that s is independent of the manner in which the plate is supported. The shear stress is a maximum when $y = 0$, and if $P = 0$ $s = \frac{3wx}{4t}$. Also if $w = 0$ $s = \frac{3P}{4\pi xt}$. This becomes infinite when $x = 0$. If P is uniformly distributed over a circular area of radius r_0, the value of s will be a maximum when $x = r_0$ and

$$s = \frac{3P}{4\pi r_0 t}.$$

The relation between z and x can be found from a consideration of equation (10). This can be written

$$\frac{d}{dx}\left(\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx}\right) = -\frac{6(m^2-1)}{m^2Et^3}\left(wx + \frac{P}{\pi x}\right)$$

hence

$$\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx} = -\frac{6(m^2-1)}{m^2Et^3}\left(\frac{wx^2}{2} + \frac{P\log_e x}{\pi}\right) + B.$$

Since $x\frac{d^2z}{dx^2} + \frac{dz}{dx} = \frac{d}{dx}\left(x\frac{dz}{dx}\right)$, then, multiplying both sides of the last equation by x and integrating,

$$x\frac{dz}{dx} = -\frac{6(m^2-1)}{m^2Et^3}\left(\frac{wx^4}{8} + \frac{Px^2\log_e x}{2\pi} - \frac{Px^2}{4\pi} + \frac{Bx^2}{2}\right) + C.$$

In the cases to be considered $\frac{dz}{dx} = 0$ when $x = 0$ and therefore the constant $c = 0$. Dividing both sides of the equation by x and integrating again,

$$z = -\frac{6(m^2-1)}{m^2Et^3}\left\{\frac{wx^4}{32} + \frac{Px^2(\log_e x - 1)}{4\pi}\right\} + \frac{Bx^2}{4} + D \quad . \quad (12)$$

This is the equation of the neutral surface, the values of the constants B and D depending upon the conditions of a particular problem. On differentiating equation (12) once and twice and inserting in equations (7) and (8), the values of the principal stresses f_1 and f_2 are obtained. Then from equations (5) and (6) the values of the principal strains can be deduced if required.

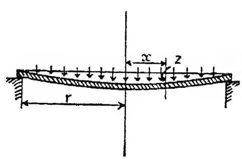


FIG. 154.

The case of a plate supporting a uniform load of w units per unit of area is shown in Fig. 154, and putting $P = 0$ in equation (12),

$$z = -\frac{6(m^2-1)wx^4}{32m^2Et^3} + \frac{Bx^2}{4} + D,$$

and if $z = 0$ when $x = r$,

$$D = +\frac{6(m^2-1)wr^4}{32m^2Et^3} - \frac{Br^2}{4}$$

and $$z=\left\{\frac{6(m^2-1)(x^2+r^2)w}{8m^2Et^3}-B\right\}\left\{\frac{r^2-x^2}{4}\right\} \quad . \quad . \quad . \quad (13)$$

If the plate is freely supported on a circular hole of radius r, then when $x=r$, $f_1=0$ for all values of y, and equation (7) becomes

$$m\left(\frac{d^2z}{dx^2}\right)_r+\frac{1}{r}\left(\frac{dz}{dx}\right)_r=0.$$

On substituting the values of $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$ obtained from equation (13) and simplifying

$$B=\frac{3(m-1)(3m+1)wr^2}{2m^2Et^3}$$

and inserting this in equation (13)

$$z=\frac{3(m^2-1)w}{16m^2Et^3}\left\{\frac{(5m+1)r^2}{m+1}-x^2\right\}(r^2-x^2) \quad . \quad . \quad (14)$$

The maximum value of z is obtained by putting $x=0$, and therefore

$$z_0=\frac{3(m-1)(5m+1)wr^4}{16m^2Et^3}$$

Grashof recommended 3 as a value for m, and inserting this value

$$z_0=\frac{2wr^4}{3Et^2}$$

In order to find the value of the principal stresses differentiate equation (14) twice and combine with (7) and (8) respectively, then

$$f_1=\frac{3wy}{4mt^3}(3m+1)(r^2-x^2) \quad . \quad (15)$$

$$f_2=\frac{3wy}{4mt^3}\{(3m+1)r^2-(m+3)x^2\} \quad (16)$$

where f_1 and f_2 are the radial and circumferential stresses respectively. These are a maximum when $x=0$ and $y=\pm\frac{t}{2}$, that is, at the centre of the plate. Hence

$$p=f_1=f_2=\pm\frac{3wr^2(3m+1)}{8mt^2}$$

and if $m=3$

$$p=\pm\frac{5wr^2}{4t^2}.$$

Curves showing the relation between f_1, f_2, and r for a freely supported plate 36 in. diameter and 1 in. thick are shown in

Fig. 155. The load is taken as 56 lb. per sq. in. and m as 3·5.

Inserting the values of f_1 and f_2 from equations (15) and (16) in equations (5) and (6) of Art. 109,

$$e_1 = \frac{3(m^2-1)wy}{4m^2Et^3}\left\{\frac{(3m+1)r^2}{m+1} - 3x^2\right\}$$

and

$$e_2 = \frac{3(m^2-1)wy}{4m^2Et^3}\left\{\frac{(3m+1)r^2}{m+1} - x^2\right\},$$

where e_1 and e_2 are the radial and circumferential strains respec-

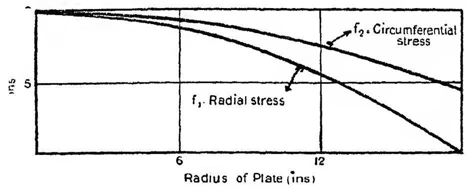


FIG. 155.

tively. These are a maximum when $x = 0$ and $y = \pm\frac{t}{2}$. Hence

$$e_1 = e_2 = \pm\frac{3(m-1)(3m+1)wr^2}{8m^2Et^2},$$

and when $m = 3$

$$e = \pm\frac{5wr^2}{6Et^2}$$

It may be noted that the value of Ee is less than p.

The magnitude of the vertical shear is obtained by putting P $= 0$ in equation (11). In flat plates of ordinary dimensions where $\frac{t}{r}$ is small, the principal stresses arising from the shear stress are smaller than the direct stresses due to bending. Also the bending stresses are a maximum on the surfaces of the plate where the shear stress is zero. The maximum stresses in the plate are therefore the bending stresses f_1 and f_2. The shear stress f_s induced by f_1 and f_2 is given by $f_s = \frac{f_1 - f_2}{2}$ where f_1 and f_2 are as defined in (15) and (16). It may be noted that this is zero in the centre of the plate.

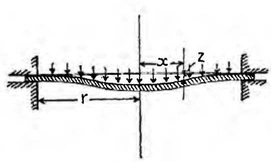


FIG. 156.

110. Circular Plate Uniformly Loaded and Fixed at its Circumference—This case is shown in Fig. 156, and it will be assumed that there is no direct force induced in the plane of the plate due to the fixing. When $x = r$, $\frac{dz}{dx} = 0$, and putting P $= 0$

in equation (12) of Art. 109, differentiating and equating to zero,

$$B = \frac{3(m^2-1)wr^2}{2m^2Eh^3}$$

and inserting this in equation (13) of Art. 109,

$$z = \frac{3(m^2-1)w(r^2-x^2)^2}{16m^2Et^3} \quad . \tag{1}$$

and the greatest deflection which occurs at the centre of the plate is

$$z_0 = \frac{3(m^2-1)wr^4}{16m^2Et^3},$$

and if $m=3$,

$$z_0 = \frac{wr^4}{6Et^3}$$

which is $\frac{1}{4}$ of that for the same plate simply supported.

Differentiating equation (1) twice and inserting in (7) and (8) of Art. 109,

$$f_1 = \frac{3wy}{4mt^3}\{(m+1)r^2-(3m+1)x^2\} \tag{2}$$

$$f_2 = \frac{3wy}{4mt^3}\{(m+1)r^2-(m+3)x^2\} \tag{3}$$

and from equations (5) and (6) of Art. 109,

$$e_1 = \frac{3(m^2-1)w(r^2-3x^2)y}{4m^2Et^3} \quad . \quad . \tag{4}$$

$$e_2 = \frac{3(m^2-1)w(r^2-x^2)y}{4m^2Et^3} \tag{5}$$

where $f_1 f_2$ and $e_1 e_2$ are the radial and circumferential stresses and strains respectively. At the centre of the plate, and putting $y = \pm\frac{t}{2}$,

$$f = f_1 = f_2 = \pm\frac{3(m+1)r^2w}{8mt^2}$$

This is not the maximum stress in the plate. The maximum stress is the radial stress at the circumference, and putting $x=r$ in equation (2) and $y=\pm\frac{t}{2}$, the maximum stress is

$$f_1 = \mp\frac{3r^2w}{4t^2}$$

At the centre of the plate, and putting $y=\pm\frac{t}{2}$,

$$e = e_1 = e_2 = \pm \frac{3(m^2 - 1)r^2w}{8m^2Et^2}$$

The maximum strain is the radial strain at $x = r$, and putting $x = r$ in equation (4) and $y = \pm \frac{t}{2}$, this is

$$e_1 = \mp \frac{3(m^2 - 1)r^2w}{4m^2Et^2}$$

becomes

$$e_1 = \mp \frac{2r^2w}{3t^2}.$$

57 is shown the variation of f_1 and f_2 with r for a plate

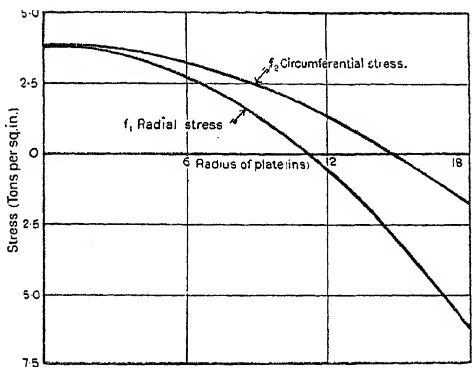


FIG. 157.

and 36 in. diameter. The plate is fixed at its circum- l carries a uniformly distributed load of 56 lb. per l m is taken as 3·5.

sults, such as those for a circular plate supporting a ributed over a concentric circular area of radius r_0, s than r, can be obtained from the general equations t. 109. If the plate is freely supported at its circumference, the maximum stress occurs in the centre of the plate, where $x = 0$ and $y = \pm \frac{t}{2}$ and

$$f = f_1 = f_2 = \pm \frac{3(m + 1)P}{2\pi m t^2}\left\{\frac{m}{m + 1} + \log_e \frac{r}{r_0} - \frac{(m - 1)r_0^2}{4(m + 1)r^2}\right\}$$

If $r_0 = r$ this reduces to the result for a distributed load over the whole plate. When $m = 3$

$$f = \frac{P}{\pi t^2}\left(\frac{3}{2} + 2 \log_e \frac{r}{r_0} - \frac{r_0^2}{4r^2}\right)$$

If the plate is clamped at its circumference, then when $x=0$ and $y=\pm\frac{t}{2}$

$$f=f_1=f_2=\pm\frac{3(m+1)P}{2\pi m t^2}\left\{\log_e\frac{r}{r_0}+\frac{r_0^2}{4r^2}\right\}$$

If $m=3$ this becomes

$$f=\pm\frac{P}{\pi t^2}\left(2\log_e\frac{r}{r_0}+\frac{r_0^2}{2r^2}\right)$$

Also when $x=r$

$$f_1=\mp\frac{3P}{2\pi t^2}\left(1-\frac{r_0^2}{2r^2}\right).$$

From a consideration of these expressions it is found that if $r>1{\cdot}7\,r_0$ the stress f in the centre of the plate is the maximum. If $r<1{\cdot}7\,r_0$, the maximum stress is the radial stress at the circumference.

111. Rectangular and Oval Plates Uniformly Loaded and Freely Supported at the Edges—For these cases simple approximate methods and results for the maximum stress will be given. For the more rigorous treatment the reader is referred to Prof. Love's treatise on Elasticity.

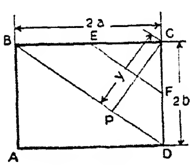


FIG. 158.

In Fig. 158 is shown a rectangular plate of length and breadth $2a$ and $2b$ respectively. If a is great compared with b the plate sensibly becomes a very broad beam supported at its sides, and the stress is a maximum along the line through F parallel to the long sides where F is the mid-point of CD. Similarly, if b is great compared with a, the maximum stress will occur along the line through E parallel to CD, where E is the mid-point of BC. Thus it seems reasonable to assume that for a plate where the ratio $\frac{a}{b}$ is not excessive the maximum stress will occur along a diagonal.

Suppose the triangular half BAD of the plate is removed and consider the equilibrium of the other half. The resultant of the supporting forces on the edge BC will act through E and similarly the resultant of those on CD will act through F. Hence the resultant of the supporting forces on the two edges will act through the mid-point of EF, that is, at a distance $\frac{y}{2}$ from BD. Also the resultant of the load on the triangular half will act through the centre of gravity of the triangle, that is, at $\frac{1}{3}y$ from

BD. There is thus an unbalanced couple M about the line BD of amount $M = 2wab\left(\frac{y}{6}\right)$ where w is the load per unit of area. But $\frac{y}{2b} = \frac{2a}{BD}$ or $y = \frac{2ab}{\sqrt{a^2+b^2}}$. Hence $M = \frac{2wa^2b^2}{3\sqrt{a^2+b^2}}$. Also if f be the average stress on the surfaces of the plate across the line BD, then

$$f = \frac{6M}{BDt^2} - \frac{3M}{t^2\sqrt{a^2+b^2}}$$

or

$$f = \frac{2a^2b^2w}{(a^2+b^2)t^2}$$

If $a = b$, so that the plate is square,

$$f = a^2w$$

It may be noted that the maximum stress which occurs in the centre of the plate is, by Grashof's theory, $f = \frac{1 \cdot 16\, a^2w}{t^2}$, where Poisson's Ratio is taken as $\frac{1}{3 \cdot 5}$ and $2a$ is the length of the sides of the square.

In the case of elliptical and oval plates the maximum average stress may be assumed to occur along the major axis. Considering half the oval with one side bounded by the major axis, the positions of the resultants of the force round the edge and of the weight on the plate can be found graphically. Let c be the distance between these two resultants. Then if M be the bending moment about the major axis $M = \frac{Wc}{2}$ where W is the total load on the plate. Also $f = \frac{6M}{2at^2}$ where $2a$ is the length of the major axis and substituting for M, then $f = \frac{3Wc}{2at^2}$, where f is the average stress along the major axis. In the case of a circular plate of radius r,

$$c = \frac{2r}{\pi} - \frac{4r}{3\pi} = \frac{2r}{3\pi}$$

and

$$f = \frac{W}{\pi t^2} = \frac{r^2w}{t^2}$$

where

$$w = \frac{W}{\pi r^2}.$$

The maximum stress for this case, from Grashof's treatment,

occurs in the centre of the plate and is $f=\frac{1\cdot 23\, r^2 w}{t^2}$, where Poisson's Ratio is taken as $\frac{1}{3\cdot 5}$ and r is the radius of the plate.

112. Rectangular Plate Fixed at the Edges and Uniformly Loaded—This case cannot be solved by the preceding approximate method, and another approximate method will be applied. Let the plate be of length l and breadth b and assume that it consists of strips of unit width and length l placed on the top of strips of unit width and length b, the whole forming a plate of length l and breadth b. The central deflection of the central long strip is $\frac{w_1 l^4}{384\mathrm{EI}}$ and of the central short strip it is $\frac{w_2 l^4}{384\mathrm{EI}}$ where w_1 and w_2 are the loads per unit area carried by the two strips. If the strips are of equal thickness then $\frac{w_2}{w_1}=\frac{l^4}{b^4}$. Also $w_1+w_2=w$, the total load per unit area on the plate. Hence $w_1=\frac{wb^4}{l^4+b^4}$. The greatest bending moment M_1 on the long strip occurs at the end and is $\frac{w_1 l^2}{12}$, so that

$$\mathrm{M}_1=\frac{wb^4l^2}{12(l^4+b^4)}$$

and by symmetry

$$\mathrm{M}_2=\frac{wl^4b^2}{12(l^4+b^4)}$$

Thus the greatest bending moment occurs at the middle of the long sides. Let f_2 be the stress induced by M_2, then

$$f_2=\frac{6\mathrm{M}_2}{bt^2}=\frac{wl^4b^2}{2(l^4+b^4)t^2}$$

In the case of a square of side $2a=l=b$ this reduces to $f=\frac{wa^2}{t^2}$, and the more vigorous treatment of Grashof gives $f=\frac{1\cdot 24wa^2}{t^2}$, where Poisson's Ratio is taken as $\frac{1}{3\cdot 5}$.

113. Rankine's Theory of Earth Pressures—Consider the case where a vertical wall of height h supports a mass of earth having a horizontal surface of indefinite extent, the surface being level with the top of the wall. The vertical pressure of the earth at a depth h is wh where w is the weight of a unit volume of the earth. Let p be the horizontal pressure at the depth h. Then,

considering a point P′ in the earth at a depth h, there will be wholly normal pressures on horizontal and vertical planes of amounts wh and p respectively. These planes will be normal to the vertical plane containing the pressures and the pressure on any other plane through P′ normal to the plane of the pressures will be oblique.

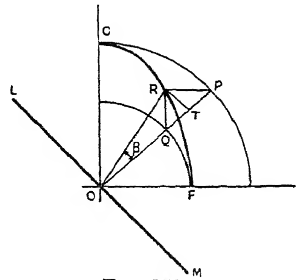


FIG. 159.

Referring to Fig. 159, let OC and OF represent wh and p respectively.

Then, as shown in Art. 19, the resultant pressure on any plane LM perpendicular to OP is represented in magnitude and direction by OR, where R lies on an ellipse of major and minor semi-axes OC and OF respectively. The angle POR $=\beta$ is a maximum when ORT is a right angle and

$$\sin\beta=\frac{\mathrm{TR}}{\mathrm{OT}}=\frac{wh-p}{wh+p}$$

In the case of earth β cannot exceed ϕ where ϕ is the angle of friction, and therefore

$$\sin\phi=\frac{wh-p}{wh+p}$$

whence
$$p=wh\left(\frac{1-\sin\phi}{1+\sin\phi}\right).$$

The total force on a unit length of the wall is $\frac{ph}{2}$. The resultant of the total pressure acts at a depth $=\frac{2}{3}h$.

The above analysis suggests a simple graphical method for finding p, which will also apply when the upper surface of the earth is inclined at an angle θ with the horizontal. In Fig. 160 let OB $= wh$ and the angle BOR $=\phi$, the angle of friction. Also the angle BOQ $=\theta$, the inclination of the surface of the earth.

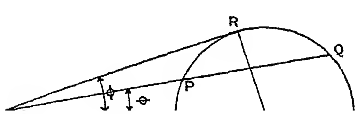


FIG. 160.

Then, if p' is the horizontal pressure at a depth h corresponding with the inclination θ,

$$p'=\frac{\mathrm{OQ}}{\mathrm{OP}}\,wh\cos^2\theta.$$

When $\theta = 0$

$$p' = \frac{\mathrm{OA}wh}{\mathrm{OB}} = \mathrm{OA} = p = wh\left(\frac{1 - \sin\theta}{1 + \sin\theta}\right).$$

When $\theta = \phi$

$$p' = wh\cos^2\phi.$$

The total pressure on a unit length of the wall is $\frac{p'h}{2}$, which acts at a depth $= \frac{2}{3}h$ from the top of the wall.

Let the surface of the earth be horizontal and suppose that a horizontal pressure p_1 greater than p is applied at P′. When p_1 is a maximum consistent with stability the angle β in Fig. 159 will be on the opposite side of OP and OF $= p_1$ will become the semi-major axis of the pressure ellipse. Hence, reasoning as before, $p_1 = wh\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)$. Also, by a similar reasoning, the greatest vertical pressure consistent with this horizontal pressure is $p_2 = p_1 \frac{1 + \sin\phi}{1 - \sin\phi} = wh\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^2$.

Thus, this is the greatest vertical pressure consistent with stability which can be applied to the earth at a depth h. It will be noted that p_2 is proportional to h, so that the bases of the foundations of heavy buildings rest at an appreciable depth below the surface of the earth.

The theory applies fairly well in the case of sand and similar material, but in most practical cases it cannot be expected to give reasonably accurate results because of the lack of homogeneity of the earth, caused, for instance, by the presence of rocks, moisture, etc. Experiments on the horizontal pressure of sand have been conducted by P. M. Crosthwaite,* who includes an account of other theories of earth pressures in his paper.

114. Stability of Dams—A dam can be defined as a type of wall used to close up the opening at the end of a natural valley which is to be used as a reservoir. Since dams are not, as a rule, designed to withstand tension, it is important that the line of resistance, both when the reservoir is empty and full, shall lie within the middle third of every horizontal cross-section. The most efficient type of dam, from the point of view of stability, is that with a triangular cross-section. Referring to Fig. 161, let ABC be the triangular cross-section of the dam and assume first that the reservoir is empty. If D be the mid-point of the base BC, the centre of gravity G of the dam lies on AD at a

* "Min. Proc. Inst. C.E.," vol. ccix, 1920.

distance $\frac{h}{3}$ from the base. Hence the resultant of the weight acts along the line GE where BE $=\frac{b}{3}$. Similarly, the resultant of the weight above any other horizontal section will act through a point distant one-third the breadth of the section from AB. Thus the line of resistance when the reservoir is empty is the straight line AE where BE$=\frac{b}{3}$.

Now let the reservoir be full. The total horizontal pressure on a unit length of the dam is w_1h where w_1 is the weight of a unit volume of water. Let R be the resultant of the weight of the dam and the pressure on the vertical face. Then for no stress at B the line of action of R must cut the base at F where CF$=\frac{b}{3}$.

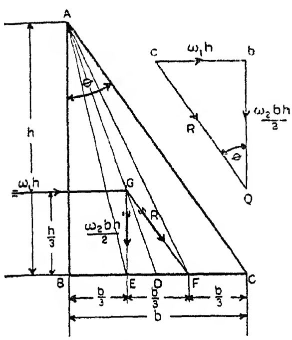


FIG. 161.

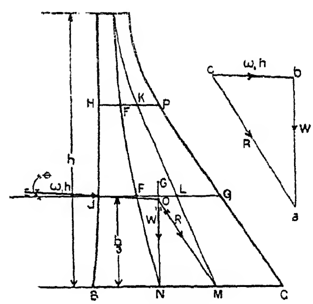


FIG. 162.

Taking moments about C

$$\frac{Rb\cos\theta}{3}=\frac{w_2bh^2}{3}-\frac{w_1h^2}{3}.$$

Also if abc be the triangle of forces for the forces acting at G, then

$$\frac{w_2bh}{2h}=\frac{R\cos\theta}{h} \text{ or } R=\frac{w_2bh}{2\cos\theta}$$

Hence $w_2b^2h=2w_2bh^2-2w_1h^2$ and

$$^2-2bh+\frac{2w_1h}{w_2}=0,$$

from which the value of b can be obtained if the other quantities are known.

Actual dams are designed so that the lines of resistance lie definitely within the middle third. A typical cross-section of a masonry dam is shown in Fig. 162, and it will be noted that

the section is substantially based on the triangular form. Consider the base BC. The centre of gravity of a unit length of the dam is at G and the line of resistance when the reservoir is empty passes through N, where N is vertically below G.

When the reservoir is full there will be forces per unit length of the dam of amounts w_1h and W respectively, which intersect at O. Denoting the resultant of these forces by R, the point M, where the line of action of R cuts the base, is a point on the line of resistance. The magnitude and direction of R is found by the triangle of forces indicated by *abc* in the figure. The points E, F, L, and K are found in a similar manner by considering the portions of the dam above the horizontal sections JQ and HP. Thus the lines of resistance when the reservoir is empty and full are denoted by SEFN and SKLM respectively.

The stresses on any horizontal section such as BC are (1) a normal stress due to the dead weight of the dam, (2) normal stresses induced by bending about the mid-point of BC, and (3) shear stress caused by the horizontal component of the resultant water pressure on AB. Considering a unit length of the dam the normal compressive stress due to its weight is $\frac{W}{BC}$. The bending moment both when the reservoir is empty and full is obtained by taking moments about the mid-point of BC. The direct stresses induced by bending can then be obtained and the actual normal stress at any point is the algebraic sum of the bending stress at the point and the stress due to the dead weight of the dam. (Art. 66).

It may be noted that the normal stress on BC is more correctly $\frac{W}{BC}+\frac{w_1h\sin\theta}{BC}$, but since $\sin\theta$ is usually very small, the stress is sensibly $\frac{W}{BC}$.

The shearing force across the plane BC is $w_1h\cos\theta$. Also the total force normal to the section is $W+w_1h\sin\theta$, and if μ is the coefficient of friction, then for equilibrium $w_1h\cos\theta$ should not exceed $\mu(W+w_1h\sin\theta)$. Since θ is usually small, the term containing $\sin\theta$ can be neglected.

EXAMPLES. X

(1) A load of 560 lb. falls through a height of 1 in. on to a collar at the lower end of a vertical tie bar 10 ft. long and $1\frac{1}{8}$ in. in diameter. If Young's Modulus is 13,000 tons per sq. in., find (*a*) the maximum instantaneous stress per sq. in. and (*b*) the instantaneous stretch of the bar. (S. & A.)

(2) A ship of 4,000 tons is towing by means of a steel hawser another vessel of 1,000 tons. The hawser is 350 ft. long and its cross-sectional

area is 5·52 sq. in. If at an instant when the towing vessel has a speed of $9\frac{1}{2}$ knots the hawser, which has been slack, becomes again taut, and if at the instant before this occurred the speed of the towed vessel was 7 knots, find the common speed as soon as the hawser is taut and the total amount of kinetic energy in foot tons lost. If this lost kinetic energy is absorbed in stressing the hawser, calculate the instantaneous stress set up in the hawser. Young's Modulus equals 30×10^6 lb. per sq. in.; 1 knot equals 6,080 ft. (S. & A.)

(3) A carriage spring 36 in. long is made of steel plates $2\frac{3}{4}$ in. wide by $\frac{1}{4}$ in. thick. Determine the number of plates which will be required to carry a central load of 10,000 lb. if the maximum stress is not to exceed 10 tons per sq. in. Find also the deflection under the load, taking Young's Modulus as 30×10^6 lb. per sq. in.

(4) A tie rod 1 in. diameter is fixed at its ends by means of nuts. The diameter at the bottom of the thread is $\frac{3}{4}$ in. The distance between the supports is 20 ft. and the screw thread at each end extends over 6 in. of this length. Compare the strength of this rod with the strength of one in which the diameter over the 19 ft. between the ends of the screw threads has been reduced to $\frac{3}{4}$ in., (*a*) when the rods support dead loads; (*b*) when they support live loads and are subjected to impact. (Victoria.)

(5) Concrete exerts on earth at the bottom of a trench a downward pressure of 2 tons per sq. ft., the earth weighs 130 lb. per cubic ft., and its angle of repose (in Rankine's theory) is 30 deg. What is the least safe depth below the earth's natural surface of the bottom of the concrete? Why is it not possible to make much practical use of the theory of earth pressure?

(6) A retaining wall trapezoidal in cross-section 30 ft. high and 4 ft. thick at the top is subjected to earth pressure on its vertical face. The wall is not surcharged. The concrete of which the wall is built weighs 140 lb. per cubic ft., the earth weighs 120 lb. per cubic ft., and the angle of repose of the earth is 30 deg. Find (*a*) the magnitude of the thrust exerted by the earth on the back of the wall per ft. length of wall, (*b*) the necessary thickness of the wall at its base.

(7) A circular steel plate 3 ft. 6 in. diameter and $1\frac{1}{2}$ in. thick is subjected to a normal pressure of 50 lb. per sq. in. Find the maximum induced stress and the central deflection (1) if the plate is freely supported; (2) if it is fixed at the edges. Young's Modulus equals 30×10^6 lb. per sq. in. and Poisson's Ratio $= \frac{1}{3}$.

(8) A tempered steel ring 3 in. mean diameter is formed from a round bar 1 in. diameter. Find the maximum stress induced in the ring by a diametrical tension of 5 tons.

(9) A clock spring is made from steel $\frac{1}{2}$ in. wide and $\frac{1}{50}$ in. thick and is 10 ft. long. If the maximum stress is not to exceed 50 tons per sq. in., find the maximum twisting moment required to wind it up. How many complete turns are required to wind it up, and what is then the strain energy stored in the spring? If the maximum radius of the spring is 1 in., find the greatest force between the arbour and its bearings.

(10) A bar of steel 2 in. square is bent in a plane parallel to a pair of sides to a mean radius of 3 in. If it is subjected to a bending

moment of 5,000 lb. in., tending to increase the curvature, find the maximum induced stresses on the inside and outside of the bend, (1) neglecting the initial curvature, (2) taking account of it.

(11) A steel ring 6 in. mean diameter and of circular section $1\frac{1}{2}$ in. diameter is subjected to a diametrical compression of 2 tons. Find the decrease in diameter in the direction of the load and the increase in the direction perpendicular to it. Take Young's Modulus equals 30×10^6 lb. per sq. in.

(12) A steel tyre 4 ft. mean diameter 4 in. wide and $1\frac{1}{2}$ in. thick is subjected to a diametrical compression. Find the greatest decrease in diameter corresponding with an induced maximum stress of 10 tons per sq. in. Young's Modulus equals 30×10^6 lb. per sq. in.

ANSWERS TO EXAMPLES

EXAMPLES. I

(1) 2·23 tons per sq. in., $24{\cdot}0\times 10^6$ lb. per sq. in.
(3) 9,950 lb. per sq. in.
(4) Copper 22·6 lb., steel 38·7 lb. each.
(5) 1·25 and 1·76 tons per sq. in. respectively.
(6) 0·0169 cu. in.
(7) 23·987 in.
(8) 5·31 tons per sq. in.
(9) 1 ton per sq. in. inclined at 45 deg. to longitudinal direction.
(10) 8,000 lb. per sq. in. tensile, 12,000 lb. per sq. in. compressive, 260 deg. F.

EXAMPLES. II

(1) 18·2 tons per sq. in., 28·7 tons per sq. in., 19·5 per cent., 5·0 in. tons.
(2) 19·6 tons per sq. in., 28·8 tons per sq. in., 26·25 per cent.
(3) 36 per cent. on 3 in., 35·2 per cent. on 1·72 in.
(8) 12,800 tons per sq. in.

EXAMPLES. III

(1) Arc of circle 53·2 ft. radius, 0·451 in., 11,730 lb. per sq. in.
(2) 1,580 lb. per sq. in.
(3) 3·43 tons per sq. in.
(4) $\sqrt{2}$ to 1.
(5) 15·6 tons, 2·37 tons per sq. in.
(6) 28·3 cwt.
(7) 135 lb. per sq. in.
(8) 500 lb. ft.
(9) $\frac{5}{4}\pi a^4$, $\frac{3}{2}\pi a^4$, $\frac{bd}{12}(b^2+d^2)$.
(10) 4·45 ft. from each end.
(11) $y \propto \sqrt{Lx-x^2}$, $y \propto (Lx-x^2)$.
(12) 7·5 tons per sq. in., 1·4 tons per sq. in., 8·53 tons per sq. in. at 5·33 ft. from one end.
(13) 1·19.
(15) 2·23 tons per sq. in., 1·96 tons per sq. in. at 11·75 ft. from end farthest from load.

EXAMPLES. IV

(1) 184·3 (in.)^4 units.
(2) 2·55 tons per sq. in.
(3) 16 sq. in., 2·4 in.
(4) 1·4 in., 15 sq. in.
(5) 1,610 lb. per ft. run.
(6) 1,880 lb. per ft. run.
(7) $bd^2 = 7{,}100$, $r = 0{\cdot}675$ per cent.
(8) 25,100 lb. per sq. in., 532,000 lb. in. or 500,000 lb. in.
(9) 7,850 lb., 16,670 lb. per sq. in.
(10) 4 sq. in.
(11) 480 lb., 60 lb., 333·3 lb. per sq. in., 166·7 lb. per sq. in.

EXAMPLES. V

(1) $\sqrt{3}$ to $\sqrt{2}$.
(2) Bending Moments and Reactions at end nearest load, centre and other end : $-\frac{5}{32}Wl$, $-\frac{1}{16}Wl$, $\frac{1}{32}Wl$ and $\frac{19}{32}W$, $\frac{1}{2}W$, $-\frac{3}{32}W$ respectively.
(3) 343 lb. ft. (8 ft. from body), 850 lb., 1,280 lb.
(4) (*a*) 9,600 lb. ; (*b*) 25,600 lb. ft., 16,000 lb. ; (*c*) 2 ft. from wall.
(5) 16·2 ft.
(6) 4·95 tons, 42·4 tons, —4·41 tons.
(7) 0·248 in. or 0·174 in. approx.
(10) $\frac{7wl^4}{1{,}052EI}$.

EXAMPLES. VI

(1) 4·25 ft.
(2) 700 and 280 lb. per sq. in. compressive and tensile respectively, —11·3 deg. and 78·7 deg. respectively with longitudinal direction.
(3) 0·154.
(4) 245 lb. per sq. in.
(6) 8·25 tons per sq. in.

EXAMPLES. VII

(1) 1·55 in., 0·85 in.
(2) 4,740 lb. per sq. in. compressive in steel, 2,910 lb. per sq. in. tensile, 1,650 lb. per sq. in. compressive.
(3) 2·875 in., 5·4.
(4) 5·5 tons per sq. in.
(5) 8·72 in. × 2·18 in.
(6) 1·18 in., 1·08 in.
(7) Yes. Rankine : $\frac{}{32{,}500}$, Gordon : $\frac{1}{2{,}710}$.
(8) 16·2 tons.
(9) 2·12 in.
(10) 12·3 tons.
(11) (1) Concrete, 150 lb. per sq. in. ; steel, 2,250 lb. per sq. in. (2) Concrete, 223 and 77 lb. per sq. in. ; steel, 2,600 and 900 lb. per sq. in. (3) Concrete, 291 and 9 lb. per sq. in. ; steel,

3,400 and 105 lb. per sq. in. (4) Concrete, 632 lb. per sq. in. compressive; steel, 6,900 lb. per sq. in. compressive and 5,360 lb. per sq. in. tensile.

EXAMPLES. VIII

(1) $28 \cdot 9 \times 10^6$ lb. per sq. in., $11 \cdot 0 \times 10^6$ lb. per sq. in., $25 \cdot 9 \times 10^6$ lb. per sq. in.
(2) 3·54 in., 4·75 in.
(3) 2,100 per sec., 4,200 per sec.
(4) 0·675 per sec.
(5) 13·3 per sec.
(6) 13,900 lb. per sq. in., 9·80 in., 0·362 radian.
(7) 7·27 in. external diameter, 5·08 in. internal diameter.
(8) 14,500 lb. per sq. in.
(9) 0·783 sec., 0·582 ft. from starting position.
(10) 3·65 in.
(11) 10·25 radians per sec., 0·0048 ft., 0·00004 ft., 0·00001 ft.
(12) (1) 313 per minute, 313 revs. per minute; (2) 296 per minute, 332 revs. per minute.

EXAMPLES. IX

(1) 124·5 ft. per sec.
(2) Radial pressure = 2,000, 930, 350, and 0 lb. per sq. in. Circumferential tension = 3,940, 2,870, 2,290, and 1,940 lb. per sq. in.
(3) $1 \cdot 68 \times 10^{-4}$, 5,040 lb. per sq. in.
(4) 3,230 lb. per sq. in.
(5) 6·50 tons per sq. in. compressive, 8·60 tons per sq. in. tensile, 8·50 tons per sq. in. compressive, 6·46 tons per sq. in. tensile, 0·00639 in.
(6) 1·57 in.
(7) (1) 5·19 tons per sq. in; (2) 11·0 tons per sq. in.; (3) 12·8 tons per sq. in.
(8) 0·438 in.
(9) On surface of shaft in plane of bending, 4·24 and 3·06 tons per sq. in. respectively, 33·7 deg. with axis of shaft and on planes inclined at 78·7 deg. and 186·7 deg. with axis of shaft.

EXAMPLES. X

(1) (*a*) 0·0705 in.; (*b*) 7·65 tons per sq. in.
(2) 9 knots, 221 ft. tons, 55·3 tons per sq. in.
(3) 14 plates, 0·97 in.
(4) 1 to 1, 0·764 to 1 (for equal resiliences).
(5) 3·83 feet.
(6) 18,000 lb., 10·1 ft.
(7) (1) 5·45 tons per sq. in., 0·064 in., 3·28 tons per sq. in., 0·016 in.
(8) 31·6 tons per sq. in.
(9) 1·87 lb. in., 3·57 turns, 20·9 in. lb., 1·87 lb.
(10) 3,750 lb. per sq. in. tensile and compressive, 4,830 and 3,030 lb. per sq. in. tensile and compressive respectively.
(11) $5 \cdot 48 \times 10^{-4}$ in., $3 \cdot 78 \times 10^{-4}$ in.
(12) 0·263 in.

PART II

CHAPTER XI

TESTING APPLIANCES

115. THE best criterion of the suitability of a material for a given purpose is undoubtedly the degree of satisfaction it gives in its use, but, as vast quantities of material have to be used without previous trial, a system must be adopted by which desirable qualities may be recognized without actual use. This is the object of "testing." When materials have exhibited certain qualities under test and have afterwards proved satisfactory under working conditions, it is assumed that other material with the same test qualities will prove equally satisfactory under similar conditions. For purposes of design it is this plan of which use is actually made. The result is that the majority of iron and steel works possess testing laboratories, so that they may be at all times fully informed as to the qualities of the material which is being turned out. Testing is necessary for both the manufacturer and the user of a material. Consumers' tests are most often made in independent testing laboratories.

116. Qualities revealed by Tests—The qualities, important from an engineering point of view, which may be revealed by the tests commonly available, are the following :—

Tensile strength, at elastic limit, yield point, and fracture.

Crushing strength, either elastic or ultimate.

Shearing strength, elastic or ultimate.

Torsional strength, elastic and ultimate.

Transverse strength, elastic and ultimate.

The modulus of elasticity, direct and in shear.

Relative hardness.

Strength to resist repeated applications of stress.

Strength to resist blows.

Power to resist the action of chemical fumes or liquids, including the rusting action of moisture.

Power to resist abrasive action.

The tests mentioned above come under the head of "Commercial tests," and, as a rule, are readily available. In addition, there

are many physical experiments carried out largely for the purpose of information on new materials or for purposes of research.

The most common test of engineering material, and the one which yields the most valuable information, is the Tensile Test. It can be easily and quickly made, and the information which it yields is very definite within its own limits. Moreover, it is the kind of test which is, as a rule, most generally available. For these reasons the machine of greatest importance in any testing laboratory is the Tensile Testing Machine. Many tensile testing machines may also be used for tests other than tensile.

117. Testing Machines—There are two main functions to be performed by a tensile testing machine. Firstly, it must be possible to grip the ends of the specimen in question in such a way that they can be pulled apart, so as to stretch the bar and probably break it; the second essential is that it may be possible to measure the load so applied with a sufficient degree of accuracy, in other words, to apply and control a pull and at the same time to measure it.

On Fig. 163 is a sketch representing the simplest form of

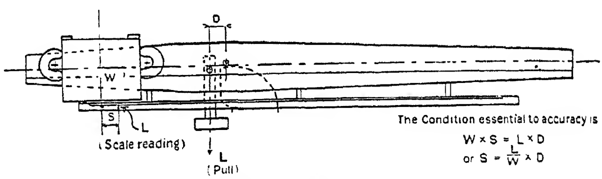
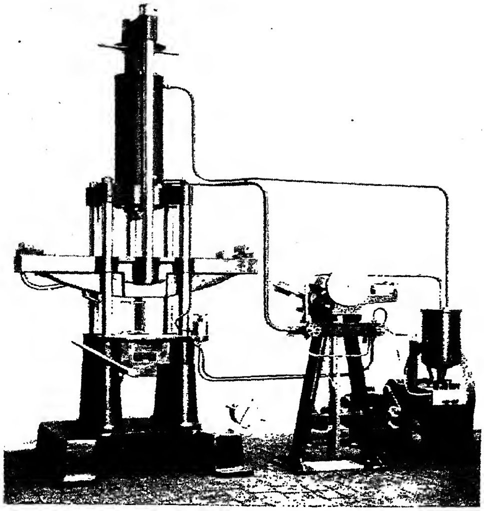


FIG. 163.—Essential Measurements, etc.

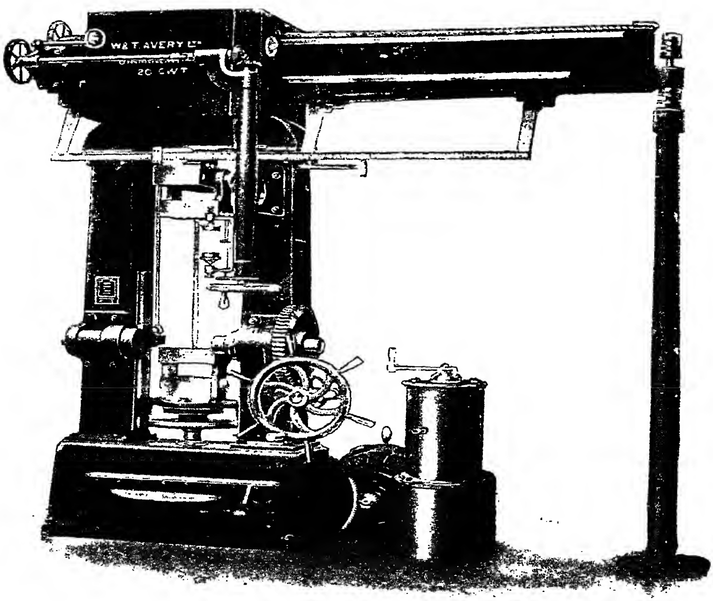
(1) The fulcrum distance, D.
(2) The weight, W, of the poise weight.
(3) The length, S, of the scale divisions and their load value, L.
(4) The load upon the knife-edges should not exceed 5 tons per in. length of bearing.
(5) The knife-edges and beds should be in good condition.

lever testing machine. This is a view of the Buckton Wicksteed Vertical testing machine to be described later. This machine is representative of a class. The lever balances on a knife-edge, which is represented by a small circle. The shackle which holds the top end of the specimen is hung from a second knife-edge placed at a distance D to the left of the first knife-edge. If the pull on the specimen be called L, this is brought about by a poised weight, W, which can be moved from one end of the beam to the other. As shown in the figure, W is so placed that its weight balances that of the right-hand side of the beam itself. The main knife-edge rests on a plate fixed on the top of the main standard of the machine. As W is moved towards the right the pull on the specimen is increased, and when it reaches its extreme right-hand position the maximum pull is attained.

PLATE III.

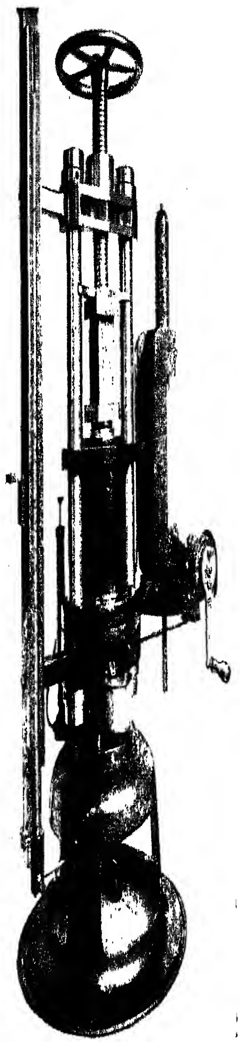


(*a*) ALFRED AMSLER TESTING MACHINE FOR TENSION, COMPRESSION AND BENDING

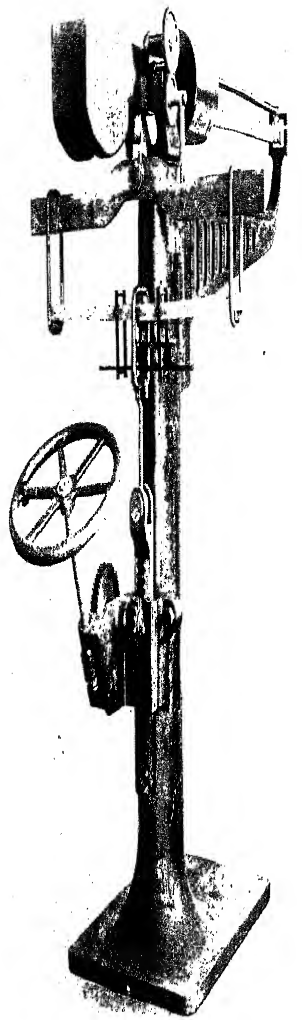


(*b*) AVERY SINGLE LEVER TESTING MACHINE

PLATE IV.



(a) AMSLER LAFFON COMPRESSION TESTING MACHINE



(b) DENISON 1,500 LB. TESTING MACHINE ARRANGED FOR TRANSVERSE TESTS OF TIMBER

Application of the Load—This is effected either by screws or by hydrostatic pressure. Generally the former plan is used in the machines of small power and the latter plan in the larger machines.

Measurement of the Load—Most frequently this is done by balancing the pull against an accurate standard weight acting through a system of levers. In the single-lever machine the specimen is vertical and the lever horizontal. This is the commonest type in Great Britain (as in the single-lever machines of Wicksteed, Denison, and Avery).

In some machines (Wicksteed and that of Greenwood & Batley) the specimen is kept horizontal by using a bell-crank lever between specimen and weighing mechanism.

In one type of machine (notably American) the load is measured by being applied to the main plate of what is nothing more than an ordinary weighing machine, with a poise weight which is very small compared with the load to be measured (Riehlé and Olsen). In most of the above lever machines the actual magnitude of the load is read off from the scale, which runs the length of the long arm, l. The short arm, S, is known, so that the pull on the specimen $L=K\frac{SW}{l}$ (W being the poise weight and K a constant.)

As W moves it pushes a vernier scale along the graduated lever, so that it is possible to read L at once without multiplication. In some few cases S and l remain constant and W is made to vary. An example of this plan is found in Bailey's cement machine, where W is increased by running water or shot into a container.

Hydrostatic Type—The most notable machines of this class are those of Messrs. Amsler of Schaffhausen. In these the application of load is effected by forcing oil beneath an easily fitting ram working in a cylinder which is supported on pillars rising from the bed plate or main casting. There is a slight leakage of oil expected and allowed for by the provision of a drain. For a given size of ram the load depends on the oil pressure. This is measured by allowing it to force from the vertical a heavy pendulum and then measuring the resulting angular displacement, as seen by the position of a pointer on a scale. See Plate III (*a*).

In contrast with this modern plan, the older Amsler-Laffon machines of nearly twenty years ago depended for the load indication on the height of a mercury column acting through a differential piston (Plate IV (*a*)). This last is given a partial rotary movement to reduce friction and the pressure is intensified to the required extent. As the oil is pumped below the ram it is pushed upwards, and with it a short crosshead. From this

crosshead hang a pair of rods supporting the shackle, in which is held the upper end of the specimen. The lower end is held in grips fixed to the bedplate. For compression the same general plan is followed. The authors have found it better, in the case of a transverse bending machine, to replace the mercury column gear by a large pressure gauge placed close to the ram cylinder, in order to obviate lag in pressure. Extension is provided for by the use of screw tension rods. When the user has become quite familiar with a machine of this type, especially if external power is used to rotate the differential piston, he will find it very sensitive.

An instance of the successful employment of an Amsler-Laffon Compression Machine is that of Mr. Carrington, when experimenting on short tubular struts of high tensile steel. In

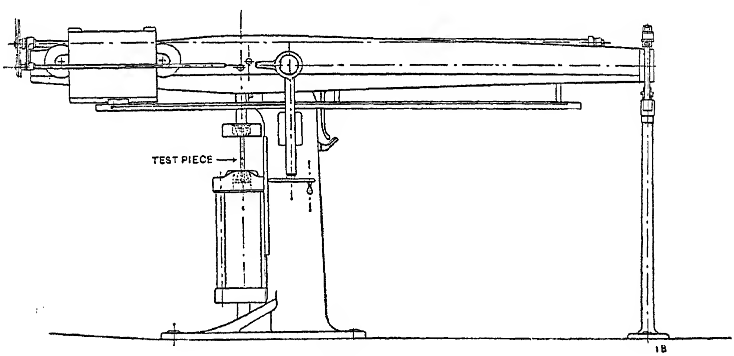


FIG. 164.

this case a small electric motor was used to give rotary movement to the differential piston.

118. Single-lever Machine of Messrs. J. Buckton & Co.— This, probably the best known of the British testing machines, is shown on Fig. 164. The specimen to be loaded has its upper end attached to the shackle which hangs from the left-hand knife-edge which forms part of the lever of the machine. There is a second knife-edge a little to the right of the first. This is fixed in the lever so as to point downwards, and rests on a plate fixed in the top of the main standard. Its edge forms the line about which the machine beam is allowed to balance when the specimen is quite unloaded. The lower end of the specimen is held by the second shackle shown. This lower shackle can be forcibly drawn downwards by a screw and nut worked from a large hand-wheel or by power (electrical or belt), or its rod may be attached to a ram working in a hydraulic press cylinder.

The jockey weight weighs 1 ton in the largest machines and can be moved outwards towards the free end of the beam by a hand-wheel attached to the vertical shaft shown. When balanced and with the autographic gear not in use, the poise rests well to the left of the knife-edges, with the vernier which it carries set at the zero of the main scale. To increase the load the poise is moved to the right by turning the hand-wheel and the beam is brought up to a horizontal balance by applying the straining gear. Then, as the poise is moved still further to the right, the straining gear must be kept constantly employed so as to counteract the stretch of the bar. During the elastic stage this use of the straining gear is small, but when the yield load has been passed, the amount of strain which needs to be taken up is very much greater. When the yield load is reached, the sudden stretch allows the beam to fall upon its lower stop and the straining gear needs a good deal of adjusting before the beam floats again. It should be the object of the operator to keep the beam at all times floating horizontally. It is not possible to do this completely, but it should be approximated to. This machine is made in various capacities from 5 to 100 tons.

119. Avery Single-lever Machine (Plate III (*b*))—In its general scheme this is a similar machine to the one by Messrs. J. Buckton & Co. just described. Its capacity is up to 30 tons. Straining is effected by an electric motor acting through a screw with a quick return motion. The speed of straining is 1 in. per minute. The steel yard is divided directly to 0·10 ton, and by vernier to 0·01 ton.

120. The Buckton Horizontal Machine (Fig. 165) will be seen to consist of a lever and poise, with a bell-crank lever added for the purpose of converting vertical pull into horizontal pull. After many years' experience with a machine of this type, the writers have formed a very high opinion of it as being a convenient and easy machine to use, for all kinds of work, especially where long specimens have to be dealt with. The length variation is made by sliding a notched block along the sliding frame of the machine and fixing it in the right position with a key.

Machines of the horizontal type are also made by Messrs. W. & T. Avery, Ltd., Messrs. Greenwood & Batley, and Messrs. Adamson of Hyde.

121. Machines for Light Testing, as in the Testing of Wire—Three of these are by Messrs. S. Denison & Son, of Leeds, and are shown on Plate V (*a*) and (*b*) and Plate IV (*b*). In Plate V, both single-lever machines, the load is increased by running the poise towards the outer end of the beam. Strain is taken up by screws and nuts or by a rack-and-spur pinion. The gripping

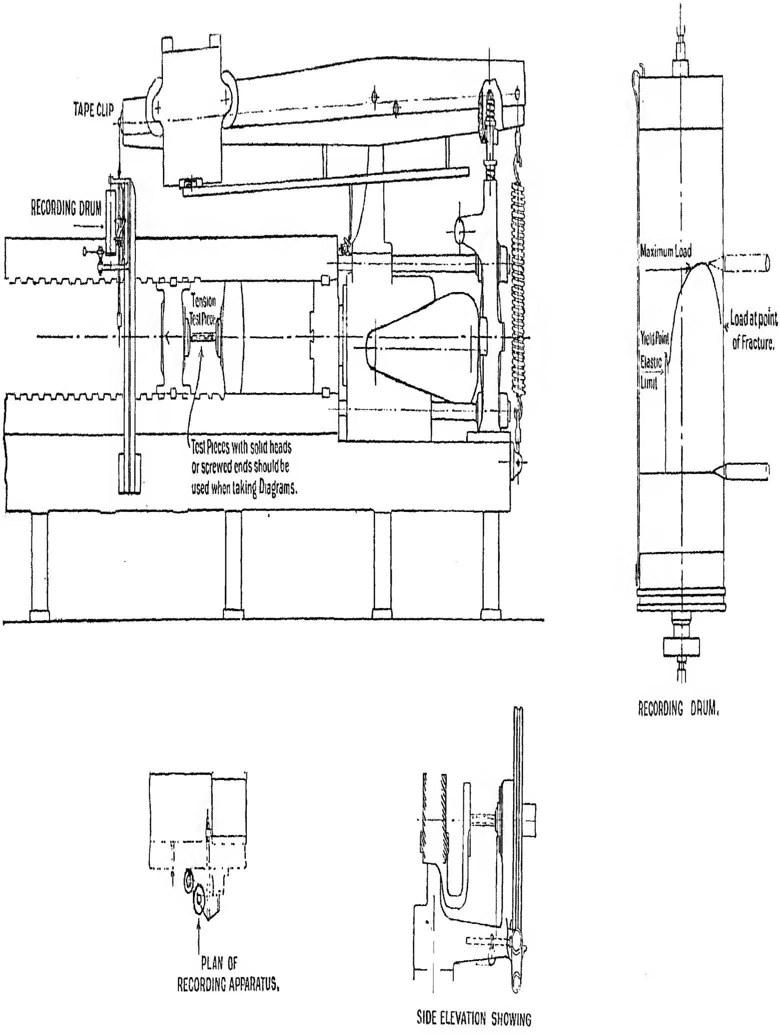
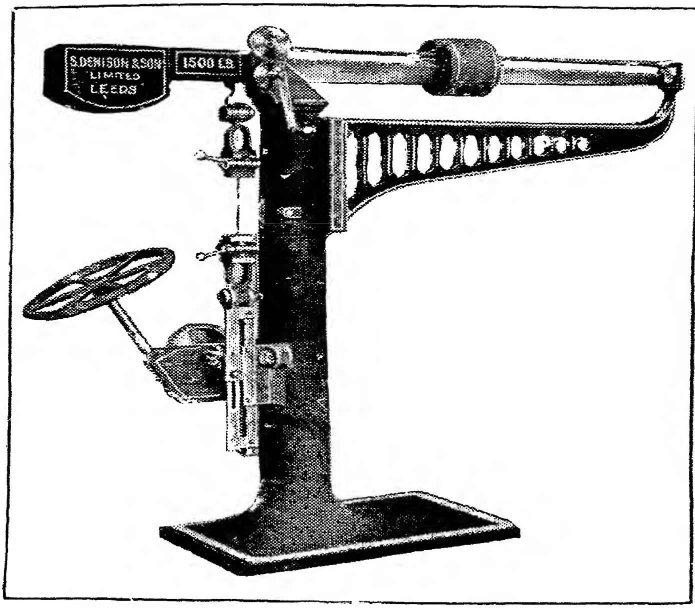
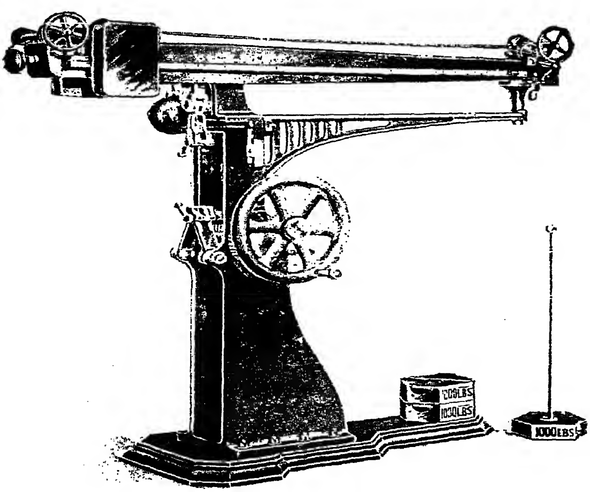

TAPE CLIP
RECORDING DRUM
Tension Test Piece
Test Pieces with solid heads or screwed ends should be used when taking Diagrams.
Maximum Load
Load at point of Fracture.
Yield Point
Elastic Limit
RECORDING DRUM.
PLAN OF RECORDING APPARATUS.
SIDE ELEVATION SHOWING TORSION APPARATUS

FIG. 165.

PLATE V.

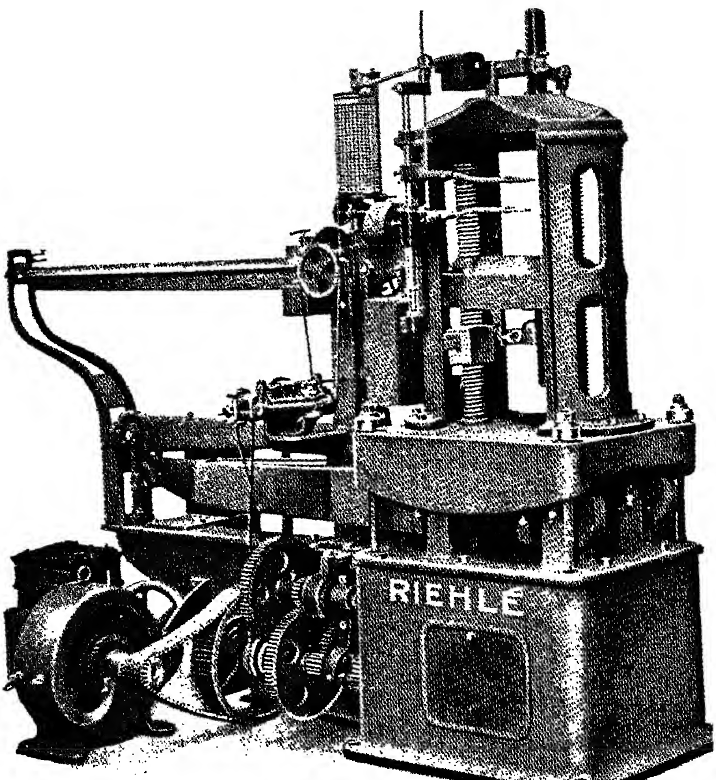


(*a*) DENISON SINGLE LEVER TENSILE TESTING MACHINE



(*b*) DENISON SINGLE LEVER TENSILE TESTING MACHINE, CAPACITY 6,000 TO 8,000 LBS.

PLATE VI.



RIEHLÉ TESTING MACHINE

wedges, which in this case are cut finely like a file, are mechanically controlled by turning a handle. Their capacities are from 800 lb. to about 3 tons. Another view of the Tension Machine is shown on Plate IV (*b*).

122. Riehlé Type (Plate VI).—These machines are much favoured in the United States of America, and of late years a few have been installed in Britain. The main features of these American machines are that the specimen is placed vertically, mechanical or electrical driving of straining gear (consisting of spur wheels acting as nuts working on tension screws) and weighing mechanism of the multiple-lever type similar to that of a weighing machine.

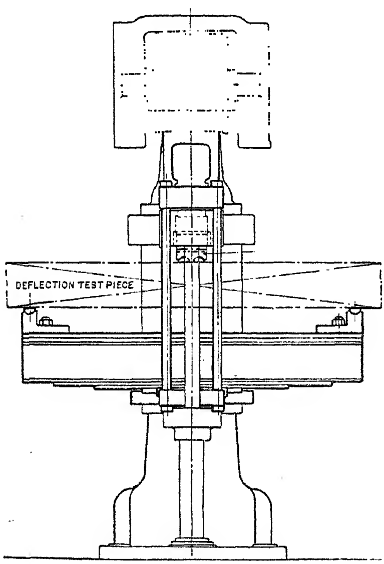


FIG. 166.

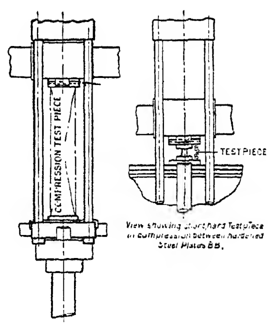


FIG. 167.

In machines of this type calibration is easy, as it is only necessary to place standard weights on the lower of the compression tables or platens.

Besides those machines which have been mentioned, there are many other types in use, such as the Diaphragm Machine of Emery, used in some of the American laboratories, and the countless Continental machines, but those already given should suffice for the present purpose.

123. Compression Testing—Most machines designed primarily for tension have attachments by means of which the

pulling of two points apart may be made to cause one hardened platen forcibly to approach a second one. Fig. 167 shows how this is done in the Buckton-Wicksteed machines and in many others. The compression platens should be of hardened steel, planed and polished and so set that the planes remain parallel.

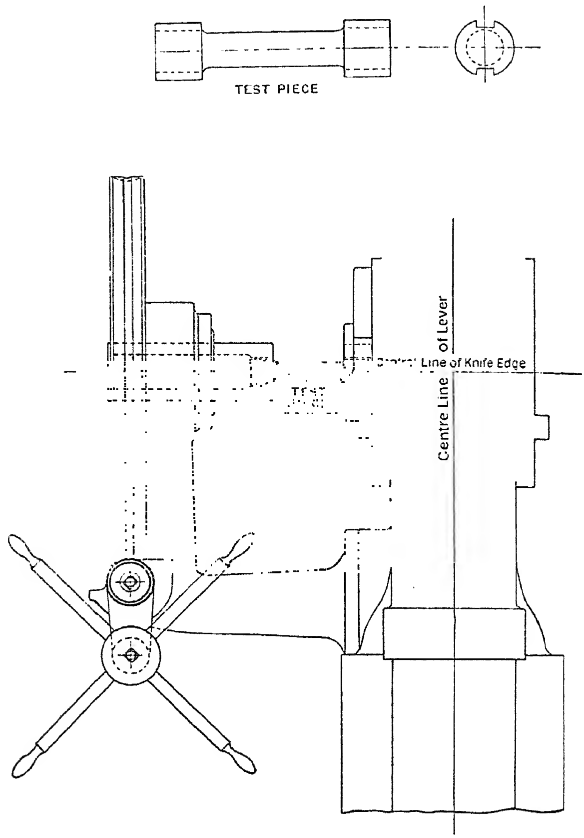


FIG. 168.

In addition to the above compression gear, an arrangement for carrying out transverse tests is also on Fig. 166. An attachment for torsion is shown on Fig. 168.

Where the straining mechanism depends on screws, the speed of strain depends on the speed with which the screws or nuts can be revolved. When hydrostatic pressure is employed, the

speed of ram movement depends on the velocity of flow of the water into the cylinder behind the ram. If the accumulator pressure is very high, it is difficult to control the inrush of water with a single valve, and, for this reason, it is convenient to have a fine needle valve placed between the main valve and the ram cylinder. This needle valve may be set nearly closed, so that, no matter how far the main valve is opened, the inflow cannot exceed a certain speed, which may be prearranged. This imposes a definite limit on the speed of straining, and the authors have found the plan of the greatest use for most kinds of work. In this case the accumulator pressure is about 1 ton per sq. in. There is no jerking and the loads reach the specimens with perfect smoothness.

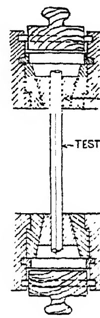


Fig. 169.

124. The Tension Gripping Dies—In nearly all types of machine the ends of the tension piece are held between wedges, as shown on Fig. 169, with roughly cut faces resting in sockets formed in the shackles. For the larger and rougher commercial test pieces the cutting of the teeth on the wedge faces is coarser than for the finer specimens, such as wires. In many cases heads are formed to rest in special holders.

125. Test Specimens—Typical flat bars with enlarged ends

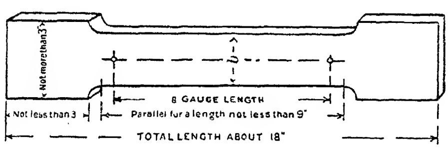


Fig. 170.—For Plates and Similar Material.

Should the strength of the machine not exceed 50 tons, the width "b" should not exceed the following

Test-bar thickness . . .	Under 0·375 in.	0·375 in.–0·875 in.	Over 0·875 in.
Maximum width of "b" .	2·5 in.	2 in.	1·5 in.

These figures are maxima; 1·5 may be used for all plates.
The above dimensions agree with the recommendations of the Engineering Standards Committee.

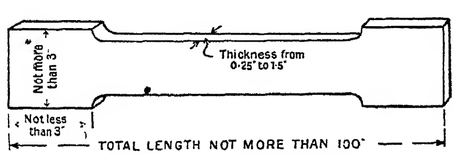


Fig. 171.—Long Flat Specimen.

are shown on Figs. 170 and 171, while a long parallel bar without enlarged ends is seen on Fig. 172. These are all intended to be

held in wedge grips. Figs. 173, 174, 175, and 176 show turned bars with enlarged ends. Three of these have screwed ends to fit specially formed dies.

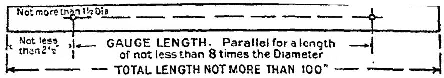


FIG. 172.—For Bars, Rods, and Stays.

Long specimens, where it is not feasible to use enlarged clamping ends

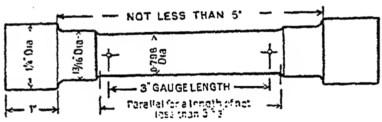


FIG. 173.—For Area = 0·5 sq. in. Enlarged Ends.

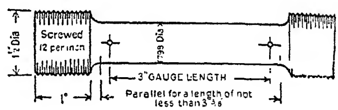


FIG. 174.—For Area = 0·5 sq. in. Screwed Ends.

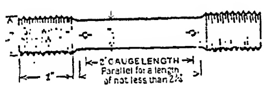


FIG. 175.—For Area = 0·25 sq. in. Screwed Ends.

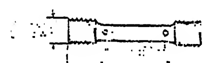


FIG. 176.—Special Test Bar, 0·309 in. diameter.

The above represent a few types of specimens as used by the authors in the Materials Testing Laboratory of the Manchester College of Technology. They are typical of the sort of commercial test bars which may be required in any test laboratory, and are suitable for a large range of materials.

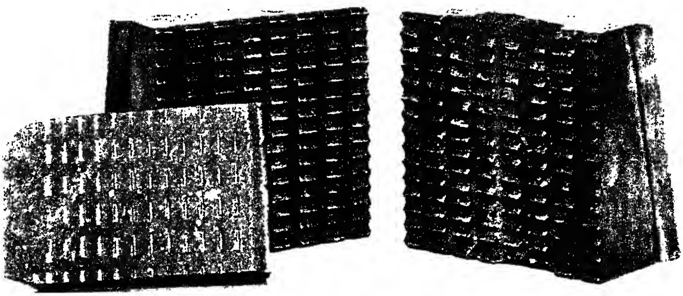
On Plate XI (*a*) are photographic views of two compression pieces, one of wrought iron (*a*) and the other of steel (*b*), showing them before and after compression.

A set of wedges used for average commercial specimens is seen on Plate VII. In using these wedge grips it is generally only necessary to press them into position against the specimen with the two quite level and then apply the straining gear. A small load should be set on the beam to give something to pull against. When the strain is applied the wedges are pulled forward on their sloping sockets, close in on the bar, and so grip it.

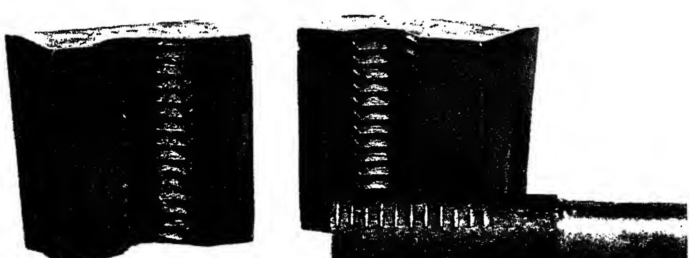
In some machines an attempt is made to provide a mechanical gear to control the movement of the wedges when coming into grip. Some of these are helpful. Others are apt to go wrong at the psychological moment and cause delay and annoyance.

126. Calibration—Precision and Accuracy—When the poise is placed so as to balance perfectly the pull on the specimen, the

PLATE VII.



(*a*) BUCKTON'S WEDGE GRIPS FOR FLAT BARS



(*b*) BUCKTON'S WEDGE GRIPS FOR ROUND BARS

magnitude of the load is indicated by the position of the poise on its lever, as shown by a pointer or a sliding vernier moving on a scale. The precision relates to the fineness of the division of the scale along which the vernier slides. For example, a certain testing machine may have its load scale divided into equal spaces, each corresponding to one-tenth of a ton. Each of these spaces may be further subdivided, by means of a vernier, into ten equal parts. All the spaces are thus divided into one hundred parts, or it is possible to read to the nearest one-hundredth of a ton, or 22·4 lb. That is to say, with such an arrangement it is possible to read a load to within the nearest 20 lb. or thereabouts.

It does not necessarily follow that such readings give true loads, that is, although it may be possible to read loads with a certain degree of precision, such are not necessarily accurate. Precision limits the fineness of a reading but does not guarantee its accuracy. Accuracy can be tested only by dead loads which are known to have true values. In a vertical machine with a horizontal lever a series of loads may be suspended from the end of the short arm of the lever in place of the specimen, and the scale indications compared with the actual test loads. In small machines this is not difficult. The specimen may be replaced by a temporary tray or pan on which the standard weights are placed.

In all vertical machines, especially the smaller ones, the friction of the knife-edges is very small and hardly appreciable when the machines are in good condition, but in the larger horizontal machines there are a greater number of knife-edges. There is also the friction which acts against the horizontal movement of the long frame, whose function it is to transmit force from the specimen to the weighing levers. This frame, in the Wicksteed Horizontal Machine, rests on "rockers" and rollers. Suppose that an accurate test load be hung from a specially placed right-angled lever in such a way as to impose a push on the frame, which has precisely the same effect on the levers as would have the pull of a specimen, W. When W is acting, the poise lever is raised, and *vice versa*. Let this lever be down and the poise moved along the scale very gradually until the test load is allowed to raise the poise. In this case the lever reading will be more than the test load, the difference being made up by the friction w. If now the poise is moved well to the other side of this last position and moved until the poise falls and moves the load, then the friction w acts against the load. w is one half the difference between the two extreme positions of the vernier. These processes ought to be repeated for as many loads as possible throughout the whole range of the scale. By so doing a definite idea will be gained as to the accuracy of the machine, especially as to the extent of the error due to the

friction of the weighing mechanism. This minor friction is the worst sort of error, though leverages may not be correct. It is best to make an over-all determination of accuracy and not merely to check one section. Thus, when the poise is moved a certain distance from its zero position, its vernier indicates a certain load. This is, or should be, the load supported by a specimen when the beam is raised by the straining mechanism. By making a direct check with dead weights the total error at each load becomes apparent.

It sometimes happens that, when there are two machines in the same laboratory, elastic load strain graphs from the same bar in the different machines show angles of slope which are not quite the same. This indicates that the leverage of one of the machines is not correct, and means must be adopted to get rid of the defect. Leverage tests with known dead loads on both ends of the lever or its equivalent set of levers are of little use unless the lever-mechanism friction has been reduced to a minimum. For this purpose all knife-edges should occasionally be removed and examined. Every knife-edge should be sharp, clean cut, and straight, and should never be allowed to wear to a rounded or curved angle.

In a general laboratory devoted to the testing of materials there should be at least one machine which is reliable and reasonably free from error. The other machines can be checked from it by means of a steel test piece and extensometer.

An interesting case of the calibration of a large compression machine in the authors' laboratory is worth noting. The machine (Plate VIII (*a*)), of 900 tons capacity, was designed as a diaphragm machine, but difficulties at the diaphragm end prevented its completion in that form and it remained simply as a hydraulic press whose water pressure was indicated on a large Budenberg gauge. Acting against the water load was the friction of the main cup leather and that of the auxiliary cylinders used for bringing the ram home after a test. A steel compression bar about 3 in. in diameter and 12 in. long was placed in a horizontal Wicksteed machine and submitted to increments of load of one-fifth of a ton. A pair of Martens' mirrors were used and the loads applied through a pair of steel balls. The testing machine used and the results were assumed correct. The bar was next placed in the hydrostatic press machine and loaded by increments of pressure shown by the gauge. Using the former results in conjunction with the latter, it was possible to construct a formula which would give the load in tons on a specimen, in terms of the observed pressure. Its form was—

$$P = k(p - K)$$

Actually $$P = 0{\cdot}1392\,(p - 34)$$

where P is the total load in tons, p the water pressure in lb. per sq. in., and k and K empirical constants.

As the friction constants vary with the age and condition of the packing rings, the constants are liable to change also, and frequent calibration is desirable.

In every testing laboratory a steel "spring" bar which has been carefully calibrated, preferably with dead loads and a reliable extensometer, will prove useful as a standard for reference as well as a means of calibrating new machines. A testing machine, at one time fairly accurate, may change in course of time; and, in a general laboratory, where there are several machines, a means of checking and comparing is most desirable.

127. Dead-weight Calibrating Gear—The scheme for calibrating the testing machines of the Manchester College of Technology is represented on Figs. 177 and 178. The more important machines are placed in line parallel to the long sides of the room. Of these, A is the large press machine whose calibration has just been referred to; B is a 50-ton horizontal Buckton machine; C is an Amsler-Laffon 30-ton compression machine; D is a 60-ton transverse Amsler-Laffon machine; and E is a 30-ton Amsler-Laffon machine for tension. The intention of the arrangement is to be in a position to check the readings of B, C, D, or E at any time, and for this purpose two deep corridors or trenches were made, one down each side of the line of machines. Along these corridors are laid rails on which run bogies. Each bogie carries a set of five weights, 1, 2, 3, 4, and 5 tons. These weights are arranged so that they may be hung from suitable suspension rods. In testing a machine, say C, the bogies are moved so as to be one on each side of the machine, a horizontal short girder is set with its middle point resting on the top of the ram of the machine, and a pair of suspension rods hanging from

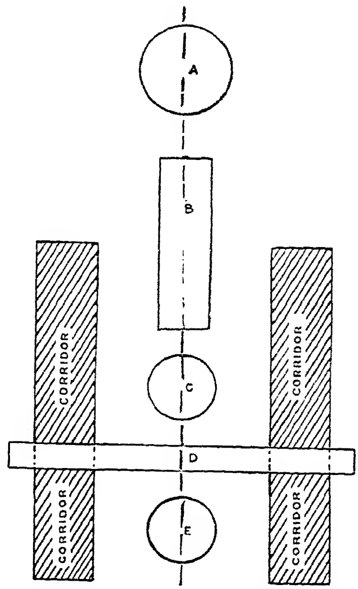


FIG. 177.—Rough general plan of the principal machinery in the Material Testing Laboratory at the College of Technology, Manchester.

each end. In making a calibration, first one ton is hung from each rod, making two tons altogether, then 4, 6, 8, 10 tons. When all slack has been removed, the ram of the machine is pumped up so as just to lift the test load and the mercury column reading is noted, and so on for all loads up to 10 tons.

In the case of the horizontal machine, B, a 1-to-1 bell-crank lever converts vertical pull into an equal horizontal push on the weighing mechanism.

These tests take up much time and require additional help, besides putting the particular machine out of use for the moment, but they are always a check on the machines and serve as a means of settling disputes as to test results. Of course, the pits are boarded over when not in use.

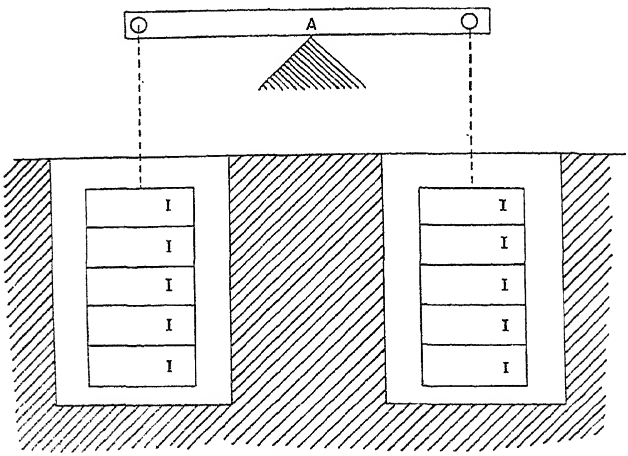
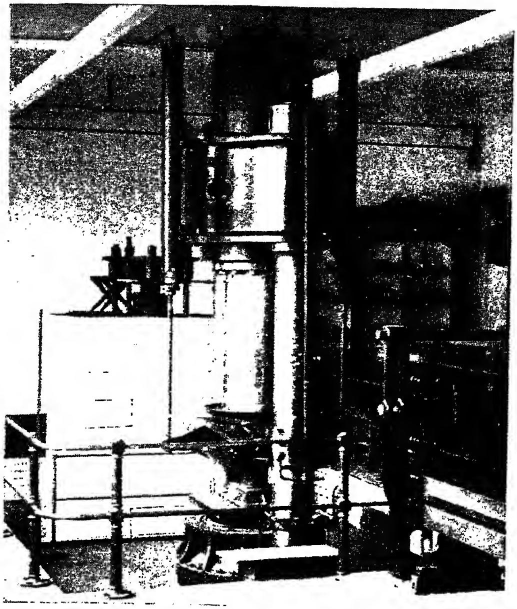


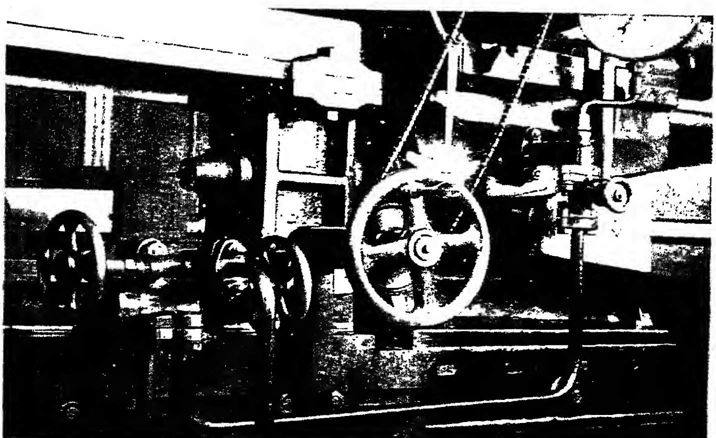
FIG. 178.

Accuracy of 50-ton Buckton Machine—The dead-weight scheme for calibration has been utilized to ascertain the accuracy of the 50-ton horizontal testing machine in the above laboratory. A 1-to-1 knee lever was used to transmit the force of the dead weights to the crossheads of the machine (Plate IX). First the arms of the knee lever were carefully measured in every way available and were adjusted to be so nearly equal that not the slightest difference could be detected. When the knee lever had been put in place with its links attached, the test loads were applied, first 2 tons, then 4 tons, and, by successive increments, up to 10. When any given load had been applied, the poise weight was adjusted until the lever was in perfect balance, as indicated by the central position of the pointer connected to the

PLATE VIII.



(a) HEAVY COMPRESSION MACHINE AT THE COLLEGE OF TECHNOLOGY, MANCHESTER. CAPACITY 900 TONS



(b) VALVES FOR BUCKTON HORIZONTAL TESTING MACHINE, SHOWING NEEDLE REDUCING VALVE

PLATE IX.

ARRANGEMENT SHOWING SUSPENSION OF CALIBRATING WEIGHTS OF BUCKTON 50-TON TESTING MACHINE

lever. It was found that, for each test load, there was a small range of position of the poise at any point in which equilibrium could be maintained. This range was found to be about 0·15 ton. Some figures obtained are as follows :—

Scale Reading.	Test Loads, tons.				
Tons	2	4	6	8	10
Upper limit	2·297	4·312	6·317	8·340	10·327
Lower limit	2·182	4·179	6·179	8·182	10·167
Average mean	2·239	4·245	6·248	8·261	10·297

Evidently the zero of the scale had been placed too far back. Mean percentage errors obtained from figures in the above and similar tables are (plus or minus):

Individual (greatest and least)	2·58	1·56	1·10	0·96	0·68
Greatest (all plus)	4·27	2·00	1·39	1·37	1·05
Least (all plus)	1·80	0·93	0·50	0·65	0·58

These errors are plotted in the diagram (Fig. 179).

These curves show that the errors are greatest at the low loads and diminish rapidly, the actual discrepancy between the scale load and the true load on a specimen very soon falling below 1 per cent. Hemming found that a certain Olsen machine working up to 30,000 lb. possessed an accuracy varying from 0·1 to 0·4 per cent. Also Martens found that the error of an oil-pressure machine (Amsler-Laffon) rarely exceeded 1 per cent.

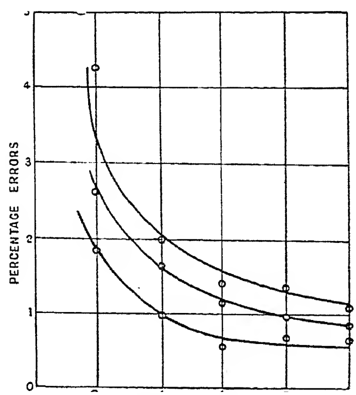


Fig. 179.

The sensitiveness of a machine is often expressed by stating what weight must be added at the shackle in order to lift the beam. From all the evidence available it appears reasonable to expect a testing-machine error to be generally about 1 per cent., and often to be well below this limit.

On the next page are Messrs. Buckton's tests for accuracy (Figs. 180, 181, and 182).

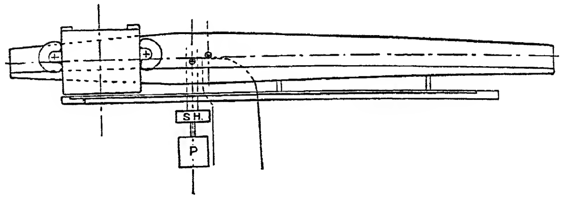


FIG. 180.—Test No. 1 (Accuracy of Machine as a whole).

A proving weight, P, is hung from the shackle, SH, of the machine.
The poise weight is then moved forward to balance the proving weight.
The scale should then indicate a load equal to the proving weight.
The proving weight is usually 1 ton or ½ ton, according to the size of machine.

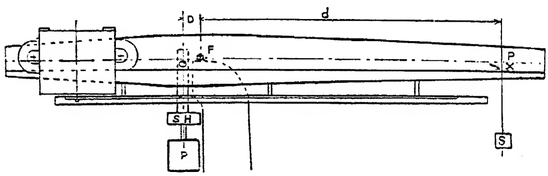


FIG. 181.—Test No. 2 (Accuracy of D).

The proving weight, P, is hung from the shackle, SH, of the machine, as in test No. 1, but the poise weight remains at zero.
A standard weight, S (e.g. 56 or 28 lb.), suspended at a certain distance, d, from the main fulcrum, F, should balance the proving weight.
A nick is usually provided in the rail of the machine (at X), for the purpose of suspending the weight S, then, d being given, $S=\frac{P\times D}{d}$. or, if S is given, $d=\frac{P\times D}{S}$.

P, S. and d being accurate, this test becomes a proof of the accuracy of D.

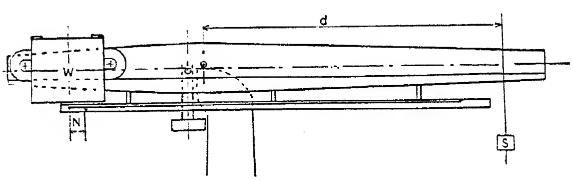


FIG. 182.—Test No. 3 (Accuracy of W).

The proving weight is removed. The weight S remains in position.
The poiseweight is moved backwards a distance N, representing a negative load equal to the weight of the proving weight removed. The lever should now be in equilibrium.
If, for practical reasons, N must be reduced, S must be reduced to correspond, according to the equation $W\times N=S\times d$ or $S=\frac{W\times N}{d}$

This test becomes a proof of the accuracy of W (if S, N, and d are exact), because $W=\frac{d}{N}\times S$.

128. A Tension Test—The testing machine is the essential appliance needed for testing of any kind. In addition, compasses, micrometers, and punches are needed. To obtain the permanent elongation it is customary to make a centre dot in the neighbourhood of one end and from it set off the gauge length and mark the second point by a second dot. The distance between the centre dots is the gauge length. Its increase after fracture is found by fitting together the broken pieces and measuring the new distance between the dots. With parallel bars it is a good plan to mark out a line of centre dots 1 in. apart, from end to end of the bar, so that the elongation may be found if fracture occurs close to one of the holders.

In making a test to destruction it is generally convenient to begin by advancing the poise so as to give sufficient load for the tightening of the wedges. The amount will be shown by experience, and depends on the size of the bar.

When the bar has been set in the holders, the straining mechanism is applied until the beam is pulled up to its horizontal position.

A pair of compasses is used to scribe the gauge length from the centre point which defines one end. The load is now steadily increased by running the poise out, meanwhile keeping the beam floating horizontally. This is where a needle controlling valve is useful. When the test has got fairly started the valve (Plate VIII(*b*)) can be left partly open and balance maintained by working out the poise. When the yield load is reached there is a sudden and rapid departure of the scribed line from its first position, and the load must be recorded.

Fracture can now be approached. The important load to find is the maximum load, and to get it the poise must be advanced and balance maintained until the maximum is reached. With ductile materials the local elongation now takes place.

With brittle materials like cast-iron there is nothing to be observed until fracture supervenes.

Test Results—The results of commercial tests required by the engineer include: The stress at the elastic limit; the stress at yield point; the maximum stress or breaking stress; elongation on some specified length, after fracture; reduction of area at fracture; appearance of fracture surface. In brittle materials the stress when the first crack appears is the yield.

The heading of a Test Report is given on the following page.

MATERIALS TESTING LABORATORY

....192

REPORT on ....

No. of samples ....

Nature of materials ....

Submitted by ....

Test No.	Mark on Specimen.	Initial Dimensions in Inches.			Initial area of cross-section in sq. inches.	Stress in Tons per Sq. Inch.		Elongation after Fracture per Cent. on			Reduction in area per cent.	Remarks.
		Width	Thickness	Diameter		Yield Point.	Maximum.	Inches	Inches	Inches		

The yield point and elastic limit as defined by the Engineering Standards Committee in 1911 are as follows :—

BRITISH STANDARD DEFINITIONS OF YIELD POINT AND ELASTIC LIMIT *

The Main Committee having been approached with the request that Standard Definitions for the terms " Yield Point " and " Elastic Limit " should be drawn up, remitted the matter to those Sectional Committees which had prepared Standard Specifications for Steel and Wrought Iron, with the result that the following definitions have been unanimously agreed upon. At the request of the Ships Committee a note is added with regard to the use of these terms in the commercial testing of materials used in the construction of Ships and their machinery.

DEFINITIONS

129. Elastic Limit—The Elastic Limit is the point at which the extensions cease to be proportional to the loads. In a stress-strain diagram plotted to a large scale it is the point where the diagram ceases to be a straight line and becomes curved.

Note—The Elastic Limit can only be determined by the skilful

* Abstracted by permission of the British Engineering Standards Association from B.S. Specification. Copies can be obtained from the Secretary of the Association, 28 Victoria Street, Westminster, S.W.1.

use of very delicate instruments and by the measurement of the extensions for small successive increments of load. It is impossible to determine it in ordinary commercial testing.

130. Yield Point—The Yield Point is the point where the extension of the bar increases without increase of load.

Practical Definition of Yield Point—The Yield Point is the load per square inch at which a distinctly visible increase occurs in the distance between gauge points on the test piece, observed by using dividers ; or at which when the load is increased at a moderately fast rate there is a distinct drop of the testing machine lever, or, in hydraulic machines, of the gauge finger.

Note—A steel test piece at the Yield Point takes rapidly a large increase of extension amounting to more than $\frac{1}{200}$th of the gauge length. The point is strongly marked in a stress-strain diagram.

Note added by request of the Ships Committee—The Sectional Committee on Sections and Tests for Materials used in the construction of Ships and their Machinery, while concurring in the foregoing definitions, desires however to make the following remarks with regard to the use of the terms "Yield Point" and "Elastic Limit" in connection with Steel and Wrought Iron for Shipbuilding.

The Ships Committee does not recommend the use of either "Yield Point" or "Elastic Limit" in the Standard Specifications for Ship Material, for the following reasons :—

Yield Point. In regard to the ascertainment of the Yield Point there is considerable divergence of opinion as to the best method of determining it, and all methods involve greater time and care than can be expected in the Works. While it is possible in Works by careful testing at a greatly reduced speed to obtain the Yield Point in ordinary mild steel and wrought iron, some of the harder steels and other constructional materials have no definite Yield Point at all, and some have no Elastic Limit.

Elastic Limit. It is quite impossible to determine the Elastic Limit in the time available for ordinary commercial testing. In its determination a specially delicate and accurate extensometer must be used, in the hands of a careful and competent observer, and the determination for each test bar would require a considerable time. It is properly a matter to be left to laboratories organized for scientific purposes.

The Ships Committee is of opinion that the present method of fixing, by experience, the working stress for any material as a proportion of the ultimate breaking stress rather than as a proportion of the Elastic Limit or Yield Point, is the best practical method, and therefore it considers that the inclusion in the British

Standard Specifications for Ship Material of tests to ascertain either the Elastic Limit or the Yield Point would not justify the dislocation of the ordinary commercial testing as carried out in the works' test rooms which would be entailed thereby.

In recent tests of high tensile steel such as that used for shell material, the following plan was used for finding the yield point: A centre pop was placed on the surface of the bar and from it was scribed an arc 2 in. away (Fig. 183). The bar was then loaded up to the specified yield-point load and the dividers again placed in the centre. If the scribed arc departed by the smallest appreciable amount from the compass point, the yield point was considered to have been passed and the material rejected. For special grips used in this work during the war see Plate X(a). With these grips it was possible to test over sixty specimens of mild steel per hour, including the recording of the yield point, ultimate stress and elongation on 2 in. The specimens were all 0·564 in. diameter and a highly skilled man operated the testing machine.

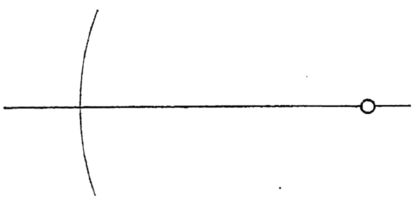


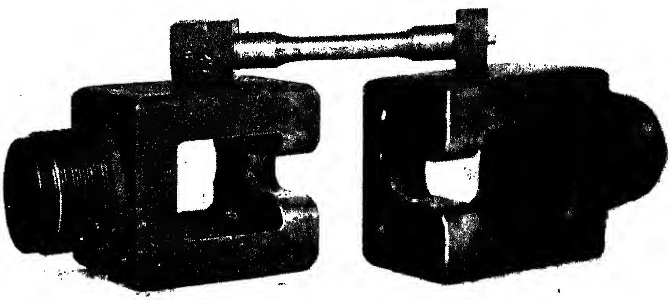
Fig. 183.

Instances of broken tension specimens of cast iron, wrought iron, and mild steel are seen in the photograph (Plate X(b)).

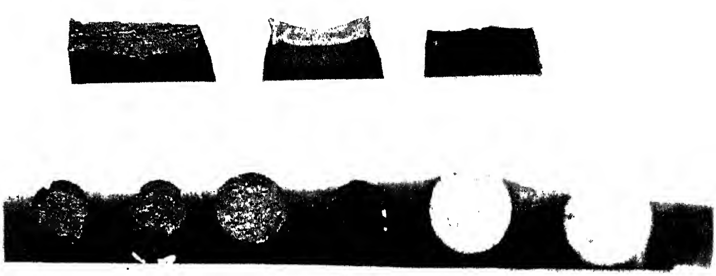
131. Measurement of Elastic Strain. Extensometers—When stretches or shortenings at loads below the elastic limit are desired for the purpose of finding the true elastic-limit load or determining the modulus of elasticity, some form of extensometer must be attached to the bar when under load. There are many designs of extensometer available, but it is safe to say that the simplest are the best. They include the simple lever extensometers of Goodman and Kennedy, in which the stretch of the bar is communicated to the short arm of a light lever of 40/1 or 50/1 ratio of arms with the point of the long arm moving over a graduated scale. Extensometers of this class may be depended upon to give extensions directly to the nearest 0·001 in. and by estimation to the nearest 0·0001 in. With care and practice very good results can be obtained with instruments of this class, which have the additional advantage of being easy to calibrate.

The microscope—combined with leverage—instrument of Ewing for 8 in. gauge length (Plate XII(a)) is a favourite extensometer in Britain, as it is convenient both for reading and calibration. The scale is divided in about thirteen numbered

PLATE X.

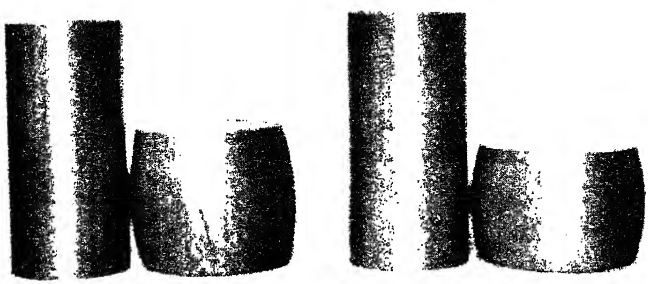


(a) SPECIAL "DROP IN" GRIPS FOR BUCKTON 50
USED IN RAPID TESTING.

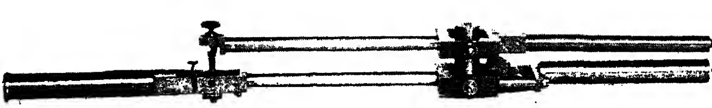


(b) BROKEN TENSION SPECIMI

PLATE XI.



(*a*) COMPRESSION SPECIMEN OF WROUGHT IRON AND MILD STEEL BEFORE AND AFTER COMPRESSION



(*b*) EWING'S EXTENSOMETER FOR SHORT SPECIMENS $1\frac{1}{4}$-INCH GAUGE LENGTH

divisions, each of which represents $\frac{1}{500}$ in. extension. Each numbered division is divided into ten equal parts, and by estimation it is possible to read to one-fifth of one of these small divisions, or to $\frac{1}{25,000}$ in. These remarks also apply to Ewing's $1\frac{1}{4}$ in. gauge length extensometer for compression shown on Plate XI (*b*). The scale is similar, but one numbered division represents $\frac{1}{2,500}$ in. shortening.

The most accurate extensometers are those of the "rocker" or "roller" class, working with scales and telescopes. The scheme of the well-known Martens' extensometer is shown on Fig. 184. In this instrument, instead of clipping it opposite to the centre of the specimen, so as to get a mean reading, it can be applied separately to both faces *d* and *d*. Against each of these surfaces lies a "measuring bar" *a* and *b*. At the lower end of each of these is a pair of points (for round bars) or a disc edge (for flat bars). At the other end is a groove. In this groove rests one edge of the "rocker," which is a piece of steel having a diamond section, and is therefore two knife-edges. Of these edges one rests in the aforementioned groove and the other against the specimen, being held there by springs.

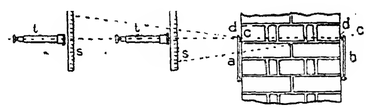


FIG. 184.

One-half of the total width of this rocker forms the short arm of the multiplying lever, its long arm being a beam of light from the surface of the mirror to the scale reading visible at the moment. Take the left-hand telescope in the figure. The sloping beam strikes the scale *s* ; thence it proceeds to the mirror *c*, and after reflection reaches the eye through the telescope *t*. If the column at *c* extends, the rocker is tilted with its right-hand edge raised. This causes the beam of light to strike the scale a little lower down, its movement being a measure of the extension. There is a fixed crosshair in the diaphragm of the telescope, and to the observer the scale appears to move over this. The scale is divided into centimetres and millimetres.

As the observer looks into the telescope, when the focus has been adjusted and the mirror set, he sees the centimetre scale with the crosshair line lying at right angles to the length of the scale. Strain is indicated by the apparent movement of the line along the scale.

In the Martens' apparatus used by the authors it was found

that each centimetre on the scale represented about 0·001 in. strain of the specimen. Each centimetre is divided into ten millimetres, each of which represents 0·0001 in., so that each of the smallest divisions on the scale gives a direct reading of $\frac{1}{10,000}$th of an inch. As the scale is very clear and distinct, each millimetre can be divided by estimation into ten parts, or, with more certainty, five; so that it should be possible to measure a strain to the nearest $\frac{1}{50,000}$th of an inch. It is absolutely essential that the scale be set at the correct distance from the mirror (in this case 46·85 in.). For correct fine readings much depends on care and on experience. As both the movement of the rocker edge and the line upon the scale are vertical, the reading should not be affected to any great extent by the inclination of the reflected line of sight. For purposes of calibration the measuring bar may be set on the extensible bar shown on Plate XII (*b*), and the differential screws turned just sufficiently to extend the bar a definite amount, say 0·001 in., and care taken to note to what distance on the scale this corresponds. Persistent readjustment will have to be made by changing the mirror and scale distance until a satisfactory correspondence is arrived at.

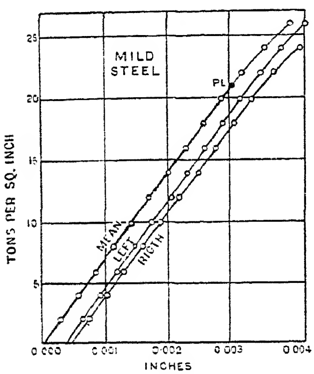


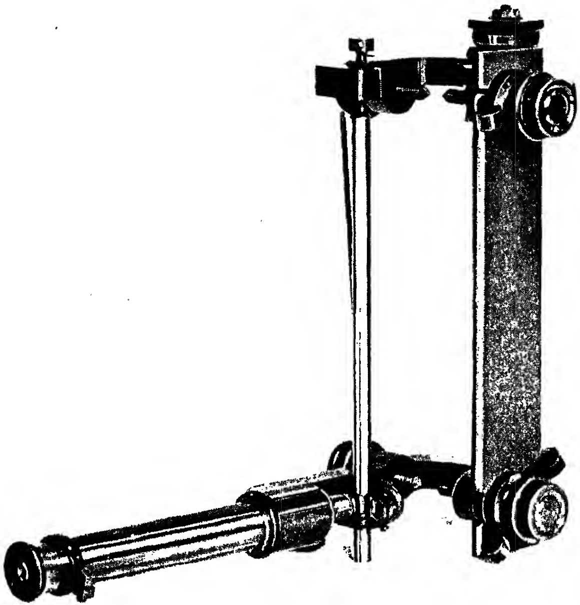
FIG. 185.

This diagram shows the elastic strains given by Martens' Extensometer when applied, say, to a steel tensional bar. The lighter lines are for the mirrors on opposite sides of the bar and the thick line represents the plotting of the mean values of these.

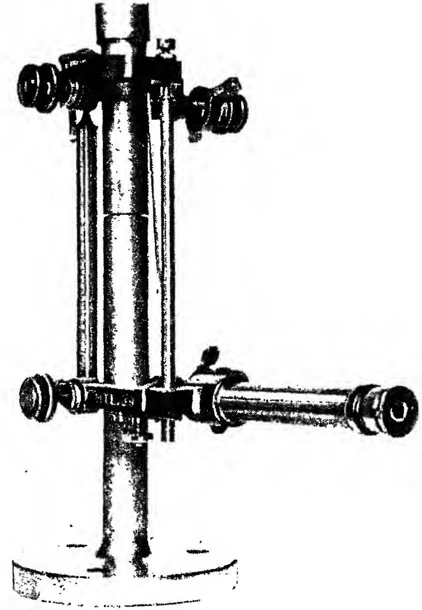
132. Autographic Diagram Apparatus—A load-strain diagram may be constructed by plotting observed and recorded results. This is often done, but for some purposes an automatic diagram drawn by the specimen itself during the test gives valuable and important information.

To design and construct a recorder which shall be capable of drawing automatically a correct load-strain diagram is a matter of extreme difficulty. Many attempts have been made, with varying success. The function of such a device is to cause a pencil, or some other device for scribing a line, to move over a sheet of paper in such a way as to form a graph showing load and the accompanying strain. In most cases the loads are represented by ordinates and the corresponding strains appear as abscissæ. The strain may be elongation, shortening, or angular twist; the first of these is far away the most usual case. Again, the strain may be elastic or plastic, or both of these simul-

PLATE XII.



(*a*) EWING'S EXTENSOMETER FOR 8-INCH GAUGE LENGTH



(*b*) EWING'S EXTENSOMETER FIXED ON CALIBRATING BAR

taneously. The case referred to here is the last-mentioned.

It has come to be the common practice for the diagram to be described on a sheet of paper fixed to a revolving drum. Furthermore, it is the usual custom for axial movement of the pencil to indicate load and rotary-movement of the drum strain, but these arrangements may be reversed.

The important questions to be answered are: Is the axial movement of the pencil truly proportional to increase in the load, and does the drum revolve to an extent which is equal to or proportional to the strain? Put in another way: Is it possible, using the correct scale, to find the load at any stage in a test by measuring from the zero line to the point in question? And is the same true for the strain?

Against affirmative answers to both questions there are reasons for doubt, and many influences are at work. It may be well to examine these in detail. In this way the causes of error will be made apparent.

In order to appreciate fully the pros and cons of autographic recorders the following concrete example may well be considered in some detail. The example referred to is the recorder designed and perfected by the late Mr. J. Hartley Wicksteed, of Messrs. Joshua Buckton & Co., Ltd., of Leeds, and represents one of the best-known appliances of the kind in use in Great Britain.

The general scheme of this apparatus as attached to a Wicksteed vertical-type testing machine is represented on Fig. 186. This shows the single-lever machine, which is provided with a moving jockey weight W. To increase the load on the specimen, the weight is moved towards the longer end of the lever, that is, towards the right. The lever balances about a knife-edge, which rests on a plate at the top of the main standard of the machine. The shackle which holds the top end of the test specimen hangs from a second knife-edge placed in the short arm of the lever. The maximum load of 50 tons normally comes on the specimen when the weight is at the other end of the lever, in the position shown. Also, normally the centre line of the lever must be kept in a horizontal position, so that a vernier, attached to the weight moving along a graduated scale on the lever, may indicate accurately loads on the specimen.

The machine is shown as the weight would be placed when arranged for use with the autographic gear. For this purpose the jockey weight is shown as having been run to the extreme end of the lever furthest from the specimen. The specimen stretches under the increasing load, caused by a screw gear which pulls on the lower shackle carrying the lower end of the specimen. The specimen itself is provided with heads which fit corresponding sockets in the shackles; thus slipping is made

impossible, and the stretch recorded is that of the portion between the heads.

When an autographic test is to be made the free end of the lever, instead of resting on a stop, is allowed to hang from the lower end of an extended helical spring, as shown ; at the same time the weight is towards the 50-ton end of its scale. The result is that, as the traction is applied to the lower end of the test piece, a part of the load of the jockey weight is relieved by the contraction

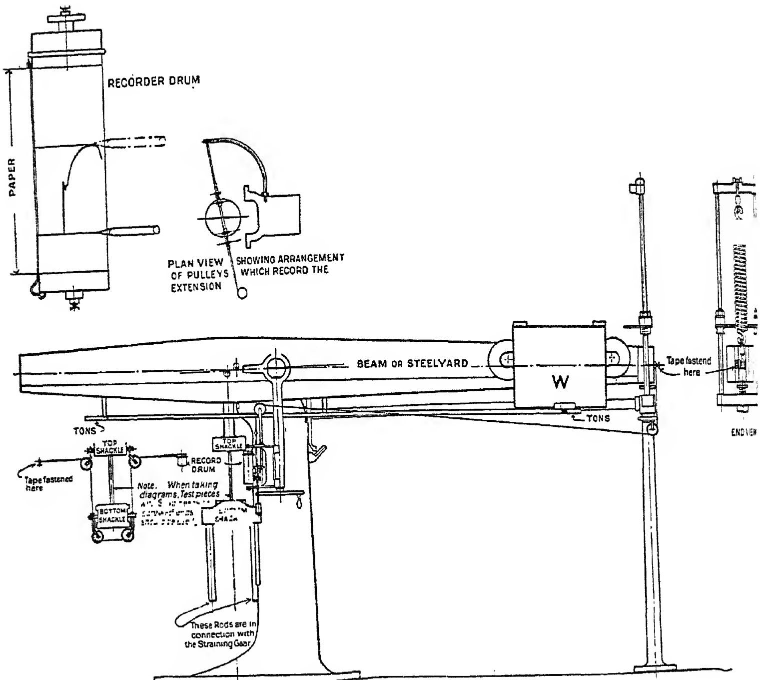


FIG. 186.

of the spring, the beam tilting slightly upwards, and the specimen being extended. At this stage the pull on the bar is equal to (W × leverage)—(upward spring tension + its leverage). As the spring contracts from its initially extended length, at which its tension is just sufficient to hold the beam in a position somewhat below the horizontal, the beam is drawn upwards by the traction on the specimen.

The extent of this movement is a measure of the load on the bar, and is transmitted by a tape to the recording pencil, which

is thus moved upwards in a direction parallel to the axis of the drum. In this way the distance moved by the pencil records the pull on the specimen.

The stretch movement is communicated to the drum in a manner similar to the one used for the load indication, namely, through a tape. This is attached to a fixed point apart from the machine; it then passes over outrigged guide pulleys on the lower and the upper shackle and is taken thence direct to the drum. The drum is rotated by the tape from the shackles, and this movement, combined with the up-and-down motion of the pencil communicated from the end of the lever, results in the desired load-strain curve.

The view (Fig. 186) shows the complete appliance as ready for making a test and at the same time drawing an autographic curve showing the essential features of the test.

Starting with the test piece, the details may be traced out. Its lower end rests in the bottom shackle and its upper end in the top shackle. The specimen terminates in solid cylindrical ends and these fit corresponding recesses in the shackles. The result is that the pull on the bar is absolutely positive and in no way dependent on friction or the bite of toothed wedges; the consequence of this is that the amount by which the shackles are separated during a test is identical with the stretch of that portion of the specimen which lies between the heads. By means of the tape this movement is transmitted so as to cause a partial turning of the drum, and as the tape passes over a pair of pulleys on each shackle, the movement of the tape is double that of the shackles.

The frame which carries the rotating drum is fixed on to the main standard of the machine, close to the specimen. The hand-wheel seen near the drum is for the purpose of moving the poise weight during an ordinary test, but is not used in the present instance.

The vertical travel of the pencil is effected as described, by the movement of the hard woven tape which is fixed to the right-hand end of the beam or steelyard and passes over two pulleys attached to the fixed part of the machine.

The use of this flat surveyors' tape for both transmissions represents one of the most important improvements on the original design made some years ago, and has made it possible to draw more accurate diagrams than heretofore. Formerly wire, light chain, and whipcord were tried, but none of these was quite satisfactory. The wire if fine was too breakable, or if of copper too extensible. If thicker, it was found to be liable to kinks and too stiff for bending easily round the small guide pulleys. The whipcord did not break, but was too elastic. By

using a tape similar to that of the surveyor, great flexibility is attained, with ample strength. In this way it is possible to get transmission between steelyard and weighted pencil carriage which will reverse its direction without lost motion, and thus be delicate enough to record very small differences in the load.

In the end view will be seen a light frame which is bolted to the top of a pillar, and is used to carry the spring. In the lower part of this frame is seen a view of the end of the beam, to the centre of which the end of the tape is attached. This opening in the frame limits the movement of the beam end to within a few inches, up or down.

The view also shows the beam end floating upon the spring, which is suspended from a knife-edge. The beam end must be clear of the lower stop plate, when the poise weight is set. When the poise weight is at zero on the scale, the beam end should be clear of the upper stop plate.

During an experiment the probable amount of the spring's stretch must be so prearranged that the beam will be likely to rise as far above the lower stop at the end of the test as it was below the upper stop at the beginning. By doing this the operator will keep the average position of the beam as nearly as possible horizontal.

Carrying out an Autographic Test—The *modus operandi* is as follows: The specimen to be operated upon, having been turned to such diameter as will give a cross-section of $\frac{1}{4}$ sq. in. or $\frac{1}{2}$ sq. in., is placed in the holders of the shackles. The poise weight is now run to the outer end of the beam to such a position that the scale reading is 50. This means that if the spring were not present, the pull on the specimen would be 50 tons when the straining was tightened, so as to raise the beam from its stop; but the end of the beam actually floats on the spring to such an extent that there is no pull whatever on the specimen, the pull due to the poise being relieved by the spring. It must be observed that, under these conditions, the beam occupies the lower of its two extreme positions. Weaker springs can be used for the weaker specimens, so that the load scale shall not be unduly small.

When all tape adjustments have been made, the test may proceed. With the long end of the lever and its stationary poise floating on the extended spring, the test is carried forward by the application of a steadily applied pull by the straining wheel. It is most essential that this pull be steady and continuous. The result will be that the spring automatically adjusts itself to the conditions, and the upward movement of the beam end becomes a measure of the tension on the bar. It must be understood that this pull is (the difference between the weight of the poise and

the diminishing pull of the spring) multiplied by (the leverage of the machine). In other words, as the beam rises under the applied pull, the spring takes less and less from the full effect of the poise. The upward movement of the beam end is utilized in giving vertical movement to the pencil carriage as previously described. For this purpose one end of the tape is fixed to the end of the beam; it then passes round guide pulleys, and so direct to the carriage which carries the marking pencil. This carriage is weighted, so that there is at all times a constant pull on the tape to keep it tight.

The extension of the specimen is recorded by the rotary movement given to the drum through a second tape. This is fixed by a bracket from the standard, passes round a guide pulley on the top shackle, one on the bottom shackle, a second on the bottom shackle, over a pulley on the top shackle again, and thence to the sheave portion at the top of the drum. This arrangement of guide pulleys results in a drum movement which is twice that of the stretch.

133. Sources of Error—The information yielded by any appliance of the kind described is only accurate up to a point which is controlled by all the conditions under which the apparatus works. Like all autographic and semi-autographic gears, the present appliance is liable to certain errors due to known causes. These are:

1. Stretching of the tapes, by which the drum falls short of its full rotation and the pencil fails to attain a height which represents the correct load.

These are both minus errors, giving results below the true values. They have, however, now been reduced to minima, and, fortunately, are not likely to vary from time to time.

2. Incorrect calibration of the spring.

This error depends on personal skill in carrying out the work of calibration, which, presumably, is done with dead weights. The error is plus or minus, but the total amounts should be very small.

3. Want of horizontality of the beam.

This means that the given load is higher or lower than its true value, and that the true result is only obtained when the beam is perfectly horizontal.

This error is due to the small amount of friction, which interferes with the rocking of the beam to or from its horizontal and stable position, and to the very small moment resulting from the balance weight. These errors, due to obliquity, have been investigated, and found to be very small in amount, but they are still there and must not be forgotten. Their total effect is reduced when the beam moves in one direction only, and, as this autographic gear works on these lines, the error will be in this

way reduced. The nearer the beam is to its mid-position the smaller is the error.

4. Thickness of the graph line, which limits precision in reading.

The combined result of these errors is largely to preclude the use of the diagrams for quantitative work ; that is, the distance from the horizontal zero line to a given point on the curve is only approximately proportional to the load on the specimen, and its horizontal distance from the vertical zero is a measure of stretch only so far as the errors will permit.

For these reasons it is better to take direct observations, in

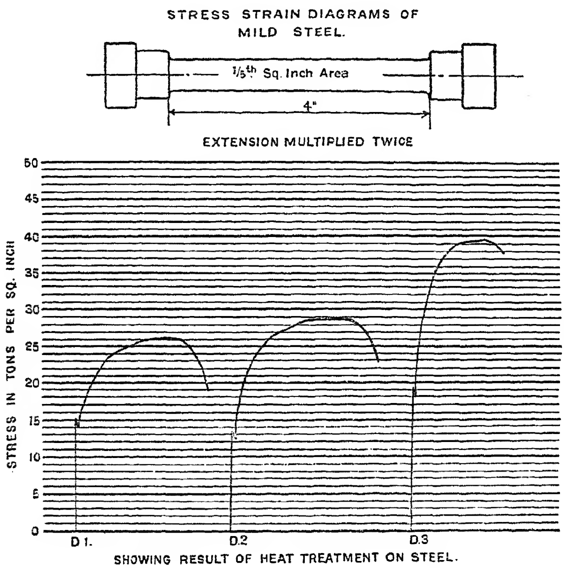


Fig. 187.

most commercial testing, in order to fix the salient points of yield load, maximum load, ultimate load, and so forth.

But, from a purely qualitative standpoint, they provide information in such a way and of such a kind as is not possible by any other means, and in this way they are extremely helpful for exhibiting the general qualities of a given material when resisting tension.

134. The Records—Examples of the records obtained by the Wicksteed autographic gear are shown on Figs. 187, 188, and 189, as photographed from the actual curves obtained during the experiments. On the last figure there are eleven examples of

what this apparatus will accomplish. Their general form is similar to that already given. Each one is complete from the commencement of the load application to the breaking of the bar. In order to make each record continuous and complete it has been necessary to limit the stretch magnification to get in the curve complete, and this has resulted in the elastic line being a nearly straight vertical line. This is the case for all the curves.

If the records are numbered in order from left to right, the different materials of which the corresponding bars are made are :—

1. Nickel chrome steel (heat-treated).
2. Chrome molybdenum steel.
3. Nickel chromium crank axle steel.
4. Carbon steel.
5. Mushet steel.
6. Carbon steel.
7. Manganese steel.
8. 25 per cent. nickel steel (not heat-treated).
9. Mild-steel casting.
10. Mild-steel (heat-treated).
11. Mild steel (not heat-treated).

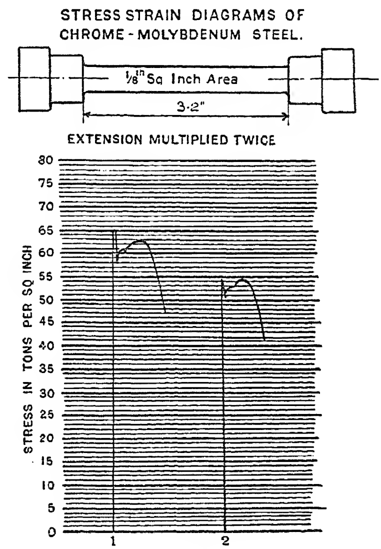


FIG. 188.

The normal curves obtained are represented by Nos. 10 and 11. Both exhibit the same general characteristics, namely, an almost straight line to the yield load, a quick drop in the load at this point, a rapid increase of load, succeeded by several diminishing fluctuations of the same kind, becoming a wavy horizontal line until general plastic elongation sets in and continues up to the highest point in the curve, where the maximum load occurs, and, lastly, the downward plastic curve to the ultimate or breaking load.

The main fact to be noted in these curves is the almost sudden drop in the load sustained by the bar just when the yield point has been passed. This feature is noted in commercial tests of most mild steels and is generally shown by the rapid drop of the beam on to its lower stop, and its remaining there until the instability has passed and the bar is once more undergoing uniform elongation.

Of the curves given, the most typical case is that of No. 11. Here the yield is near 16 tons per sq. in. and the tension sustained falls quickly to 14 tons per sq. in. when the yield has begun.

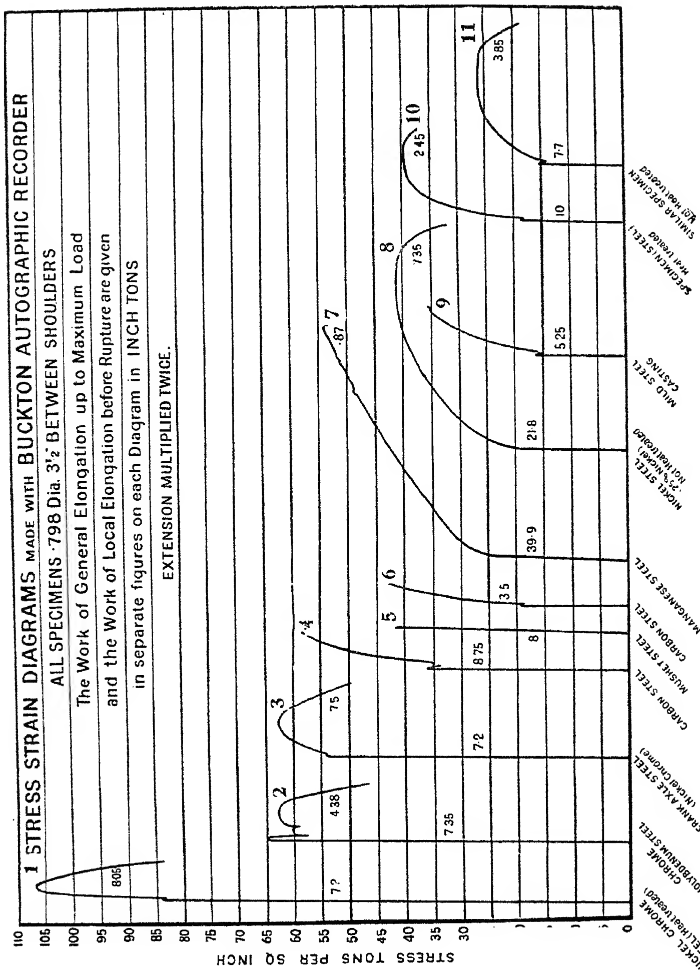


Fig. 189.

After this the tension recovers to 16 and so goes on increasing up to the maximum at about $26\frac{1}{2}$ tons.

The same phenomenon is visible in all the curves shown, with

the exception of Nos. 5, 7, and 8. In the latter two the stress carried by the bar goes on steadily increasing from zero to the maximum, and there is no sort of relapse following yield. In No. 7 there is little turning down of the end of the curve near the ultimate load, thus indicating the absence of much local contraction at fracture. In the nickel steel bar, No. 8, the turning down is more marked than in the manganese steel, No. 7. In the normal mild steel, No. 11, this local elongation and contraction effect is very well marked, as would be expected from the usual results of testing this material. In No. 5 there is no evidence of yield point, the load going right on to the maximum. In what is evidently steel containing higher carbon, No. 4, the turn back is absent, the material breaking square off at the maximum load. This is also true of the steel casting, No. 9.

The turning down of the curves near fracture in some of the steels is due to the setting in of local contraction when fracture is approached, by which the diameter is reduced, and consequently the area of section also, with the result that, though the load is less, the stress per sq. in. may go on increasing.

A noteworthy case is that of No. 2. Here the yield is reached near 65 tons per sq. in., after which the stress falls to $57\frac{1}{2}$ tons per sq. in., and does not rise again to the original yield.

In No. 7, for manganese steel, there is no reversal of the direction of the curve. The resistance of the bar increases up to rupture, and although set begins to take place almost from the beginning of the extension, the resistance of this material is stable throughout. If traction be stopped at any part of the curve there is no drop of the lever. In other words, the material takes all its permanent sets under increasing load, and this means that there is no actual yield point.

135. Effect of Speed on the Diagrams—The two curves on Fig. 190 were taken from similar bars of the same steel, but the traction was applied at a different speed in each case. The notes on these diagrams say that in the first case the speed of pull was $\frac{1}{25}$th of an inch in one minute, while the second one was pulled out much more quickly at $\frac{1}{6}$th of an inch per minute.

There is no very striking difference between the two cards; but in the quick case both yield point and maximum loads are higher than in the slower test. The general shapes of the graphs are alike, and there is little difference in the ultimate loads. It seems, therefore, that a higher speed of traction shows greater strength, both at yield and maximum, but the breaking loads are very nearly alike. This is what one finds in ordinary testing, the effect being due to the fact that there is not ample time

for extension to develop and allow equilibrium to be attained.

136. Significant Facts of the Wicksteed Curves—There are three outstanding facts connected with these curves which strike the observer immediately. These are :—

(*a*) The load as shown at any part of the curve is the actual load at the moment, and not simply the position of the poise. It is as truly the actual load as would be an increasing dead weight hung from the free end of the bar. This is very important.

(*b*) The only sort of control exerted on the test is the steady pull applied through the straining gear. The regulation of the magnitude of the load is quite automatic and cannot be influenced by the operator.

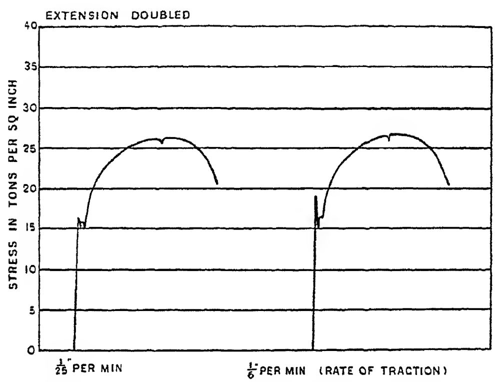


FIG. 190.—Autographic Record of Tension Tests on Mild-steel Test Pieces. $\frac{1}{4}$ sq. in. area, 2 in. datum length. Both pieces cut from same bar.

(*c*) The most striking fact in the curve itself is the sudden momentary collapse of strength in many of the steels immediately following the yield.

137. Utility and Limitations of Autographic Diagrams—It is important to realize these two points:

For what may be called *quantitative work*, that is, when it is desired to read actual numerical values from the curves at various important points, of both load and stretch, the autographic diagram is only very roughly to be depended upon. It is nearly always better to rely on direct observation.

For *qualitative* results the case is otherwise. In an autographic diagram the complete result of a test can be seen at a glance; significant facts can be appreciated without recourse to tables of figures. For example, suppose a steel maker to be developing

a new steel; he wishes to know how his method is working out, and takes autographic tests of a number of successive pieces of his material, and is able to see at a glance how his plan is succeeding and in what directions he ought to modify the process. The same thing would apply to a constructive engineer who is anxious to find a material with certain known qualities.

138. Significance of the Sudden Yield—Once more referring to the auto curves on Fig. 189, the carbon steel, No. 4, is a typical case. Here the elastic part of the line remains quite straight up to about 30 tons per sq. in., where, at the P.L., a slight curving towards the right begins. This continues until the first peak is reached, when there is a rapid fall in the load, with a continuance of the extension. When stability has been attained the curve again rises, and then follows a wavy horizontal step, where the pencil movement chiefly indicates extension without increase in load. At the end of this step the plastic stage begins and persists to the end of the test, when fracture occurs.

The same series of happenings is followed out in most of the bars, in a more or less modified degree. First comes P.L. and Y.P., to be followed by an up-and-down wavy movement, until the plastic part is reached, and so on until the ultimate load is reached.

In the curves yielded by some of the more crude auto gears the yield is masked by friction and the curve beyond the Y.P. runs as a horizontal line to the commencement of the plastic portion.

In the curve for manganese steel, No. 7, as already pointed out, there is no break in the line to indicate a Y.P., showing that in this material the tensity is never relaxed, the elastic being very gradually merged into the plastic state. The material takes set from P.L. to the end and becomes more tense as the test proceeds. Also, in the nickel steel bar there is no evidence of relaxation. The curve, however, bends down near the fracture, thus giving evidence of local contraction.

With an appliance of the Wicksteed type the complete life of the specimen is exhibited from the beginning of the loading up to fracture, and can be appreciated at a glance.

139. Meaning of the Yield—It is likely that each of the crystals constituting the specimen is made up of countless particles of extremely small size and that these are arranged in the form of a regular geometric pattern, being kept so by their mutual attraction. Failure of the steel is known to take the form of shearing along a series of parallel planes. This will at first distort the material and finally destroy the pattern. This may be followed by fracture or the particles may be re-assembled under the stress in a different and looser formation, so that there is greater freedom for the particles to roll into new positions, and plastic

elongation to be possible. At the same time, the mutual attraction between the particles will continue to be great enough to give the material elastic strength. This is shown to be the case by carrying out an elastic experiment on a piece of steel stressed far beyond its yield point, almost as far as fracture. Its modulus of elasticity is found to be very little below its original value when the stresses were well below the elastic limit. That is, steel in the plastic state may at the same time be perfectly elastic under certain stresses.

It may be suggested that the fall of stress following the yield load is not so much a weakening of the material as a temporary re-arrangement of the ultimate particles; that this must be the case is shown by the fact that after the yield, the stress goes on steadily increasing up to the maximum load. Instead of going steadily on as in manganese steel, the material breaks down at the yield load, then pulls itself together and goes on as before.

The rapid yield effect can hardly influence the suitability of the steel as a load carrier, for the simple reason that "working stresses" are always kept well below the elastic limit.

The same Wicksteed autographic gear applied to a horizontal machine has already been shown on Fig. 165.

140. Other Autographic Gears—Besides the appliances described there are many others of the same type, making use of many interesting devices. For his load movement Goodman magnifies the shortening of the testing machine standard. In the Kennedy gear the load movement is obtained by magnifying the stretch of a "spring bar" interposed between the grip and the specimen end. In the old Wicksteed "autographic recorder" the load movement was obtained by the water pressure in the machine cylinder pressing on a spring-controlled piston. An appliance, which was in a sense a forerunner of the above, has been in use for a number of years, and is made by Messrs. Riehlé of Philadelphia. Professors Unwin and Hele-Shaw, and Mr. (now Sir John) Aspinall have used autographic gears in which the load movement of the recording pencil comes from the travel of the jockey weight. The Amsler-Laffon machine uses the float on top of the mercury column for the same purpose. These last—Unwin, Hele-Shaw, Aspinall, and Amsler-Laffon—can only be described as "semi-autographic," inasmuch as the load recorded is not necessarily the actual load on the bar at the instant in question.

141. Sankey's Machine for rapidly testing Metals by Means of Repeated Bendings—This is one of a number of machines for testing small bars of any material by repeatedly bending the bar through a given angle until fracture takes place.

A machine of this type can be worked by hand in an office in such a way as to provide some information as to the comparative strength of a number of different materials. This does not entirely do away with tensile testing on a bigger scale, but provides an approximate substitute which is quite reliable within certain limits.

A plan of the machine is shown on Fig. 191. In this the specimen to be tested is marked D; one end of this is inserted into

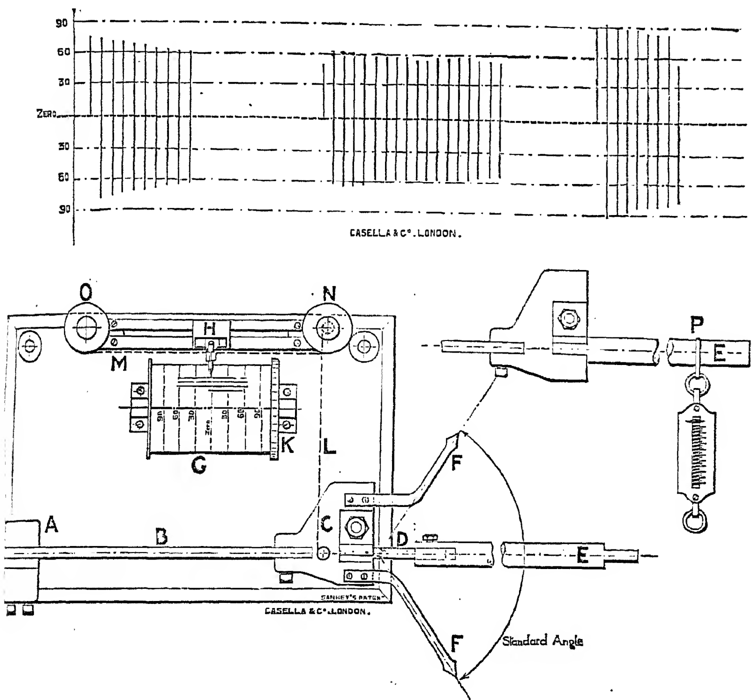


FIG. 191.

the grip C, and the other end into the rod marked E. The total length of this rod is about 3 ft., and it forms the handle by which the bar is bent. The grip C is attached to the end of a spring bar B, which itself is held at A. As the rod E is pushed away from the operator, D is bent and the force used is transmitted to C, so that B is also bent. The greater the force applied the more is B bent; so that the amount by which B is bent forms a measure of the bending moment applied to the test bar. A wire L is attached to C at one end and at the other end is taken round a

pulley at N to the pencil carriage H, so that the above-mentioned operation will cause a movement of the pencil axially to the drum, the length of the line drawn being proportional to the force applied.

The rotary movement of the drum is controlled by a spring and ratchet. After each bend either forward or backward one of the ratchet teeth is released, so that the paper moves round to the next line. The standard angle through which the specimen is bent is indicated by the two pointers F, F.

When a test is to be carried out one end of the test piece is

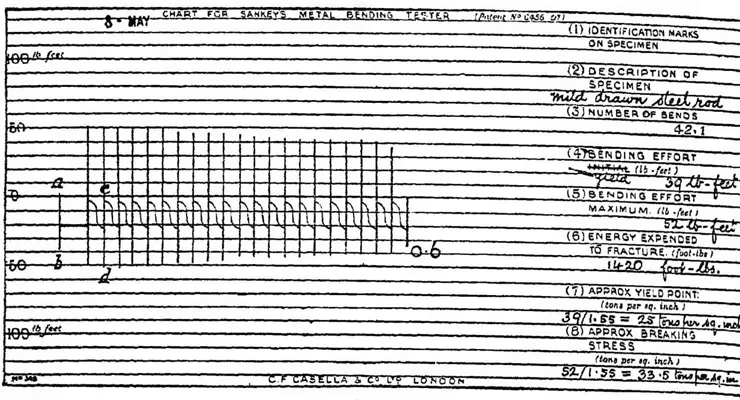


FIG. 192.

fixed into the handle E by means of a set screw, after which the other end is slipped into a hole at the grip C. The free length of the specimen is $1\frac{3}{4}$ in. First the handle is pulled towards the operator until the specimen is felt to be yielding, the angle at which this takes place being noted. The bend is then finished and reversed completely. This bending back and forth is continued until the bar breaks.

The resulting diagram consists of a number of lines which are straight and horizontal, as drawn in the first figure. The length of these lines shows the force required for each bend and the number of them gives the total number of bends (Fig. 192).

The machine is calibrated by the spring balance shewn, which can be applied to the lever at a point P a known distance from the face of the grip C. Known couples can thus be applied which are represented by the lengths of the lines scribed on the drum.

CHAPTER XII

PIG IRON AND CAST IRON

EACH of the three materials broadly known to engineers as cast iron, steel, and wrought iron, consists mainly of pure iron with which are mixed or combined small quantities of some other of the elements, these including carbon, silicon, manganese, sulphur, phosphorus, and copper, as well as minute percentages of several others. The precise quantities of these foreign ingredients vary with the material and with different grades of the same material. Their presence in the iron or steel has very definite effects on its properties, and these are still further dependent on the processes of manufacture and on the heat treatment which it undergoes. Of these secondary ingredients the most important is undoubtedly the carbon. Of this element there is from 4·6 to 3 per cent. in crude pig iron, about 3 per cent. in cast iron, from 1·5 to 0·1 per cent. in steel, and less than 0·1 per cent. to little more than a trace in wrought iron.

142. Classification of Iron and Steel—Several ways of classifying the irons and steels in general use have been used, but the one by which they are placed in the order of their carbon percentage is in many ways the most convenient, and will be used in what follows.

The intention is to consider the various classes of iron and steel in this order, with the addition of so much relative to methods of manufacture as may seem necessary, as well as some reference to typical qualities and uses. Brief mention will also be made of the influence which the constituents other than carbon have upon the properties of the finished materials.

Thus, taking pig iron, which contains not only more carbon than any of the others but is the crude material from which all the varieties of iron and steel are manufactured, as the starting-point, the various classes and grades appear as follows:

1. PIG IRON PREPARED FROM THE ORE contains from 4·6 per cent. to 3 per cent. C.

2. CAST IRONS OF VARIOUS GRADES are mixtures of quantities of pig iron of differing qualities suitably proportioned and remelted together. Contains about 3 per cent. C.

Cast iron is a comparatively hard and, unless subjected to

special treatment, brittle material. Its compressive strength is about four and a half times that of its tensile strength. Its strength is not affected by heating and quenching. It is rendered quite fluid at a relatively low temperature, when it can be run into moulds and thus cast into any desired form. The appearance of its fracture suggests a coarse crystalline structure.

3. Malleable Cast Iron—This is first cast into moulds of the desired forms, and afterwards exposed to the action of a decarburizing substance at a high temperature. The effect of this is to remove the greater part of the carbon from the iron and thus render it less brittle and more nearly malleable. Contains about 2·7 per cent. C.

4. Steel—Produced by cementation process from wrought iron, or by the Bessemer process or by one of the open-hearth processes from pig iron. In the various grades the carbon may vary from 1·5 to 0·1 per cent. Requires a melting temperature higher than that of cast iron but not so high as wrought iron. In its different varieties it possesses different degrees of hardness extending over a very wide range. The same applies to the strength properties.

The main varieties are given below, still in the order of carbon content:

(a) *Hard or High-carbon Steel*—Used for cutting tools and springs. Can be hardened by heating and quenching, but cannot be welded. Strong, but not ductile. Shows a fine crystalline fracture.

(b) *Medium Steel*—Used in engine and machine parts and in some structural work. It cannot be hardened by heating and sudden cooling, neither is it possible to weld it except with difficulty (unless by electric welding). Fracture is finely crystalline or silky.

(c) *Mild or Soft Steel*—Chiefly used in structural work. It cannot be hardened by heating and sudden cooling, but welding is possible with a reasonable amount of care.

5. Steel Castings—Used for the same purposes as cast iron, when greater strength is required. It cannot be hardened by heating and sudden cooling, nor can it be welded. The strength is fairly high, but the ductility low, 1 to 0·25 per cent. C. The appearance of the fracture is coarsely crystalline.

6. Wrought Iron and Ingot Iron—Forged or rolled from "puddle ball" obtained by the action of oxidizing flame on molten pig iron, or, more directly, eliminating from pig all but the iron, in a converter. Used in the process of steel manufacture by cementation and in cases where moderate strength and ductility are needed, especially when the material is to be hammered into difficult forms or has to be welded. Cannot be

hardened by heating and rapid cooling. Fairly strong in tension and compression. Fracture is of a dark fibrous appearance. Contains practically no carbon. The surface may be hardened by being subjected to the process of case hardening.

It is not easy to give a concise list which includes all the different irons and steels in common use. There is a good deal of overlapping of the different classes, and an imperceptible merging of one into the next; and in the above divisions there are also many subdivisions devoted to classes of the materials which possess special qualities.

143. Pig Iron—All irons are obtained from natural ores, found chiefly as oxides or carbonates of iron. In the case of the carbonates the ores are " roasted " to reduce them to oxides. These are then treated in the blast furnace by exposing them at a high temperature to the action of carbon and the carbon gases from the coke which is used as fuel. In charging the furnace the fuel and the ore, along with a certain quantity of lime which helps to form a flux, are placed in alternate layers. The chemical reactions which take place are of a complicated nature, but the essential result is that the carbon from the fuel combines with the oxygen contained in the ore. The combination of carbon and oxygen thus effected passes away in a gaseous form, the iron is set free, and, being in a fluid condition, falls to the bottom of the furnace. At the proper moment it is allowed to flow in a stream from the furnace and run into open sand moulds in the floor. The iron thus moulded cools down in the form of rough bars having a semicircular section and called "pigs."

The " pig iron " thus got from the ore is the first iron product obtained, and forms the basis from which all the other forms of manufactured iron and steel are produced.

Pig iron is the form of iron in its original and crudest state as obtained from the blast furnace. When this has been remelted at least once it undergoes changes, not in chemical composition as shown by analysis, but in those qualities which are important to the engineer and whose causes are not always at once apparent. When pig iron, colloquially referred to as "pig," has been remelted it is spoken of as "cast iron."

When the iron founder wishes to produce cast iron having given qualities both of strength and of facility in working, he mixes quantities of pig iron having certain properties which are well known, in such proportions as he knows will give him the particular quality of metal desired. The proportions required are known from experience, the properties of the available kinds of pig being either known previously or found at the time by chemical analysis and test.

Of the carbon which exists in any sample of pig iron or cast

iron, part is chemically combined and part is mechanically mixed and has a separate existence as "graphite." The existence of the graphite is clearly visible in the blackening of anything which is rubbed among filings or cuttings taken from a metal with a high percentage of "free" carbon.

In the following table are given the compositions of a number of typical samples of pig iron.

TABLE I. COMPOSITION OF PIG IRONS OF VARIOUS GRADES *

	Grey Cinder Pig.	Common White Pig.	Mottled Iron.	Best Mine Pig.	Foundry Pig.	Grey.	Mottled.	White.
Iron . .	93·55	95·27	93·29	94·56	93·53	90·24	89·39	89·86
Combined Carbon .	2·80	2·42	2·78	0·04	—	1·82	1·79	2·46
Graphite .	—	—	1·99	3·10	3·44	2·64	1·11	0·87
Silicon . .	1·85	0·36	0·71	2·16	1·13	3·06	2·17	1·12
Sulphur .	0·14	0·87	trace	0·11	0·03	1·14	1·48	2·52
Phosphorus.	0·66	1·08	1·23	0·63	1·24	0·93	1·17	0·91
Manganese .	—	—	—	0·50	0·43	0·83	0·60	2·72

144. Cast Iron, for Use in the Foundry—Pig iron when remelted in a "cupola," which is really a miniature blast furnace, and afterwards run into moulds so as to form castings when cool, is spoken of as "cast iron." The composition of cast iron is similar to that of the pig iron from which it has been produced, the precise quantities of carbon, combined and free, silicon, sulphur, manganese, and phosphorus contained in the iron being dependent on the kinds of pig used in the mixture, and to some extent on the number of times of remelting and upon the temperatures employed.

TABLE II. TURNER'S ANALYSIS †

No. of Melting.	Total Carbon.	Combined Carbon.	Silicon.	Sulphur.	Manganese.	Phosphorus.
1	2·67	0·25	4·22	0·03	1·75	0·47
8	2·97	0·08	3·21	0·05	0·58	0·53
12	2·94	0·85	2·52	0·11	0·33	0·55
14	2·98	1·31	2·18	0·13	0·23	0·56
15	2·87	1·75	1·95	0·16	0·17	0·58
16	2·88	—	1·88	0·20	0·12	0·61
18	—	2·20	—	—	—	—

* From Spretson's "Casting and Founding."

† Taken from Johnson's "Materials of Construction," page 101.

The effect of remelting appears to result in some changes in composition. The remelting has apparently an oxidizing effect, as shown by the steady diminution of the manganese and silicon. At the same time the sulphur content is seen to increase steadily, the additional amount being apparently taken up from the furnace gases. The further effect of the decreased silicon and increased sulphur appears to be to eliminate the graphite and cause the whole of the carbon to become combined. Thus from an iron whose carbon was in its original state almost wholly free was obtained by the simple process of continual remelting a metal whose carbon was almost wholly combined. The result was a white iron extremely hard and at the same time brittle, and which tests carried out by Sir W. Fairbairn showed to have a low tensile strength. The effects of remelting on the strength properties of cast iron are largely the results of changes in chemical composition brought about by increase of the carbon due to oxidization of some of the iron, and also to the fact that the free carbon becomes combined carbon. Those effects which are due to oxidization are largely dependent on the type of furnace used for the remelting, that furnace which allows an abundance of oxygen to reach the hot fuel producing the more marked effects.

The effects of the composition of cast iron on its strength properties are shown very forcibly in the figures which are given later. Here the compositions and strength properties of three typical kinds of cast iron in use at the present time for machine parts are given (see p. 320).

145. The Constituents of Cast Iron—CARBON—Molten cast iron is really a nearly saturated solution of carbon in iron, added to which are several minor constituents whose presence has considerable influence on the physical properties of the metal. The pure iron when fused is capable of taking up about $3\frac{1}{2}$ per cent. of its weight in carbon, though this amount may be considerably increased by the presence of some other constituents, such as manganese and chromium. The average value of the carbon content may be conveniently put at 3 per cent. dissolved in the iron, when molten. As this solution of carbon in iron cools to a point approaching solidification, the whole of the carbon may be retained in combination, when the metal is hard and brittle and called *white iron*; or as the metal cools the greater part of the carbon may be separated out as small particles of graphite which remain mechanically mixed with the iron, throughout which it is uniformly distributed, in which condition the metal is referred to as *grey iron* or *soft iron*, this latter term suggesting a relatively soft and easily tooled material. A third condition is reached when part of the carbon is combined and the remainder distributed throughout the white as patches of

grey, in which state it is called *mottled iron*. The condition of the iron which results on cooling depends partly on the constituent elements and in part on the rate at which the mass is cooled.

Turner says that "probably no other constituent in cast iron is of importance equal to that of the combined carbon, and the influence of the other elements is largely due to the effect they produce in increasing or diminishing the combined carbon."

SILICON—The quantity of silicon present in cast iron or in pig iron varies enormously in different samples. A reference to the figures will show a variation between 0·36 and 4·22 per cent. in the samples quoted. These figures are fairly representative of the minimum and maximum percentages of silicon present in iron, though specially prepared "silicon pig" contains as much as from 10 to 18 per cent. The presence of more or less silicon in cast iron is of importance in so far as it influences the condition under which the carbon is held, whether free or combined, and consequently the softness or hardness of the iron. An excess of silicon renders the iron weak and brittle.

Experiments by Prof. T. Turner on the influence of silicon show that as the percentage of silicon increased from 0 to 10 per cent., the *strengths* in tension, compression, and cross-breaking, as well as the modulus of elasticity, attained their maximum values with from 1 to 2 per cent. of silicon, after which they continually diminished. The *hardness* diminished to its lowest value at about 2½ per cent. silicon and then steadily increased. These figures are given on p. 314. They are confirmed by the results of other experiments. The precise effect of the addition of silicon seems to be, according to Turner, first, in small quantities, to produce sound castings by eliminating blowholes; then, when this soundness has been attained with the graphite at a minimum, it is found that the material attains its greatest compressive strength, and at the same time its greatest density. With a still greater percentage of silicon the combined carbon diminishes, with a corresponding increase in graphite; this tends to diminish brittleness and increase tensile strength. When the maximum strength has been attained, further silicon and its consequent graphite renders the cast iron soft and weak. It is found that increase of graphite leads to increase of deflection in cross-breaking tests, but when the maximum deflection has been attained by the addition of silicon, any further addition produces an opposite effect. The effect of silicon on the amount of shrinkage of cast iron is important, and the general effect appears to be that the shrinkage increases as the hardness increases. The hardness depends on the amount of combined carbon present,

and as this may be regulated, within certain limits the shrinkage can also be controlled by the addition of silicon.

MANGANESE—Just as the effect of silicon in cast iron results in a softening tendency, manganese may be regarded as a hardener. When the percentage of manganese present does not exceed 1 per cent. the effect of its presence is small, but when 1·0 per cent. is exceeded the effect becomes more marked, and with 1·5 per cent. the iron is much harder and may be relied upon to take a high polish. When 1·5 per cent. is exceeded and, at the same time, the amount of the silicon is small, the cast iron is found to have a glistening fracture and is then known as "spiegeleisen." With a still larger amount of manganese up to, say, 20 per cent., the metal ceases to be the white iron of spiegeleisen and becomes a light-grey brittle material which may easily be broken up in a mortar. It is then known as "ferro-manganese," a form of iron greatly prized by the steel-maker. Ferro-manganese contains, besides its iron and manganese, a certain amount of carbon, and it is from this carbon contained in the ferro-manganese that the certain definite proportions of carbon are added to the steel at certain stages of one or two of the steel-making processes. Although in general the effect of manganese is to act as a hardener, cases have been observed where the addition of manganese has resulted in an increase of softness, this being due probably to a partial neutralization of the silicon and sulphur.

Turner holds that the presence of manganese in foundry practice is beneficial in two ways, for its own presence results in hardness and closeness of grain, and, indirectly, it retards the absorption of sulphur during remelting.

SULPHUR—The amount of sulphur in foundry iron ought not to be more than 0·15 per cent., and, if possible, should be less than this amount. The presence of sulphur in appreciable quantity renders the iron, in the form of pig, undesirable for either steel-making or the manufacture of wrought iron in the puddling furnace, and, in foundry iron, the sulphur increases the proportion of combined carbon, and the material is weak, hard, and brittle. For the above reasons sulphur in iron is not only useless but positively deleterious.

PHOSPHORUS—The presence of this element in iron is less objectionable than that of sulphur, and for some purposes it may be distinctly useful. About 0·55 per cent. is considered a reasonable quantity for ordinary castings. More than this, say up to 1 or 1·5 per cent., renders the metal very fluid and conduces to fine and clear castings, but in obtaining these, strength has to be sacrificed. With more than this, say up to 5 per cent., the excellence of the impressions of the moulds increases but the metal becomes almost too weak and brittle.

Where the pig is to be used for the manufacture of steel, either from wrought-iron bars by the "cementation" process or in one of the acid processes, it is important that the percentage of phosphorus be kept very low, and, in any case, it ought not to exceed 0·06 per cent., and it is still better if kept below this limit.

146. Influence of Various Constituents: Summary—The above may be briefly summarized as follows:—

Carbon—About 3 per cent. total on an average. The greater the proportion combined, the harder, more brittle, and weaker the iron.

Silicon—From 0·36 to 4·22 per cent. Increase of Si increases the free carbon, tending towards more strength and elasticity, which attains its greatest value with from 1 to 2 per cent. of silicon. With greater free carbon the iron becomes softer. The iron reaches its maximum of softness with about 2½ per cent. silicon. On the other hand, when the iron has reached its minimum of graphite, the compressive strength is greatest and also its hardness.

Manganese—With more than 1·0 per cent. the iron grows harder as the manganese increases up to about 1·5 per cent. In general, the effect of manganese is the reverse of that of silicon; manganese is a hardener, while silicon acts as a softener.

Sulphur should not exceed 0·15 per cent. in foundry iron, and, if possible, should be less. It increases the proportion of combined carbon, thus tending towards weakness, hardness, and brittleness.

Phosphorus—A reasonable percentage in ordinary castings is 0·55 per cent. From 1 to 1·5 per cent. renders the metal very fluid and tends to fine castings, with increased weakness. If increased beyond this the casting improves but the metal becomes very weak and brittle.

147. Cast Iron in the Foundry—In most respects the manner of preparing the moulds for foundry use and the materials used in their formation are matters which concern the foundryman rather than the engineer for whom the castings are made, but there are one or two essential points affecting the engineer which ought to be mentioned here.

Moulds are made in either "green sand," "dry sand," "loam," or in "chills." The great majority of the moulds used are of the first class, the sand being contained in moulding boxes, which are generally made in two halves. These are much cheaper to make than dry-sand or loam moulds, where a number of castings are required, but "patterns" or "models" are necessary. In dry-sand moulds the sand is mixed with earth of a loamy nature and is of such consistency that the mould can be formed without a pattern and afterwards allowed to dry and be finished with

some sort of cutting tool. The loamy character of the sand allows it to be cut after drying and at the same time to remain sufficiently hard to contain the incoming molten metal without distortion. Dry-sand moulds are supposed to give especially sound castings. Loam moulds are made without patterns and are generally employed for large castings whose shapes are those of more or less regular curved surfaces, such as pipes, large cylinders, screw propellers, and the like.

In cases where it is desired to produce a hard surface on part or the whole of the casting, the mould, or that part of it where the hardness is required, is made of some material that will cool the molten iron quickly as it runs into the moulds. To do this, what are called "chills" are used; these may be thick or thin, according to the degree of hardness required. When, for example, it is desired to cast a roller with the main part of the cylindrical surface sufficiently hard to give a polished surface when finished by grinding, the end portions which do not come under wear are cast in sand and remain soft, whereas the part which is to be hard is cast in chills. These may consist of rings of iron whose thickness will depend on the degree of hardness desired. The particular kind of iron used where chilling is resorted to is generally a grey iron of a close-grained structure. The part of the casting which comes under the effect of the "chills" has its carbon, which in the grey iron is largely graphitic, changed into the combined form of "white" iron; this is extremely hard.

It should be noted that the molten iron must not be allowed to come into contact with cold metallic surfaces in the chills; therefore these must be warmed to 150 to 200° Cent. before the metal is poured. The function of a chill mould is to cool the outer surface of the casting as it is setting more rapidly than would be the case normally in a sand mould. By bringing the new metal under the influence of metallic surfaces the heat is conducted away more rapidly; and the degree of hardness depends largely on the rapidity of the cooling, which again is dependent on the bulk of the cooling metal.

148. The Chief Mechanical Properties of Cast Iron—For the purposes of the engineer the mechanical property which is the most important is "softness" or facility in being turned, planed, or otherwise machined. This quality is often of more importance than that of mere strength. Whether or no a particular casting has the property of soft-cutting is found out most readily by the man who does the work, whether by hand or in a machine; but samples of a particular iron may be subjected to one of the various hardness tests, such as the Scratch test, Martens or Brinell indentation test, the "Chisel" type of indentation test, or the schleroscope rebound test. The degree of hardness required

depends on the purposes for which the finished casting is to be used. For large castings where only rough planing has to be done on seatings and a few large holes have to be drilled, the degree of hardness matters little so long as the metal can be taken off quickly and cheaply, but in engine parts, while the metal must be soft enough to allow of clean and accurate machining, wearing parts must be sufficiently hard to give good wearing surfaces. There are cases where the power to resist wear is of paramount importance, and a hard iron must be used which at the same time can be cut or ground into form. The degree of hardness of a particular iron is largely bound up with the quantity of silicon present.

The following table from the "Journal of the Chemical Society" contains some important and interesting figures relative to the influence of silicon on hardness.

TABLE III, SHOWING THE INFLUENCE OF SILICON ON TENACITY AND HARDNESS OF CAST IRON *

No.	Silicon per cent.	Tensile strength per sq. in.		Hardness. Schlero-scope.	Working qualities found in the shops.
		Lb.	Tons.		
1	0·19	22,700	10·13	72	Very hard indeed.
2	0·45	27,600	12·32	52	Very hard. Not so hard as No. 1.
3	0·96	28,500	12·72	42	Hard. Softer than No. 2.
4	1·96	35,200	15·71	22	Good, sound, ordinary soft-cutting iron of excellent quality.
5	2·51	32,800	14·64	22	Rather harder than No. 4.
6	2·96	27,400	12·23	22	Like No. 4.
7	3·92	25,300	11·25	27	Like No. 6, but rather harder.
8	4·75	22,700	10·13	32	Rather harder than No. 7, though not unusually hard.
9	7·37	12,000	5·357	42	Still harder, cutting very like No. 10.
10	9·80	10,600	4·732	57	Hard-cutting iron, though still softer than No. 1.

Reference to the figures in the above table shows that, as the percentage of silicon increases from 0·19 per cent. to nearly 10 per cent. by gradual increments, the metal, which with 0·19 per cent. silicon is extremely hard, becomes gradually softer until

* Turner, "Journal of Chemical Society," 1885, and Johnson's "Materials," page 107.

with 1·96 per cent. what may be called normal cast iron is obtained suitable for engine and machine castings which have to be

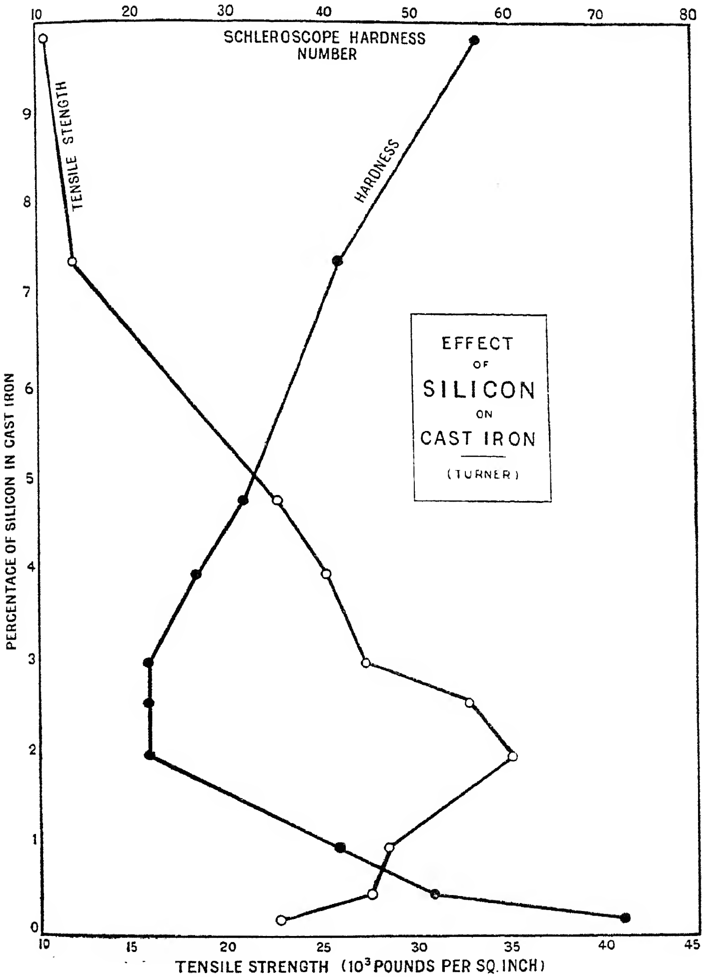


FIG. 193.

machined. Beyond this point, increased silicon means a return of hardness, but the metal is never so hard as in its original condition. Further, with this increase of silicon, producing first

a diminution, then an increase of hardness, there is a change in the tensile strength of the metal. As the hardness becomes less the tensile strength increases until it attains a maximum value with 1·96 silicon, or at the point of minimum hardness. These results are plotted on Fig. 193.

149. Results of Tests of Strength Properties of Cast Iron—Apart from chemical analysis, the tests which are usually carried out for the purpose of fixing the strength properties of a given sample of cast iron are :—

Tests for—

1. Tensile strength.
2. Compressive strength.
3. Transverse or cross-breaking strength.
4. Torsional strength.
5. Hardness.

Of these, the most usual are Nos. 3, 1, and 2, in that order, for the simple reason that cross-breaking tests are easy to make, do not necessarily require special apparatus, and can be made on castings without previous machining.

150. The Tensile Strength of Cast Iron—In order to find the strength of a given sample of cast iron to resist tensile stress, a specimen specially prepared for the purpose is exposed to a gradually increasing pull, in one or another of the tensile testing machines described, until it breaks. Before the test the cross-dimensions of the specimen are measured, and from these the area of the original cross-section calculated. Call this A. The load P causing fracture is observed and noted. Then the load per unit area or ultimate tensile stress will be $f=\frac{P}{A}$, generally expressed in lb. per sq. in. or tons per sq. in.

As there is practically no permanent elongation after fracture in cast iron, this does not require to be measured.

The most important points to be noted when making tensile tests of cast iron are (1) that the load shall be steadily and gradually applied, without any suspicion of jerk or jar, and (2) that the specimen tested is symmetrical in form and that the load be applied through its geometrical axis. This latter is very important, as an eccentric load may break a bar sooner than a larger but axial load. It is for this reason that wedge grips are not suitable for cast iron (or, indeed, any other brittle material). A cast-iron tensile test bar should be made as long as the material will allow, so as to give a maximum of flexibility, and it should be turned with enlarged ends either screwed or made to fit into sockets which are provided with spherical joints or some similar manner of ensuring the axial direction of the pull. There are several ways of doing this; examples are shown on Figs. 194,

195, 196, and 197. Of these, the grips on Figs. 194 to 196 are provided with spherical bearings and are more especially suitable for long than short specimens. That at Fig. 194 is provided with a solid nut into which the end of the specimen may be quickly screwed, the nut being held from behind. This form of holder is quickly adjusted and little time is wasted in setting the specimen in place.

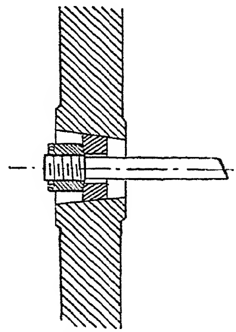


FIG. 194.—Cast Iron Tension Holder.

In the form shown in Fig. 195, suitable for large specimens, the ends bear against a split ring provided with a spherical surface. This form takes a little longer to adjust than the last.

The grip shown on Fig. 196 is really the most convenient where a number of medium-sized specimens are to be tested, especially where the work has to be got through quickly. In setting the specimen it is only necessary to slip its end into the split screw die and screw this into the holder.

The grip of Fig. 197 has been employed by the authors for use with very small specimens tested in a light testing machine provided with wedge grips. Here the flat portion of the holder

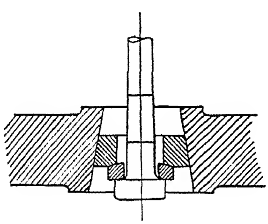


FIG. 195.

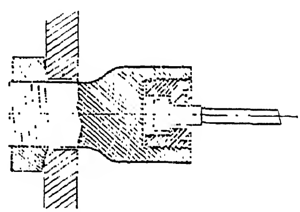


FIG. 196.—Tension Test Bar for Cast Iron.

is held between the wedges, and the screwed end of the specimen is fitted into the shank part. The wedge grips are so held that the pull is transmitted through two pin joints placed at right angles and it may be expected to be reasonably central.

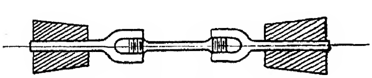


FIG. 197.

The tensile testing of cast iron presents difficulties of two kinds. In the first place, this material is always more or less brittle, and for this reason care must be exercised when turning the specimens to form them with gentle curves and with no sudden changes

in diameter ; and for the purpose of helping to keep the pull axial they should be made as long as the material will permit. Secondly, a difficulty is often experienced in getting a suitable specimen out of the particular piece submitted for testing. Where the test specimen is turned from a special casting made as part of the main casting, or specially cast for the purpose, there need be little difficulty, as the casting can be made of such a size as will suit the grips available ; but where specimens have to be cut out of existing castings, such as the arms and rims of flywheels or the frames of light textile machinery, the dimensions are often very limited, and for this reason it is absolutely necessary in a general testing laboratory to possess a large number of grips which may be expected to accommodate specimens from all possible sources and of any size.

Besides the specimens which have been given, the one shown on Fig. 198 possesses all the merits of a good cast-iron tension piece. This is a long specimen, with a parallel portion in the

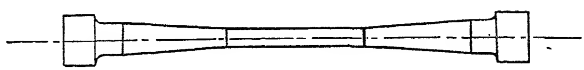


FIG. 198.—Long Cast Iron Specimen.

middle, gradually approached from a larger diameter nearer the ends, and its ends are turned to rest in pairs of spherical or conical holders. This particular form is rather costly for ordinary commercial work, but, at the same time, it may be expected to give the best results and should be approximated to as far as possible. It is also a suitable form to be used in careful elastic experiments.

Another point which must be taken into account when selecting specimens and grips is that of speed. Where large batches are to be tested, it becomes extremely important that the specimen ends and the holding grips should be such that the specimen can be placed quickly in the machine and quickly removed after it has been tested. From this point of view some types of machines are far more convenient than others, and this is a point to be watched when selecting a machine.

151. Carrying out a Tensile Test of Cast Iron—The process of making a tensile test of a sample of cast iron is a fairly simple one and the points for observation few in number. When the bar has been fixed in the machine, with the grips suitably adjusted and a slight initial pull on the bar, the only thing to be done in the case of a commercial test is gradually to increase the load until the bar breaks, which it will do quite suddenly. When this happens the breaking load is noted, and this, divided by

the original area of cross-section, gives the breaking stress per unit area.

In English-speaking countries the above is given as so many lb. or tons on the sq. in.

As cast iron is a brittle material and breaks off quite short, there is no permanent elongation capable of being measured with rule and compasses, and no attempt is made to say what this is. The results in the following table may be noted as typical.

Table IV.—Tensile Strength of Cast Iron

Description.	Tensile strength per sq. in.		Authority.
	Lb.	Tons.	
Engine castings	28,450	12·7	Popplewell.
Textile machinery	25,760	11·5	,,
Sewing machine	31,580	14·1	,,
Flywheel	19,040	8·5	,,
Pipes	31,140	13·9	,,
,,	28,450	12·7	,,
Flywheel	25,540	11·4	,,
High tension	31,360	14·0	,,
,, ,, in sand	35,840	16·0	,,
,, ,, in chill	44,800	20·0	,,
,, ,,	35,840	16·0	,,
Case hardened	23,070	10·3	,,
Rollers	21,730	9·7	,,
Various engineering castings	23,520	10·5	,,
,, ,, ,,	32,930	14·7	,,
,, ,, ,,	35,840	16·0	,,
,, ,, ,,	29,790	13·3	,,
Rough castings	19,260	8·6	,,
Ordinary machine castings	26,430	11·8	Secundo and Robinson.
,, ,, ,, (9)	22,620	10·1	Kirkaldy.
,, ,, ,, (10)	28,000	12·5	,,
,, ,, ,, (10)	24,860	11·1	,,
,, ,, ,, (15)	23,070	10·3	,,
,, ,, ,, (13)	21,730	9·7	,,
,, ,, ,, (850)	21,170	9·45	Anderson.
Hydraulic cylinder	15,680	7·0	Popplewell.
Combined C.O. 83 per cent., Graphite, 2·17 per cent.	21,730	9·7	Popplewell and Coker.
Constructional cast iron	16,580	7·4	Hodgkinson and Fairbairn.
,, ,, (81)	15,230	6·8	,, ,,
Good pig iron	19,940	8·9	Thurston.
Tough cast iron	24,976	11·15	,,
Hard ,,	30,020	13·4	,,
Good tough gun iron	29,950	13·37	,,
Remelted cast iron—			
1st melting	14,000	6·25	,,
2nd ,,	22,960	10·25	,,
3rd ,,	30,240	13·5	,,
4th ,,	35,730	15·95	,,

TABLE IV.—TENSILE STRENGTH OF CAST IRON—*continued*

Description.	Tensile strength per sq. in. Lb.	Tons.	Authority.
Varying times of fusion—			
Fused for ½ hour	18,820	8·4	Thurston
,, ,, 1 ,,	20,160	9·0	,,
,, ,, 1½ ,,	24,420	10·9	,,
,, ,, 2 ,,	35,170	15·7	,,
Soft C.I.:			
C comb. C free Si. Mn. S. P.			
0·459 2·603 3·011 0·18 0·031 0·173	12,720	5·68	Nicolson and Popplewell.
Medium C.I.:			
0·585 2·72 2·703 0·588 0·061 0·526	25,200	11·25	,, ,,
Hard C.I.:			
1·150 1·875 1·789 0·348 1·161 0·732	17,870	7·98	,, ,,
The last set of figures has already been quoted.			
From report on "Metal for Cannon," 1856:			
Best	30,700	13·7	Wade, quoted from Unwin.
Average	20,380	9·1	,, ,,

Reference should also be made to the figures on page 314, which show the interdependence of hardness and tensile strength with percentage of silicon.

There may be reasons, not here specified, why in some cases different metals are used for apparently the same purposes. It is often the case that qualities of hardness may carry more weight than mere tensile strength.

152. The Crushing Strength of Cast Iron—It is often the case that the tensile rather than the crushing strength is specified, although the compressive strength may be of more importance in actual use. This is due in most cases to the authorities who frame the specifications, and their reason is probably due to the fact that it is, as a rule, easier to get a tensile than a crushing test carried out. However this may be, it is less difficult now than heretofore to find machines fitted with the necessary tackle for compressive tests, and now most of the recognized testing houses are provided with such machines.

Apart from the accurate indication of the load, the essential points to be noted are:—

(*a*) The compression dies must have truly plane surfaces.

(*b*) They should approach one another in a rigid straight line normal to their surfaces, with no side movement tending to cause a shear on the specimen.

The most common form for a cast-iron test piece is that of a

solid cylinder whose length is from two to two and a half times its diameter, preferably the latter. During the compression of cast iron, as with most brittle materials, failure takes place by shearing diagonally across the specimen, as indicated by the dotted line on Fig. 199. The inclination of the plane of fracture, θ in the figure, has been found in cast-iron to be inclined at about 35 deg. to the axis of the piece. In order, therefore, that the specimen shall have freedom to shear at its natural angle, the length should be fully twice the diameter. Otherwise the crushing load will appear greater than its true value. This point is of considerable importance, and, although often taken into account in cast iron, is nearly always neglected in such materials as concrete. For these the favourite form of compression specimen is a simple cube, and this does not allow of the free shearing action.

If there is ample material, the authors suggest that the crushing specimens shall be solid cylinders, 2·5 in. long and 1·128 in. in diameter. This diameter gives an area of cross-section of 1 sq. in. The writers have found this very convenient. Where size is limited by the dimensions of castings or by the power of the machine, smaller sizes must be used. Diameters of 0·798 and 0·564, giving areas of 0·50 and 0·25 sq. in., with lengths in proportion, will be found useful.

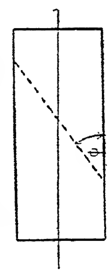


FIG. 199.

In carrying out a crushing test on a sample of cast iron the specimen is carefully placed between the platens of the machine, exactly in the centre, so that the load may be transmitted truly along the axis of the specimen.

Unless elastic measurements are desired for calculating Young's Modulus, the only actual measurement taken during the test is the crushing load. The load is gradually increased until failure takes place and the load causing this failure is noted. Failure takes place by diagonal shear, as already described, and this failure is usually accompanied by a loud report and at the same time the specimen flies into at least two parts, often more. In machines of the multiple-lever type the crushing to destruction is more violent than in hydrostatic pressure machines.

To prevent accidents from fragments it is usual to tie a piece of sacking around the specimen before the load is applied.

The precise machine to be used for crushing tests to destruction of cast iron is of little consequence so long as the load can be steadily applied and its magnitude given with accuracy. Most of the large universal machines, such as those of Buckton and Avery, can be used for compression tests, and besides these there

are several machines specially designed for crushing; one of the most convenient among these last is that of Messrs. Amsler-Laffon.

When the specimen is finally fractured, the load required to do this is observed and noted, the result being given as

$$\text{Crushing stress} = \frac{\text{Crushing load}}{\text{Area of section}}.$$

Below are given typical results of such tests:—

TABLE V.—CRUSHING STRENGTH OF CAST IRON

Description.	No. tested.	Ratio, Height Diam.	Crushing Stress per sq. in. Lb.	Crushing Stress per sq. in. Tons.	Authority.
Machinery iron . . .	12	1/1	145,600	65·0	Popplewell.
” ” . . .	6	2/1	107,500	48·0	”
“Soft” (see p. 324) .	2	2/1	60,260	26·9	Nicolson and Popplewell.
“Medium” (see p. 324) .	2	2/1	98,560	44·0	” ”
“Hard” (see p. 324) .	2	2/1	97,840	43·5	” ”
½ in. diam., various heights	2		155,200	69·3	Hodgkinson.
” ” ” ”	2	¼/½	142,200	63·5	”
” ” ” ”	2		134,400	60·0	”
” ” ” ”	2	½/½	123,200	55·0	”
” ” ” ”	2	¾/½	119,400	53·3	”
” ” ” ”	2	1½/½	119,400	53·3	”
” ” ” ”	2	2/½	111,100	49·6	”
” ” ” ”	2	3¾/½	77,160	34·4	”
Average cast iron. . .	8	1¾/1 to 2/1	126,100 to 87,580	56·3 to 39·1	Prof. C. A. M. Smith.
Light machinery iron, from 4 in. sq. bars to ½ in. sq. bars . . .	27	—	154,560 to 57,000	69·0 to 25·4	American Foundry-men's Association.
Analysis of the last material gave 3·52 per cent. free C, 2·04 per cent. Si, 0·39 per cent. Mn.					
Specially strong iron:					
Cast in sand . . .		2/1	143,400	64·0	Popplewell.
Cast in thin chill . .		2/1	170,200	76·0	”
Cast in thicker chill .		2/1	177,000	79·0	”

The above results make two things fairly clear, namely, that for a specimen of given diameter the crushing strength will be lower for the greater height, and that specimens cut from small castings yield higher results than similar bar specimens taken out of large castings.

TABLE VI. EXPERIMENTS WITH VARIOUS RATIOS OF HEIGHT TO SECTIONAL AREA

Description.		No. tested.	Ratio, Height/Area.	Crushing strength, per sq. in.		Authority.
Area, sq. in.	Height, in.			Lb.	Tons.	
0·500	$\frac{1}{4}$	1	0·50	109,000	48·7	Popplewell
,,	$\frac{3}{8}$	1	0·75	94,400	42·2	,,
,,	$\frac{1}{2}$	1	0·50	85,400	38·2	,,
,,	$\frac{3}{4}$	1	1·50	82,400	36·8	,,
,,	1	1	2·00	73,900	33·0	,,
,,	$1\frac{1}{4}$	1	2·50	76,000	33·9	,,
0·250	$\frac{1}{4}$	1	1·00	96,800	43·3	,,
,,	$\frac{3}{8}$	1	1·50	90,400	40·4	,,
,,	$\frac{1}{2}$	1	2·00	79,200	35·3	,,
,,	$\frac{3}{4}$	1	3·00	76,800	34·3	,,
,,	1	1	4·00	73,440	32·8	,,
,,	$1\frac{1}{4}$	1	5·00	72,000	32·2	,,
0·125	$\frac{1}{4}$	1	2·00	88,000	39·3	,,
,,	$\frac{3}{8}$	1	3·00	83,500	37·3	,,
,,	$\frac{1}{2}$	1	4·00	75,200	33·6	,,
,,	$\frac{3}{4}$	1	6·00	73·600	32·8	,,
,,	1·00	1	8·00	71,200	31·8	,,
,,	$1\frac{1}{4}$	1	10·00	74,000	23·0	,,
0·0625	$\frac{1}{2}$	1	8·00	74,500	33·3	,,
,,	$\frac{3}{4}$	1	12·00	70,500	31·5	,,
,,	1	1	16·00	68,000	30·7	,,
,,	$1\frac{1}{4}$	1	20·00	66,500	29·7	,,

The figures given above are shown plotted on Fig. 200, and serve to emphasize the fact that the result of a compressive test is largely dependent on the ratio of height to sectional area.

The following results of test on the samples of the three grades of cast iron operated upon in Dr. Nicolson's classic experiments on "rapid cutting tool steel" were obtained by Mr. Popplewell. Three grades were used, called respectively "soft," "medium," and "hard." These three grades of cast iron were found to have chemical compositions as given in Table VII, p. 324.

Before attempting a cast-iron crushing test care must be taken that the machine that it is proposed to use is sufficiently powerful for the purpose. The crushing strength of cast iron is somewhere in the neighbourhood of 45 tons per sq. in.—sometimes more than this—so that with a section of 1 sq. in. a machine of at least 45 tons capacity will be required. Where the maximum load of the machine is likely to be insufficient, the specimen must be smaller in cross-section. This may seem a somewhat unnecessary precaution, but the crushing strength of cast iron is high, and mistakes are sometimes made by which specimens fail to crush.

TABLE VII

	"Soft."	"Medium."	"Hard."
Carbon (combined)	0·459	0·585	1·150
Graphite	2·603	2·720	1·875
Silicon	3·010	1·703	1·789
Manganese	1·180	0·588	0·348
Sulphur	0·031	0·061	0·161
Phosphorus	0·773	0·526	0·732
TENSILE:			
Maximum stress, tons sq. in.	5·68	11·25	7·98
,, ,, lb. sq. in.	12,720	25,200	17,870
Young's Modulus, tons sq. in.	3,752	6,413	6,571
,, ,, lb. sq. in.	8,405,000	14,366,000	14,720,000
COMPRESSIVE:			
Maximum stress, tons sq. in.	26·92	44·00	43·50
,, ,, lb. sq. in.	60,300	98,560	97,440
Young's Modulus, tons sq. in.	3,812	6,259	6,491
,, ,, lb. sq. in.	8,540,000	14,000,000	14,540,000

The above figures represent the strength qualities of three typical samples of grey cast iron used in making castings for modern machinery.

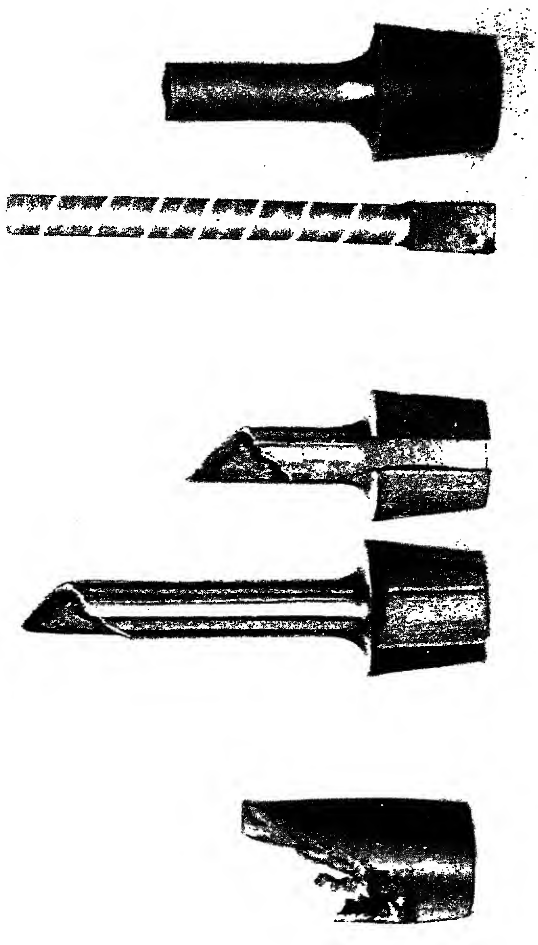
As the load increases it is observed that the specimen becomes slightly bulged near its middle. It is further seen on close observation that the surface of the specimen is covered with countless lines indicating planes of slide, which make angles of 45 deg. with the axis, covering the greater part of the bar with a fine network of lines. These can only be seen when the load is being slowly applied by hydraulic pressure. When fracture is approached a number of these cracks become merged in the one main plane of slide, which is inclined to the axis at a smaller angle than 45 deg.

On Plate XIII (*a*) is one-half of a broken cast-iron compression specimen.

153. Transverse or Cross-breaking Tests of Cast Iron—These appeal most strongly to the ironfounder and to the user of cast iron. The tests are simple and may be carried out in the absence of a testing machine if necessary. The results arrived at are definite, and, with the same kind of iron, are likely to be uniform; also, the loads are relatively small.

The form of test bar used in these is a rectangular beam supported at the two ends and loaded in the middle. The most common size of bar is 42 in. long, 2 in. deep, 1 in. wide, resting upon supports 36 in. apart.

PLATE XIII.

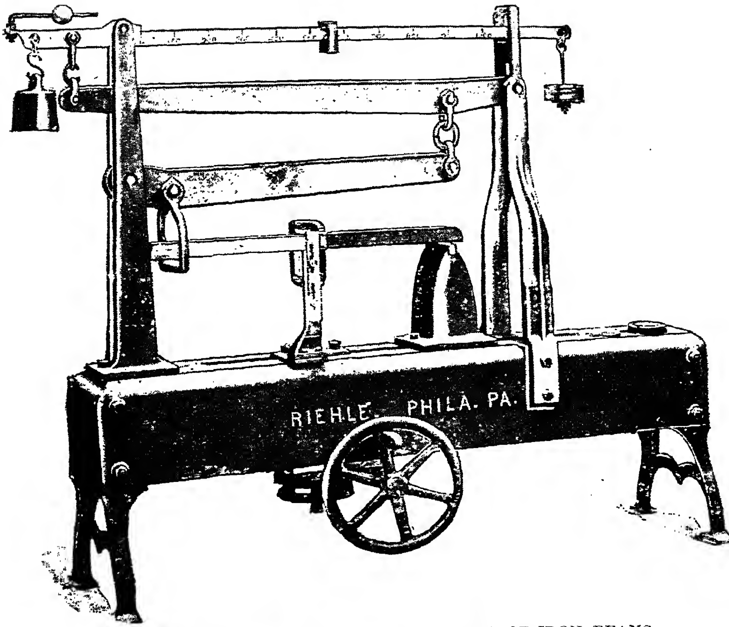


(a) BROKEN CAST-IRON COMPRESSION SPECIMEN

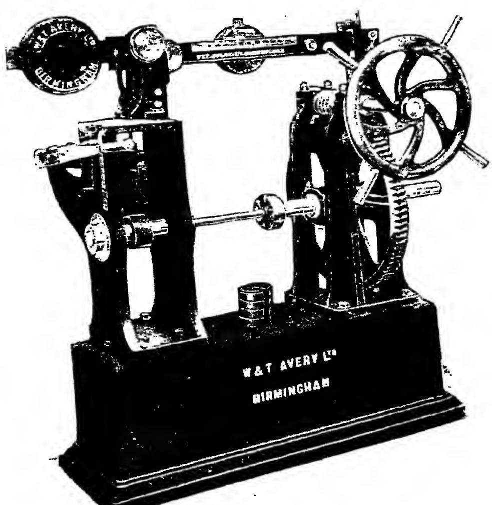
(b) BROKEN CAST-IRON TORSION

(c) MILD STEEL TORSION, SHOWING TWIST

(d) MILD STEEL TORSION, SHOWING FRACTURE



(*a*) RIEHLÉ TRANSVERSE MACHINE FOR CAST-IRON BEAMS



(*b*) AVERY TORSION TESTING MACHINE

Some authorities prefer a bar 1 in. square in section, tested on a 12-in. span.

In ordinary cases the bars are cast as nearly as possible to size and left with the scale on, while some authorities require them to be planed to the exact size.

There are occasions when it becomes necessary to use a size of bar other than the above, but such cases are comparatively rare.

In carrying out a transverse test of a cast-iron bar, two main

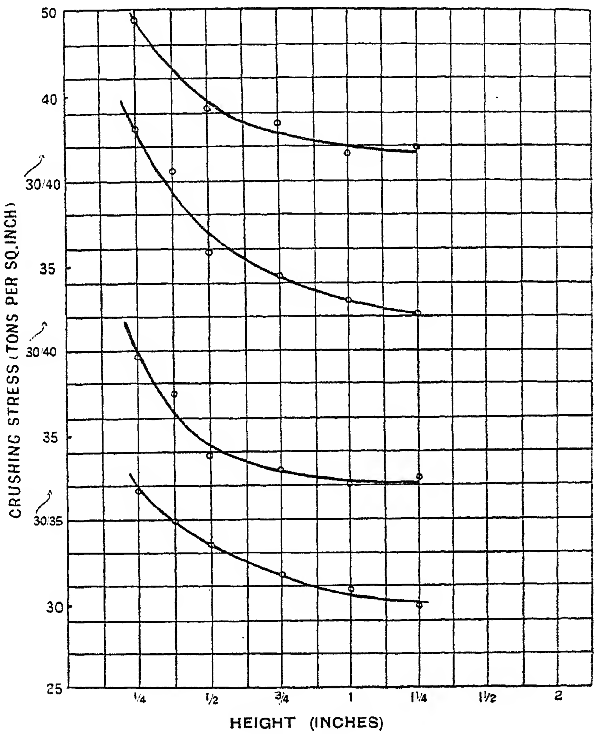


Fig. 200.

results are aimed at, namely, the *load necessary to break the bar in the manner specified* and the *greatest extent to which it is deflected* from the straight before being broken.

These results may be attained by using the cross-breaking tackle of a general testing machine, such as a Buckton or Avery machine, as already described; or a machine may be used which has been made specially for this purpose, such as that of Riehlé (Plate XIV (*a*)).

Making the Test—*Measuring the deflection* is done in one of several ways.

(*a*) Loads may be applied by uniform increments, the number of these being such as to give about ten readings, as, for example, if a beam is likely to break at somewhere about 3,000 lb., the increments of load might conveniently be 300 lb. When getting the deflection in this way the total deflection after each increase of load must be measured, most conveniently from one edge of the centre of the specimen to a fixed point or one connected with the supports. The total deflection right up to the point of fracture cannot be found directly in this way, as fracture generally takes place during an increment; but, by plotting the deflections as measured, drawing a line through the points and carrying it through to the maximum load as found, the total deflection can be determined with sufficient accuracy for ordinary purposes.

(*b*) The scheme shown on Fig. 201 is one used by the authors

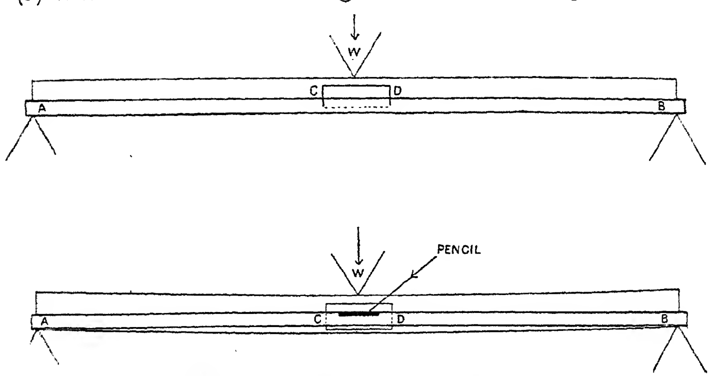


Fig. 201.—Method of Measuring Deflection of Cast Iron.

for commercial tests of cast iron where extreme accuracy is not needed and where speed is of importance.

On to the face of the beam is cemented a card CD, and a rigid straight-edge AB rests on the supports against the specimen, and close to it. As soon as the load W touches the bar and begins to deflect it, the observer draws a line, with a fine, hard pencil, against the ruling edge, and as the load goes on increasing, and with it the deflection, he continually moves the pencil point a distance of about 2 in. backwards and forwards against the straight-edge. This goes on until the beam breaks, when it is found that a band of pencil lines has been formed on the card, the breadth of this band representing the total deflection. If this plan is carefully carried out the total deflection may be expected to be given to within $\frac{1}{100}$th of an inch.

By a plan somewhat similar to the last a pencil slide is constrained to move in conjunction with the deflected beam by a wire or some rigid connexion and the pencil records the deflection on a rotating drum.

TABLE VIII. TYPICAL RESULTS OF TRANSVERSE TESTS OF CAST IRON

Description.	Width, in.	Depth, in.	Span, in.	Central breaking Load, lb.	Deflection, in.	Authority.
Special high quality .	1·00	2·00	36	4,410	0·53	Popplewell
Good cast iron . .	1·00	1·00	12	3,570	0·14	,,
,, ,, ,, . .	1·00	2·00	36	3,740	0·33	,,
,, ,, ,, . .	1·00	2·00	36	3,810	0·29	,,
,, ,, ,, . .	1·00	2·00	36	3,910	0·35	,,
Average machinery iron	1·00	2·00	—	2,470	—	,,
Good machinery iron	1·00	2·00	36	2,945	0·30	,,
,, ,, ,,	1·00	2·00	36	3,045	0·33	,,
Rough castings . .	1·00	1·00	12	3,190	0·12	,,

154. Coefficient of Transverse Strength—In the case of an elastic material under stress caused by bending, the conditions which exist are represented by $\frac{M}{I}=\frac{f}{y}$ or $f=\frac{M.y}{I}$, where M is the bending moment on the section, I its moment of inertia, and y the distance from the neutral axis to the point where the stress f occurs, generally that point where f has its maximum value. In the case of a rectangular section, d in. deep and b in. wide, the moment of inertia is $\frac{b.d^3}{12}$; here y is $d/2$ in. The modulus of the section $z = I/y$, which in this case is $b.d^2$ $M=f.z$ or $f.\ b.d^2/6$

This means that in the case of a beam of rectangular section the maximum stress in the material is $f=M/z$. This stress is tensile on the lower or convex edge and compressive on the upper or concave edge. This is quite true where the material is elastic at all stresses, and it would be possible on this basis, knowing the tensile strength of a given sample of cast iron, to calculate the bending moment required to break a rectangular beam of given dimensions, but cast iron may only be considered elastic at very low stresses and even then the elasticity is not perfect. Under small stresses the extension of the material in tension is sensibly proportional to the stress; but as the stress grows the increment of stretch grows also and the above equation fails in that it is only partially true. Therefore, in cast iron rectangular

beams the equation $f = 6.M/b.d^2$, where M is the bending moment at fracture, does not give truly the tensile stress at fracture but something which has too great a value.

For example, in the above table a certain 1 in. × 2 in. beam breaks on a 36-in. span with a load of 2,470 lb. Here by the elastic beam formula, $f = 6.M/b.d^2 = 6.2{,}470.36/4.1.4$. This works out at 36,700 lb. per sq. in. or 16·4 tons per sq. in. In this case the tensile strength found direct was 9·7 tons per sq. in. This represents the usual sort of result, the value as calculated from the beam formula in this way being generally in the neighbourhood of one-and-a-half times the strength from direct tests.

The quantity f, as calculated from the central breaking load in a transverse test, is called the "coefficient or modulus of transverse strength," or modulus of rupture, and must not be confused with the tensile strength of the material. For the reasons which have been given, its value differs from the direct tensile strength, and the ratio of one to the other is not always the same for the same kind of iron, the discrepancies being due to differences in form.

For convenience, the equation for f may be put in the form

$$f = \frac{6.M}{b.d^2} = 6.W.L/4.b.d^2, \text{ which reduces to}$$

$$= W.6.36./4.1.4 = 13{\cdot}5\ W \text{ lb. per sq. in. for 2 in.} \times 1 \text{ in. beams, and}$$

$$= W.6.12/4.1.1 = 18.W \text{ lb. per sq. in. for 1 in.} \times 1 \text{ in. beams.}$$

Equivalent Transverse Strength—Unless the specimen has been machined, which is comparatively rare, it is generally found, when the specimen comes to be measured, that it is slightly above or below the dimensions intended. Taking these dimensions into account, it is usual to calculate the load which would have broken a beam of the same material if the dimensions had been correct. It is difficult to cast test bars to the exact sizes. When a test is to be carried out, the span can be made as desired, often either 36 in. or 12 in., but the breadth and depth must be measured as they stand. It is convenient to take the measurements after fracture, and, as the pattern from which the mould has been made is necessarily taper, the breadth of the bar should be measured half-way between top and bottom. To find the equivalent strength, assume the span to be correct, call the measured breadth b', the measured depth d', the nominal breadth is 1 in. and the nominal depth 2 in. Then, as the strength is directly proportional to the breadth and to the square of the depth, the equivalent breaking load will be

$$W' = W.1.(2)^2/b'.d'^2 = W.4/b'.d'^2.$$

Below is given the heading of a usual form for reporting the results of transverse tests of cast iron.

MATERIALS TESTING LABORATORY

Date....................................

REPORT on cross-breaking tests of......................*samples of*..........................

submitted by..

Test Number	Mark on Specimen.	Initial Dimensions in inches.			Central Breaking Load.		Equivalent Load on a beam........ wide, ...deep, and ....span.		Central Deflection.	Remarks.
		Width.	Depth.	Span.	Lb.	Cwt.	Lb.	Cwt.	In.	

Also the following are the rules issued by the British Admiralty :

Admiralty Rules for Cast Iron

"Test pieces to be taken from such castings as may be considered necessary by the inspecting officer.

"The minimum tensile strength to be 9 tons per sq. in., taken on a length of not less than 2 in.

"The transverse breaking load for a bar 1 in. square loaded in the middle between supports 1 ft. apart, is not to be less than 2,000 lb."

Observations—When making cross-breaking or transverse tests of cast iron, especial care ought to be taken to have the load gradually and steadily applied. In cases where a machine with a heavy beam is being utilized for the purpose, some difficulty may arise by reason of the oscillation of the beam in conjunction with the springiness of the test bar, which may cause forces to come on the bar larger than those indicated, and for this reason the movements of the beam should be damped or controlled.

It is often convenient to compare the strength of cast iron as obtained from tests of 2 in. × 1 in. beams with that from 1 in. × 1 in. specimens. This is best done by comparing the coefficients

calculated for each. Thus for the 2 in. × 1 in. × 36 in. the coefficient is

$f = 13.5W$ (for 2 in. × 1 in.), and $= 18.W$ (for 1 in. × 1 in.) beams.

For the same material these two should be equal.

155. The Strength of Cast Iron in Shear and Torsion—When a bar of cast iron (see Fig. 202) has its end parts, B, B, held fast by being rigidly clamped in a fixed holder and the middle portion A is similarly clamped in a holder which is capable of being forcibly moved in a direction at right angles to the axis of the bar, its middle part will be cut out by the shearing of the metal when a load is applied. Such a test is, however, not altogether satisfactory in the case of brittle materials. With ductile materials and where the appliance is well made and accurately fitted, a reasonably correct result for the shearing strength of the material may be obtained. With cast iron it is impossible to avoid a certain amount of bending, and this militates against the validity of the result, as throughout the test the material cannot be said to be in a state of pure shear. Mr. E. G. Izod has carried out a number of shear tests, using an apparatus somewhat similar to that used by the author, on a number of different materials, including cast iron.*

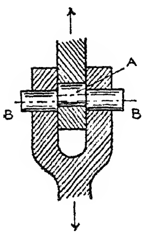


Fig. 202.

This apparatus, designed by Professor J. B. Johnson and also used by the authors, is shown on Plate XV (*a*). It will be seen that the middle portion is separate from the main block. The specimen is bolted so as to be held by the two halves of the main block and by the middle part. When a load is applied to the apparatus the middle portion is pushed downwards and cuts out a short length of the specimen.

The only way to subject a sample of any material, especially cast iron, to a stress which is one of pure shear is by twisting it. This is shown on Fig. 203.

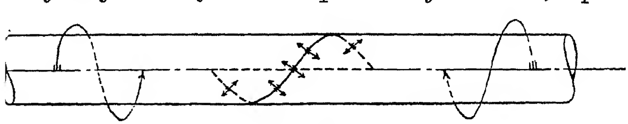
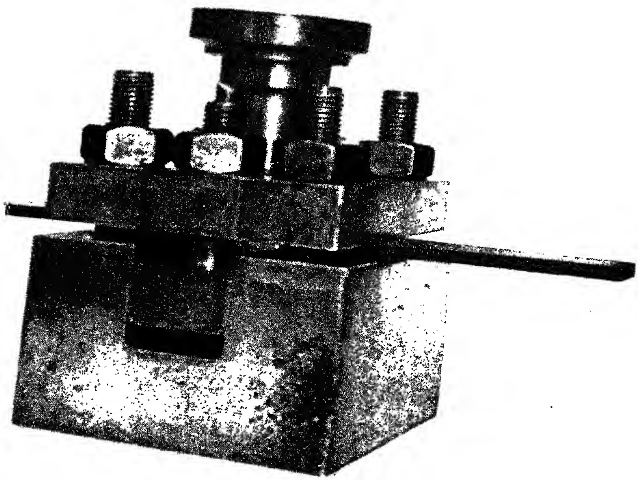


Fig. 203.

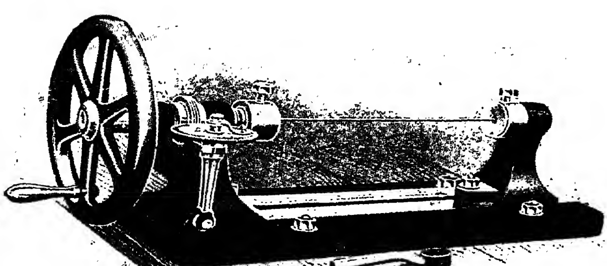
In this case the shearing effect results in tensile stresses across a helical surface, as shown by the arrows in the figure. The tensile stress is marked by arrows. As cast iron is capable of resisting

* Izod: Inst. Mech. Engs., 1906.

PLATE XV.



(a) ARRANGEMENT FOR TESTING IN SHEAR (JOHNSON'S DEVICE)



(b) MACHINE FOR TESTING WIRE UNDER TORSION

compressive better than tensile stress, fracture always occurs by the material tearing across spiral surfaces inclined at 45 deg. to the axis of the shaft. A photographic view of the two halves of such a specimen after being broken is shown on Plate XIII (*b*). The same kind of fracture occurs in all fairly hard brittle materials; the spiral fracture can be obtained very easily by holding a piece of blackboard chalk with one end in each hand and carefully twisting it until it breaks. For much the same reason that the transverse coefficient calculated is found to be greater than the direct tensile strength, so the coefficient of torsional strength is greater than the direct tensile strength.

156. Appliances—For shear tests the Johnson tackle (Plate XV,*a*) may be used, but apparatus of this kind suffers from the disadvantages mentioned above. To minimize these as far as possible, the middle slider should fit very closely between the fixed parts. All the cutting edges should be of hardened steel, and both side and central clamps should be capable of being screwed up very tight on the bar.

For torsion tests any one of the standard machines may be used, such as that of Buckton or of Avery. The torsion attachment to the former has already been mentioned in Chapter XI (see Fig. 176).

On Plate XIV (*b*) is seen a general view of the torsion machine made by Messrs. Avery & Co. In this the twist is applied through a worm wheel in whose boss one end of the specimen fits. The torsional couple at the other end of the specimen is measured through a system of levers, the end of the specimen being fixed into the small end of the first of these.

The best way of measuring elastic twist is explained by an inspection of Fig. 204 and Plate XVI. To the right of the figure is shown the circular shaft which is being twisted, the plan of it above and the end view of it below. The length of shaft which is being twisted is marked l; at one end of the length, at a point marked A, a mirror is fixed to the front of the shaft, and at the other end, B, is a second mirror. Martens' telescopes and scales are used. What is required to be found is the twist of B with respect to A. Looking through the upper telescope towards the mirror a reflected view of the scale is seen with the cross-hair of the telescope lying across it. As the shaft twists the line appears to move up the scale of the B mirror and down the scale of the A mirror. The distance from each mirror to each scale is carefully measured and called h, and the distance subtended by the angle between the rays of light from A and B is marked b, the angle of which b is the base in circular measure is $\frac{b}{h}$ radians. This can easily be converted into degrees. Larger angles beyond the yield

point can be measured by noting the movement of the hand-wheel attached to the worm.

157. Making the Tests—*Shear*—It has already been pointed out that in a shearing test of cast iron the specimen must be firmly held in the grips by tight clamping, and also the specimens should be accurately machined. The specimens are generally made with a rectangular cross-section, and it would appear that the best results may be expected from bars whose width is at least four times the depth. By making the specimen wide in this way it is able to bend the more easily, so as to accommodate itself to irregularities in the dies without cracking.

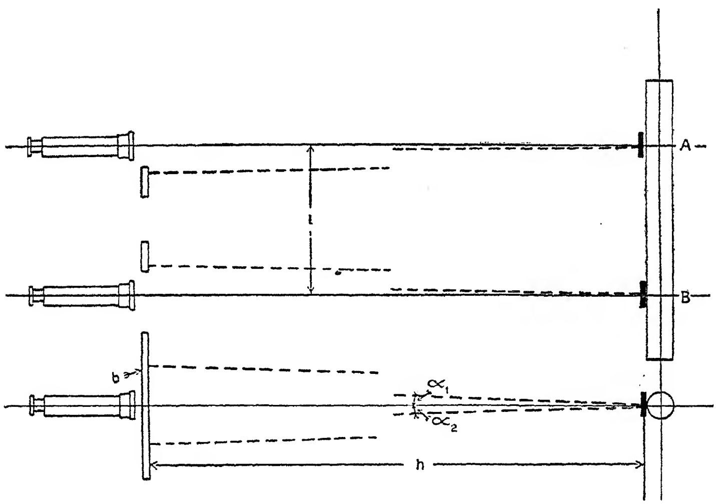


FIG. 204.

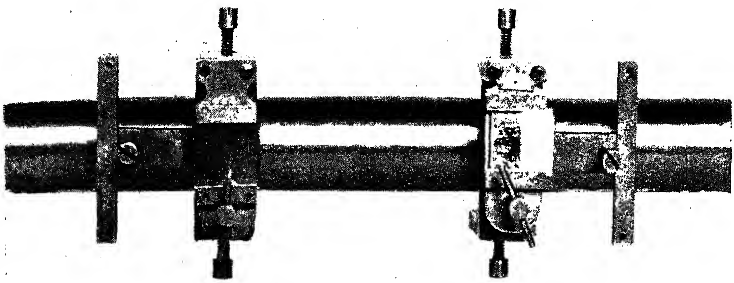
In carrying out the test, it is only necessary, after the specimen has been placed in its shackles, to place the tackle complete in a compression testing machine and to increase the load until the specimen is sheared. Then the ultimate shear stress,

$$f_s = \frac{\text{load causing shear}}{\text{total area of section.}}$$

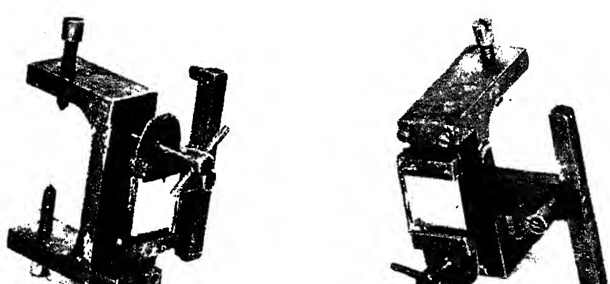
In the most usual form of shackle—the one described—the specimen is in double shear and the total area is twice the area of cross-section of the bar. In the rarer cases where the specimen is subjected to single shear the total area is simply the cross-section of the bar.

Torsion—When tested in torsion the only thing to be done is to place the turned specimen in a torsion machine and then

PLATE XVI.



(*a*) MARTEN'S MIRRORS FIXED ON TORSION SPECIMEN



(*b*) MARTEN'S MIRRORS SET IN HOLDERS

steadily increase the torque until sudden failure takes place, by tearing across a spiral surface.

What was said about the partial failure of elasticity in the case of transverse tests is true also for torsion. On the assumption of perfectly elastic conditions throughout, the coefficient of torsional stress,

$$f_{T1}=\frac{16T}{\pi D^3} \text{ (for solid shafts)} \quad . \quad . \quad . \quad . \quad (1)$$

This assumes the stress to be proportional to the distance from the centre of the shaft. When the stress is assumed to be uniform over the section at fracture the equation becomes

$$f_{T2}=\frac{12T}{\pi D^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The authors have found that by using equation (2) results are obtained which are nearer to those from direct shear and tension results than by using equation (1). Also this is more evident in hollow shafts than solid ones.

This would seem to indicate that the ideal test to give the ultimate shear strength of cast iron would be upon a thin hollow shaft specimen with the coefficient f_{T2} calculated from equation (2) above.

In the authors' elastic experiments for twist, the mirrors are fixed in adjustable holders (see Plate XVI (*a*)). The holders themselves are shown on Plate XVI (*b*).

Plate XIII (*c*) is a torsion specimen of mild steel which has been twisted a great number of times; the spiral white line was before the test an axial line on the surface of the bar.

TABLE IX. RESULTS OF TESTS OF CAST IRON IN SHEAR AND TORSION

Material.	Ult. tensile, tons per sq. in.	Direct shear, tons per sq. in.	Calculated stress max.		Authority.
			Stress max. at surface.	Stress uniform.	
Cast iron, A . .	9·7	14·8	—	—	Izod.
„ „ B . .	13·4	17·4	—	—	„
„ „ C . .	11·3	13·9	—	—	„
„ „ D′ . .	13·7	16·1	—	—	„
„ „ D″ . .	13·5	14·8	—	—	„
2·17% Graphite, 0·83% Combined C	9·7	9·5	Solid, 14·6	Solid, 10·9	Popplewell and Coker
C. A. M. Smith, p. 322	—	—	Hollow, 10·38	Hollow, 8·8	„

Plate XIII (*d*) shows the fracture of a mild-steel torsion specimen which is at right-angles to the axis of the bar.

Plate XV (*b*) shows a machine made by S. Denison of Leeds for twisting wire.

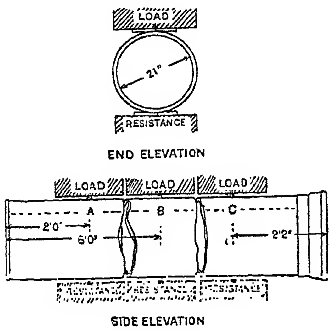
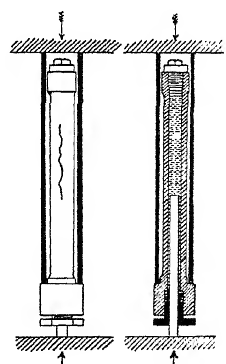


FIG. 205.

On Fig. 205 is shown a conventional test applied to cast-iron pipes.

158. Experiment on Thick Cylinder—The following case of tensile fracture caused by separation, in an approximately elastic material, should prove interesting. During 1889 this experiment, relative to a cast-iron thick cylinder, was made under the following circumstances. Soon after the completion of the 100-ton testing machine built by Messrs. Buckton for the Whitworth Engineering Laboratory at the Owens College, the ram cylinder of the hydraulic intensifier was accidentally burst. It was thought that the fracture was due to the impulsive action of the water upon the large piston, as it was stopped in its flow from the pressure tank. This sudden stoppage no doubt caused the pressure to rise above what it would have been when due to the static head alone. This would cause a similar rise in pressure to take place in the ram cylinder, and to this cause was attributed the burst, rather than to any inherent defect of the metal of which it was formed. The burst cylinder was replaced by a steel casting with satisfactory results.


ELEVATION & SECTION OF CAST IRON TEST CYLINDER.
FIG. 206.

At the suggestion of the late Professor Osborne Reynolds, under whose direction these experiments were carried out, Mr. Popplewell cut from the wall of the fractured cylinder a one-eighth-size model, similar to the original in all essential respects. An elevation and section of the model are shown in the accompanying Fig. 206.

It will be seen that the top end of the model is closed by a screw plug and several leather discs, the lower end being provided with a steel ram which was arranged to pass through a stuffing-box. In order to reproduce as completely as possible the conditions

under which the original cylinder was burst, the model was placed with the shoulder resting against one end of a length of steam pipe. The whole was placed between the compression platens of the testing machine as shown, after the cylinder had been filled with water, the ram placed in position and the gland screwed up tight. The load was gradually increased up to 0·65 ton, when the model was burst by the formation of a longitudinal crack as shown, about 4 in. long.

In addition to this model cylinder two tension test specimens were cut out of the wall of the burst cylinder at a position adjacent to the model cylinder. These broke at 6·95 and 7·29 tons per sq. in. respectively.

Using Lamé's formula to calculate the maximum tensile stress in the wall at the bursting pressure, the following data were made use of :—

Outside radius of model $= R =$ . .	0·515 in.
Internal ,, ,, $= r =$ . .	0·265 ,,
Diameter of ram = ...	0·445 ,,
Area of end of ram = . . .	0·155 sq. in.

From these data the bursting pressure is found to be

$$p = \frac{0{\cdot}65}{0{\cdot}155} = 4{\cdot}193.$$

The maximum stress as calculated from this bursting pressure is—

$$f = p\frac{(R^2 + r^2)}{(R^2 - r^2)} = 4{\cdot}193\frac{(0{\cdot}515)^2 + (0{\cdot}265)^2}{(0{\cdot}515)^2 - (0{\cdot}265)^2} = 7{\cdot}203 \text{ tons per sq. in.}$$

The mean of the two direct tensile results given above is 7·12 tons per sq. in., a result not far removed from that calculated from the bursting pressure, and well within the limits of possible accuracy.

Proceeding on the same lines with the original cylinder, it was found that, in order to cause the above stress, 7·12 tons per sq. in. would require a pressure of something greater than the available static pressure (5,400 lb. per sq. in.), which gave only about 4 tons per sq. in. It is reasonable to suppose that the increased pressure was due to inertia effects of the water.

This determination is rough, and ignores the fact that Lamé's equation remains true only so long as the material of the cylinder is elastic ; whereas cast iron is only partially elastic at the breaking points in tension. However, the approximate results yielded appear to be sufficiently near the truth for the purpose desired.

Though the quality of the iron was only moderately good, the

fracture was evidently caused by the pressure on the intensifier piston having been allowed to become excessive.*

159. Malleable Cast Iron—By malleable cast iron is meant iron cast in the usual way to the desired forms, but afterwards treated in such a manner that a large amount of its rigidity and brittleness disappear, leaving the material softer, more easy to deform permanently, capable of withstanding shocks without fracture, and with a tensile strength higher than before.

In other words, the material commonly known as cast iron is so treated as to be transmuted into a sort of wrought or malleable iron, whose strength and ductility are greater than those of cast iron but lower than those of wrought iron.

The process is especially useful in the production of small fittings such as bearings, brackets, and the like. These parts are easily and cheaply produced as castings, to be afterwards reduced to the semi-malleable state.

The Process—The most satisfactory results are obtained from castings high in combined carbon. The process used is fairly simple and consists in surrounding the material to be softened with an abundant quantity of a decarburizing substance such as black hammer scale, manganese dioxide, or red hæmatite ore. The whole is then exposed to a steady cherry-red heat for several days. On removing the castings from the furnace at the end of the required time a very definite change will have taken place both in the composition and in the properties of the iron, more especially the latter.

In a valuable paper on the "Manufacture and Properties of Malleable Iron Castings," † Mr. H. R. Stanford has given some highly useful information. The chief point raised has to do with the composition. Formerly it was the common idea that the effect of heating the castings for some days in the presence of an oxidizing substance was to remove the bulk of the carbon from the cast iron and so remove the ingredient chiefly responsible for the lack of plasticity. Mr. Stanford, however, denies the above contention, and shows by analysis of castings before and after treatment that the average loss of carbon from the forty-two cases quoted was 0·38 per cent. Considering that the original carbon was 3·04 per cent., this 0·38 per cent. of carbon lost is in reality a low percentage of the carbon in the original cast iron, equal to 0·38/3·04 or 12·5 per cent. of the carbon in the cast iron, or one-eighth.

The process by which malleable cast iron is prepared from

* The above occurs in the main in W. C. Popplewell's book on "Testing of Materials" (Scientific Publishing Co.). This was taken originally from an Owens College Calendar of 1890–91.

† From "Am. Soc. C.E.," vol. xxxiv, 1895.

castings of "white iron"—referred to by Mr. Stanford as "annealing"—consists in embedding the castings in manganese oxide or black iron scale, and packing them in annealing boxes (about 24 in. × 18 in. in cross-section). These boxes or "pots" are placed in the annealing oven and kept there for about five days. The time necessary for annealing depends partly upon the size of the castings treated, large taking longer than smaller ones. It is also influenced by the amount of sulphur in the castings. A casting containing 0·2 per cent. sulphur may take nine days, while a purer iron with only 0·04 per cent. sulphur may be completely annealed in a little over three or four days.

Mr. Stanford considers that what happens is nothing more or less than a release of the carbon combined in the white iron used and the transformation of the greater part of it into free carbon, with the result that a kind of grey iron is formed. It is further considered that the temperature employed (cherry red) is not sufficiently high to permit of the formation of the continuous mesh of carbon usual in grey cast iron, but that the carbon is distributed in minute separate particles. In this way the iron crystals will be in mutual contact as in steel.

At the conclusion of the annealing period, whose exact duration is fixed by experience, the oven is allowed to cool down. The first twenty-four hours of this cooling period is generally completed with closed doors.

160. Mechanical Properties of Malleable Iron Castings—The effect of subjecting castings of the right kind to the above annealing process is, broadly speaking, to somewhat reduce the carbon and increase the strength and the ductility. This is not difficult to appreciate when it is remembered that the original metal is a brittle and relatively weak material which by the process is transformed into something approaching low-carbon steel or wrought iron.

The following test results, taken from a paper by Mr. A. G. Ashcroft,* are instructive as showing the kind to be expected from this material.

Two brands of material were tested, denoted here as (A) and (B).

The analyses of these malleable iron castings were as follows:

	(A)	(B)
Total carbon	2·838%	1·470%
Graphite	2·300%	1·456%

* From "Am. Soc. C.E.," vol. xxxiv, 1895.

TESTS

Tension ;		
Yield point, per sq. in.	21,600 lb. (9·68 tons)	20,000 lb. (8·94 tons)
Maximum per sq. in.	46,930 lb. (20·95 tons)	46,140 lb. (20·60 tons)
Elongation on 10in.	1·47%	2·8%
Modulus of elasticity per sq. in.	$24 \cdot 53 \times 10^6$ lb. (10,950 tons)	$25 \cdot 93 \times 10^6$ lb. (11,620 tons)
Compression :		
Crushing, per sq. in.	47,380 lb. (21·15 tons)	48,380 lb. (21·60 tons)
Modulus of elasticity per sq. in.	$21 \cdot 70 \times 10^6$ lb. (9,680 tons)	$22 \cdot 94 \times 10^6$ lb. (10,240 tons)
Cross-breaking :		
Coefficient of transverse strength per sq. in.	88,030 lb. (39·30 tons)	64,500 lb. (28·80 tons)
Modulus of elasticity per sq. in.	$29 \cdot 34 \times 10^6$ lb. (13,100 tons)	$27 \cdot 12 \times 10^6$ lb. (12,330 tons)
Twisting :		
Torsion stress at elastic limit per sq. in.	20,970 lb. (9·35 tons)	20,000 lb. (8·93 tons)
Coefficient of torsional strength per sq. in.	57,340 lb. (25·6 tons)	62,030 lb. (26·8 tons)
Modulus of rigidity per sq. in.	$9 \cdot 453 \times 10^6$ lb. (4,220 tons)	$9 \cdot 064 \times 10^6$ lb. (4,120 tons)

In Mr. H. R. Stanford's paper, already referred to, figures are given which show the composition of the iron both before and after annealing, as well as the tensile strength and ductility. These are for the average of fifteen groups of forty-two samples.

The figures in the first table given refer to analyses, and the second to the mechanical properties.

TABLES SHOWING AVERAGE ANALYSES OF FORTY-TWO SAMPLES OF CAST IRON BOTH BEFORE AND AFTER BEING CONVERTED INTO MALLEABLE CAST IRON BY ANNEALING ; AND, IN THE SECOND, THE MECHANICAL PROPERTIES OF BOTH.

TABLE X.—COMPOSITION (PERCENTAGES)

Condition.	Total C.	Combined C.	Graphite C.	Mn.	Si.	P.	S.	Loss of Carbon.
Before annealing	3·04	2·85	0·19	0·21	0·73	0·154	0·050	0·38
After annealing	2·66	0·31	2·35	0·21	0·72	0·153	0·050	

TABLE XI.—MECHANICAL PROPERTIES

Condition.	Time taken to Anneal.	Max. Strength per sq. in.	Elongation per cent.	Reduction per cent.
After annealing	108 hours	49,810 lb. (22·3 tons)	6·61	6·23

These figures are important, first, as showing the changes in composition which take place during annealing in the manner already described, and, secondly, as showing the strength properties of the material so formed.

The amount of carbon lost in the process is only 0·38 in the 3·04, or 12·5 per cent. of the total carbon. The manganese and sulphur remain unchanged, and the losses of phosphorus and silicon are minute.

CHAPTER XIII

STEEL

161. Classification of Steel—The various kinds of the important material known as steel may be briefly stated as follows:

1. Crucible steels, chiefly used for cutting tools, springs, and similar uses.
2. Bessemer steel, made by the Bessemer process and used for boilers, structural work, axles, and rails.
3. Open-hearth steel, made by the open-hearth process, and forming the greater part of structural steel.
4. Steel castings, made in some sort of open-hearth furnace.

Of these the crucible process is used in the manufacture of most of what are spoken of as *high carbon* or *hard* steels; the open-hearth process for the softer steel, low in carbon, known as mild steel; and the Bessemer process for *mild* steels also, but of late years to a more restricted extent. The following brief description of the main processes used in the manufacture of steel will help to explain the chief differences in their mechanical qualities.

Crucible Steel—The word "crucible" used here in reality refers to one stage of the process. In preparing it, bars of wrought iron are taken and first converted into steel by the absorption of carbon. They are then broken up into shorter pieces, which are placed in a crucible and melted. The molten mass so prepared is run into ingot moulds to set. Finally, while still hot, the ingots are worked and rolled into their final forms of bars, sections, or plates.

The first part of the process is called *cementation*, and the iron bars so treated should be as pure as possible, flat in section, and cut into convenient lengths. These are placed in boxes of refractory material, each bar being completely surrounded by charcoal. The boxes are then placed in a cementation furnace and kept at a suitably high temperature for several days, up to six, according to the size of the bars. The iron is in this way carburized gradually from the outer surface inwards, the general rate of penetration being about $\frac{1}{8}$ of an inch in each twenty-four hours. The resulting material is called *blister* steel from the

appearance of its surface. Formerly the blister bars were heated and rolled together to form *shear steel*, which, on being again cut, heated and rolled, became *double shear steel*, which was the steel of commerce until about 1740. At that time the crucible process was introduced by Huntsman, who cut up the double shear bars, remelted them in crucibles, cast the molten metal into ingots and then rolled them. This crucible process of steel manufacture has now given place to the Bessemer and open-hearth processes for the production of all except high-class tool steels, spring steel, and steels whose ingredients have to be carefully and precisely controlled. The cementation process, though still used as part of the crucible process in the manufacture of the highest classes of tool steel, is the oldest of the methods used in the production of steel.

Bessemer Steel—In the cementation process, steel is produced by the *carburization* of wrought iron, which is practically pure iron. In the Bessemer process steel is made by the converse process of the *decarburation* of pig iron. The essential feature of the Bessemer process consists in the manner of removal of the carbon, which is effected by forcing fine currents of air through the molten pig iron.

The pig iron contains silicon as well as carbon, and the oxygen of the air currents combines with these two ingredients with the result that they are burnt out with the temperature of the molten mass raised to a very high point at the same time. The main result is that all the silicon and carbon is eliminated. The iron thus cleared is recarburized by the addition of the requisite quantity of spiegeleisen containing manganese, carbon, and silicon. In this way the exact amount of carbon is supplied which is necessary to produce a steel of the desired carbon content. It might be thought possible to carry the process to such a point that there was still contained the proper percentage of carbon in the mass and to stop there. This is difficult, and it is found better to clear out all the carbon and afterwards to re-supply the desired amount. The effect of the manganese in the spiegeleisen is to neutralize much of the iron oxide in the mass. If this were not taken away its presence would cause red-shortness* in the steel produced.

The Bessemer process is carried out in a pear-shaped vessel of steel plates provided with a refractory lining (see Figs. 207, 208). The latter is a view through the trunnions and shows the air passage. This vessel is called the converter. It can be rotated about the trunnions provided, through which the air blast is carried to the bottom end, whence it is blown into the iron. The converter is first placed in a horizontal position to

* Material is said to be red-short when it breaks with practically no ductility, at the same time being red-hot.

receive its charge, which may be taken direct from the blast furnace, or it may consist of remelted pig. When the converter has received its charge, the blast is started and the converter again turned to a vertical position. The blast is allowed to continue until the appearance of the flame at the converter's mouth indicates that the desired point has been reached. The blast is now stopped, the converter turned down, the spiegeleisen added, and the charge poured off into ladles.

These ladles containing the molten steel are taken away and their contents poured into large ingot moulds. The Bessemer process is the cheapest in use for steel production, but possesses certain disadvantages which do not belong to the other univer-

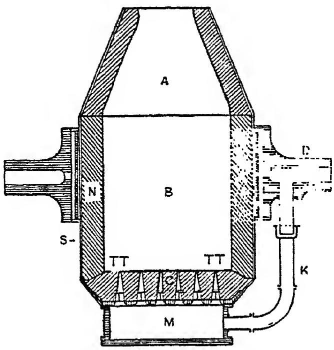


Fig. 207.—Bessemer Converter. Section through Trunnions.

A, Hood; B, Body; C, Bottom; TT, Tuyers; D, Hollow Trunnion; K, Blast Pipe; M, Wind Box; N, Lining; S, Steel Shell.

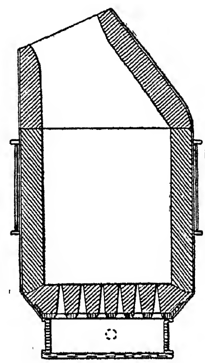


Fig. 208.—Bessemer Converter. Section at right-angles to Trunnions.

sally used "open-hearth process." It may here be mentioned that the inflow of air at the bottom of the converter is arranged to be of such pressure as is necessary to support the mass of iron in the converter, which during the process is literally resting on a cushion of air.

The combustion caused by the blast removes all the carbon and silicon, but no sulphur and no phosphorus. If there are small percentages of either or both of these in the pig iron, the deleterious impurities will pass into the steel. It is therefore necessary, where steel free from sulphur or phosphorus is desired, that an ore be used which does not contain either of these, giving a pig also free from sulphur and phosphorus. Such a pig iron is spoken of as Bessemer pig. In order to provide sufficient silicon for combustion the iron should contain from $1\frac{1}{2}$ to $2\frac{1}{2}$ per cent. of silicon. Sometimes, but only where the percentage of silicon

is $2\frac{1}{2}$ per cent. and a high temperature results, a quantity of steel scrap, from 10 to 15 per cent., is added.

The rate of elimination of the foreign ingredients is shown on Fig. 209.

It will be observed that the carbon is gradually diminished from the beginning, slowly at first and more rapidly afterwards. At the end of nine minutes the carbon has been completely eliminated, the silicon at six minutes, and manganese at the end of three. Small percentages of sulphur and phosphorus remain throughout. The times here shown are for an average typical charge.

It will be seen that the changes in the colour of the flame from the bath are indicated.

One of the later devices for making steel free from phosphorus from the pig which contains it, consists in tipping calcined lime into the converter. The lime takes up the phosphorus and goes to form part of the slag. In this arrangement it is necessary to have a "basic" lining to the converter, and this is constructed of magnesian limestone. This process is called the "basic Bessemer" process.

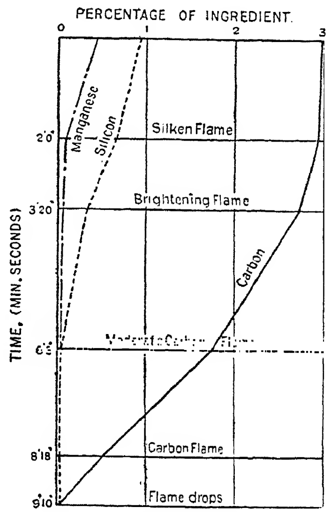


FIG. 209.—Chemical Reductions in a Bessemer Converter.
(From Howe, "Journ. Iron and Steel Inst.," vol. ii.)

162. Open-hearth Steel—By open-hearth steel is meant steel produced, not in a closed converter, but on an open hearth which is somewhat saucer-shaped in section (Figs. 210 and 211). High temperature is maintained by a flame which arises from the combustion of producer gas, and which is directed forward along the furnace and downward so as to play upon the upper surface of the iron, which is thus kept in a molten state.

The flame, mixed with an excess of oxygen from the air admitted, tends to reduce the carbon and other foreign ingredients in the iron. The maintenance of the high temperature is assisted by an arrangement of "hot blast." The air which enters with the gas and provides the oxygen needed for its combustion passes over and through a maze of hot brickwork before it enters the furnace. Combustion thus starts with the hot instead of cold, with the consequence that the

temperature of the furnace is higher than if only cool gas and air had been admitted. The brickwork which in this way heats the incoming air has been itself previously heated by being exposed to the influence of the outgoing products of combustion. The complete furnace is built in duplicate, with two inlets for the gas and air and two sets of firebrick surfaces. As the cool gas and air enter they pass over and through the heated firebrick, and the products of combustion are allowed to pass through the second nest of firebrick, which they heat until red-hot. The current is now reversed, the firebrick just heated being now used to heat the new incoming gas and air. So the process is con-

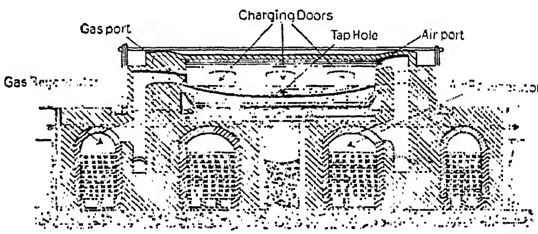


FIG. 210.—Siemens Furnace. Longitudinal Section.

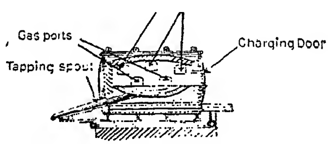


FIG. 211.—Siemens Furnace. Cross-section.

tinued, each nest of firebrick being heated and giving up its heat alternately. This arrangement is the regenerative gas furnace. The temperature maintained is as high as 4,500° F. instead of 3,500 without the regenerative arrangement.

By using the regenerative furnace much fuel is saved and at the same time a higher temperature attained. The roof of the furnace, which is depressed towards the hearth, helps to deflect the flame downwards towards the charge. In working the furnace it is a common plan to begin by placing on the hearth pig iron, which is rendered molten by the action of the intense flame, and at the same time much of the carbon and silicon is burnt out by the oxidizing flame. Following this consumption of carbon and silicon a great deal of the iron is oxidized also. The oxide resulting floats on the surface of the bath and serves as a

slag. The effect of this oxide slag is to cover the scrap iron and steel, which are now thrown into the bath, and so protect them from the oxidizing flame. The iron, wrought-iron, and steel so supplied to the mass of partly-oxidized cast iron has a diluting effect, and further reduces the total percentage of carbon. Such carbon as is left in the bath, after the flame has removed a part and the iron and steel scrap have been supplied, is removed by the iron oxide slag. A part of the oxygen in this slag unites with the unremoved carbon and forms carbonic oxide, which, bubbling through and from the surface of the bath, combines with further oxygen from the air and passes away as carbon dioxide.

Another plan employed when the quantity dealt with is large is to decarburise the pig iron by means of a quantity of iron ore in the form of oxide, preferably hæmatite ore. This plan does away with the necessity of using the artificial oxide as a slag. There may be still remaining a quantity of oxide slag not destroyed, and this is disposed of just before pouring the charge by adding a quantity of either spiegeleisen or ferro-manganese, both containing high percentages of manganese. This manganese, by uniting with the oxygen in the slag, sets free a large quantity of iron and so adds to the yield. The manganese thus added acts in reality as a deoxidizer and must not be confused with the recarburizing charge supplied to a Bessemer converter after all the carbon has been burnt out. In the open-hearth process the plan is not to destroy all the carbon with the intention of resupplying the requisite percentage afterwards. In this process there is generally some carbon remaining before the addition of the manganese, and the main function of this last is to neutralize some of the oxygen in the floating slag. One remarkable peculiarity of the manganese so added is its power of reaching and affecting all points of the bath.

It should be obvious from the above that, compared with the Bessemer process, the open-hearth process is more amenable to control. This may be expressed in the following way:

(*a*) Samples may be taken from an open-hearth bath and rapidly analysed. If the carbon percentage is not as desired there is thus opportunity for correction before tapping. This cannot be done in the Bessemer process.

(*b*) It is found that the steel in an open-hearth bath is very homogeneous, as shown by samples taken from different parts of a bath. This exceptional homogeneity is probably in a large measure due to the mechanical movement of the material caused by the bubbling of the fluid bath.

(*c*) One of the chief objections which engineers have to steel made by the Bessemer process is the lack of uniformity, as shown

by differences in the results of tests of samples cut, say, from the same plate.

It is found that the open-hearth steel which can be purchased for the use of engineers is far more uniform in quality than the Bessemer steel which comes on the open market.

(*d*) Open-hearth steel is more reliable than Bessemer steel. It is with the latter that many failures have occurred which have been difficult to explain.

Phosphorus and Sulphur in Open-hearth Steel—The same trouble is caused by these two ingredients as in the Bessemer process. The normal lining to the open-hearth furnace is siliceous or "acid." Where lime has to be supplied to destroy the phosphorus in the pig it will also unite chemically with the acid lining and so destroy it. Therefore where a pig free from phosphorus is not available, lime must be employed and a different lining used in the furnace. For this purpose magnesian limestone is employed and the process is now referred to as the basic open-hearth process. There are thus four processes used in the production of structural steel, namely :—

(i) The Bessemer acid process, for all kinds of pig.

(ii) The Bessemer basic process, used when the pig contains phosphorus.

(iii) The open-hearth acid process, for any kind of pig.

(iv) The open-hearth basic process, necessary when the pig contains phosphorus.

The above remarks show how any phosphorus in the pig iron used may be eliminated : the elimination of the sulphur by the addition of manganese is left to the steel-maker who controls the process. Of the two processes, acid and basic, the former is cheaper but possesses the disadvantages which have been mentioned. Of course bad steel high in phosphorus may be made with either process, but this is due to errors in carrying out the process and not to uncontrollable defects in the materials employed.

An engineer who is about to use the steel does not want more than a specified quantity of impurity. He finds it safer to specify the exact percentage which must not be exceeded than the process to be used in its production. The usual values which were formerly accepted for these limits for phosphorus were at one time from 0·06 to 0·08 per cent., but there has been a tendency to stiffen the specifications by insisting on not more than 0·04 per cent.

The General Mechanical Properties of Steel—There are so many varieties of what may be properly termed "steel," that it is difficult to give a comprehensive list. Without referring to other constituents whose effect on the properties will be described later,

it may be said here that the percentage of carbon present goes a long way towards defining the general properties of a given steel. As a general rule, the lower the percentage of carbon, the softer, weaker, and more ductile is the steel. With a percentage of 0·1 C the steel has a very low tensile strength, it is extremely ductile when elongated in tension, and appears very soft as judged by the cutting tool. It is incapable of being hardened in any way but by case-hardening, which is cementation in another form. In case-hardening the existing steel is not actually itself made harder, but some or the whole of it is transformed into a material of a harder nature.

Such steel with very low carbon content is often referred to as "dead soft," and is capable of being bent when cold through a large angle without exhibiting any sign of a crack: this applies to both bars and plates. This plan of bending cold a piece of steel, when some idea as to its ductility is desired, forms one of the tests often specified for fairly soft steels. One favourite way of exhibiting the great ductility of a sample of round bar steel is to tie it into a knot and then pull it so as to be partially closed up. Cases of the bending of very ductile steel without fracture are shown on Plate XVII.

Composition and Strength Properties of the Steel operated upon by Professor J. T. Nicolson—The Table XII on page 348 gives representative examples of three modern steels made by Messrs. Armstrong, Whitworth & Co. which were used in the experiments on rapid-cutting tool steel carried out by Dr. Nicolson for the Joint Committee.

The Principal Steels—When the whole range of steel is considered, from the very softest low-carbon steel to the hardest tool steel, it is possible to form some idea of the great number of uses to which steel is put, the enormous differences in the qualities required, and the extent to which most of these qualities are dependent on their composition, but not entirely so. This matter of chemical composition is chiefly a question of the extent to which small quantities of foreign ingredients are added to the iron which goes to form by far the greatest part of the steel. The range may be placed as follows: it is not by any means quite complete, but includes most of the typical varieties.

The Highest Grade of Tool Steel—Steels in this class are all made by the crucible process, preceded in most cases by cementation. The main characteristics of these are—relatively high percentage of carbon, they are hard even in the annealed state as compared with the low-carbon steels, they are capable of being hardened to a desired extent by heat treatment, their tensile strengths and yield points are high, the ductility is low, and the appearance of the fractures is that of a fine crystalline struc-

TABLE XII

Young's Modulus.	Elastic limit per sq. in.		Yield point per sq. in.		Maximum per sq. in.		Elong. on 4 in. per cent.	Reduction per cent.
	Lb.	Tons.	Lb.	Tons.	Lb.	Tons.		
SOFT STEEL. C = 0·198; Si = 0·055; Mn = 0·605; S = 0·026; Ph = 0·035%								
Tensile :								
$29{\cdot}99 \times 10^6$	14,690	6·56	25,700	11·47	59,600	26·60	28·80	54·8
$29{\cdot}86 \times 10^6$	18,370	8·20	31,200	13·93	58,000	25·93	21·2	37·1
$29{\cdot}99 \times 10^6$	16,530	7·38	29,470	13·11	58,980	26·33	24·5	44·3
Compressive :								
$29{\cdot}79 \times 10^6$	14,670	6·55	25,670	11·46	—	—	—	—
$30{\cdot}18 \times 10^6$	11,598	5·73	—	—	—	—	—	—
MEDIUM STEEL. C = 0·275; Si = 0·086; Mn = 0·650; S = 0·037; Ph = 0·043%								
Tensile :								
$30{\cdot}33 \times 10^6$	14,740	6·58	22,110	9·87	64,400	28·75	26·5	37·5
$30{\cdot}33 \times 10^6$	14,740	6·58	22,110	9·87	64,760	28·91	25·4	36·5
$30{\cdot}00 \times 10^6$	14,690	6·56	21,960	9·83	64,940	28·99	25·0	37·7
Compressive :								
$29{\cdot}66 \times 10^6$	11,000	4·91	23,000	10·24	—	—	—	—
$30{\cdot}175 \times 10^6$	11,000	4·91	23,000	10·24	—	—	—	—
$30{\cdot}175 \times 10^6$	11,000	4·91	22,000	9·82	—	—	—	—
HARD STEEL. C = 0·514; Si = 0·111; Mn = 0·792; S = 0·033; Ph = 0·037%								
Tensile :								
$30{\cdot}385 \times 10^6$	29,500	13·16	43,100	19·74	104,900	46·83	13·5	20·0
$30{\cdot}10 \times 10^6$	29,600	13·11	44,060	19·13	104,900	46·83	14·5	18·0
$30{\cdot}233 \times 10^6$	30,000	13·25	44,400	19·80	104,000	46·49	11·5	10·5
Compressive :								
$30{\cdot}86 \times 10^6$	21,930	9·79	46,080	20·57	—	—	—	—
$30{\cdot}75 \times 10^6$	21,910	9·78	45,990	20·53	—	—	—	—
$30{\cdot}61 \times 10^6$	21,930	9·79	46,080	20·06*	—	—	—	—

ture. The following are the percentages of carbon in tool steel of this kind. It is to be noted that a carbon steel having from 0·90 to 1·10 carbon content is the one which can be put to a greater number of uses than any other. The percentages of carbon suitable are as follows :

1·5 C	Extremely fine edge requiring no strength, as in razors and surgical instruments.
1·50 C to 1·25 C	Files, stocks and dies, scribers, scrapers, small drills, turning tools, and similar purposes where the cutting edges are to have a certain amount of strength as well as great keenness.
1·25 C to 1·10 C	Some tools for machine-tools and lathes, graving tools.
1·125 C	Stocks and dies.
1·10 C	Graving tools.
1·10 C to 1·00 C	Large turning tools, drills, dies, hatchets, axes, knives.

Manchester Assoc. of Engineers, October 24, 1903.

1·00 C to 0·90 C	Knives, dies, axes, drills, and similar purposes.
0·90 C to 0·80 C	Wood-working chisels, reamers, taps, dies, drills, and cold sets.
0·80 C to 0·70 C	Some reamers and taps, battering tools, and cold sets.
0·70 C to 0·50 C	Where dull edges are needed and battering tools.
0·55 C to 0·50 C	Steel for the springs of railway vehicles.

In the above-mentioned steels strength and ductility are of less importance than good cutting qualities. Engineers know that it is not always easy to give to the steel such a temper that the cutting edge is not dulled and at the same time the steel is not fractured by being chipped off at the cutting edge. The higher the carbon the keener the possible cutting edge but the more brittle the steel and, consequently, the greater the liability to chip. The above carbon percentages will help to give a good general idea of the quantity needed for most engineering requirements.

The more ordinary Steels used for Tools and Springs—In these high-carbon steels which are capable of being tempered to the required degree of hardness necessary for the work they have to do, the tensile strength is high with a low ductility. These steels are made by the Bessemer, the open-hearth, and the crucible processes, and have tensile strengths varying from 160,000 to 90,000 lb. or 71·5 to 40·2 tons per sq. in. The ductility, as represented by the elongation, depends on the degree of hardness imparted during the heat treatment, and is always very small.

Steel suitable for Rails—This steel, generally made by the Bessemer process, should have a high yield point of from 50,000 to 40,000 lb. or 22·3 to 17·85 tons per sq. in. in order that the metal may be able the more easily to resist the abrasive action of the wheels. The ultimate strength is from 80,000 to 70,000 lb. or 35·7 to 34·4 tons per sq. in. The elongation should be from 15 to 20 per cent. on 8 in., and the reduction in area from 40 to 50 per cent.

The tendency of rails, both in railways and electrical tramways, to become corrugated, has had the effect in recent years of making engineers use harder and stronger material. For this reason the percentage of silicon in the steel has increased.

In steel for rails the carbon content varies from 0·40 to 0·45 per cent.

Tyre Steel—What has been said about rail steel applies also to the steel used in the tyres of the wheels which run on the rails. Mr. Arnold, in his paper on "Bessemer Steel Tyres,"* gives some data relative to strength and composition of this steel.

* "Min. Proc. Inst. C.E.," vol. xcv.

TABLE XIII. BESSEMER STEEL TYRES

C per cent.	Si per cent.	Mn per cent.	Chr per cent.	Tensile strength per sq. in.		Elongation, per cent. on 2 in.	Reduction, per cent.
				Lb.	Tons		
0·28	0·07	1·25	—	82,880	37·0	26	47
0·25	0·03	1·75	—	94,300	42·1	18	26
0·28	0·08	1·54	0·42	111,550	49·8	15	26
0·32	0·11	1·46	0·30	112,000	50·0	16	29
0·28	0·11	1·41	0·64	112,900	50·4	10	14

In this paper the author also mentions a similar tyre steel, which contains more carbon (0·5) when in the unhardened and hardened state, thus :

Unhardened . . .	113,800 ..	50·8 ..	14·9 ..	31·4
Hardened in water .	155,500 ..	69·4 ..	10·9 ..	30·0
Hardened in oil . .	197,100 ..	88·0 ..	3·1 ..	4·9

This hardening, by which the tensile strength is greatly enhanced, could not have been effected without the almost 50 per cent. increase in the carbon.

Structural Steel—Structural steel includes all classes used in the construction of frames, such as bridges, the frames of steel-and-brickwork buildings, roofs, the steel of reinforced-concrete buildings and engineering structures, as well as many other similar purposes.

Such steel is produced by both Bessemer and open-hearth processes, the preference being for the latter. The following figures represent the chief qualities of the three rough grades into which structural steel is often divided :

TABLE XIV

	Yield point, per sq. in.		Maximum, per sq. in.		Elongation, per cent. on 8 in.	Reduction, per cent.
	Lb.	Tons.	Lb.	Tons.		
Mild or very soft	30,000 to 40,000	13·4 to 17·8	50,000 to 60,000	22·3 to 26·8	35 to 25	65 to 50
Medium .	35,000 to 45,000	15·7 to 20·1	60,000 to 70,000	26·8 to 31·2	25 to 20	60 to 50
Hard .	—	—	65,000 to 75,000	29·0 to 33·5	This grade is less used than the last, being too hard and lacking ductility.	

The following four typical examples of structural steel are given by Mr. Skelton:

TABLE XV

C.	Si.	S.	Ph.	Mn.	Tensile strength, per sq. in.		Elongation, per cent.
					Lb.	Tons.	
0·10	Trace	0·035	0·045	0·56	51,650	23·06	29
0·12	,,	0·050	0·050	0·58	61,110	27·28	23·3
0·14	,,	0·046	0·052	0·61	64,620	28·85	23·0
0·15	,,	0·043	0·060	0·60	64,670	28·87	24·5

In addition to these, some valuable figures are quoted by Mr. Campbell in his book on "Structural Steel."

TABLE XVI

Kind of Steel.	Ph.	Mn.	Tensile strength, per sq. in.		Elongation, per cent. on 8 in.
			Lb.	Tons.	
Extra dead soft, basic	0·04	0·05	49,280	22	27·5
Bridge rivets, acid or basic open-hearth	0·04	0·05	48,450	21·65	31·0
Hard bridge rivets	0·04	0·60	57,570	25·70	30·0
Common bridge rivets	0·06	0·60	57,570	25·70	29·0
Soft bridge steel	0·05	0·50	55,100	24·6	24·0
Medium bridge steel	0·05	0·60	58,910	26·3	23·0
Hard bridge steel	0·05	0·80	62,940	28·1	21·0
Extra hard bridge steel	0·05	0·80	66,980	29·9	19·0
Forging steel	0·05	0·90	75,040	33·5	17·0
Hard forging steel	0·04	0·90	87,360	39·0	12·1

Steel suitable for Boilers—This includes the steel which is used in the construction of boiler shells, steel specially suitable for fireboxes and flues, and the steel which is used for the rivets for the joints of shells and flues.

Boiler shell steel should have a fairly high tensile strength and moderate ductility. The tensile strength is usually between 65,000 and 55,000 pounds per sq. in. (29·0 tons and 24·5 tons), with corresponding yield stresses of from 44,000 to 33,000 lb. per sq. in. (19·6 tons and 14·75 tons). This steel is made by both the open-hearth and Bessemer processes, and should have an elongation on 8 in. of 25 to 30 per cent., with a reduction in area of 50 to 60 per cent.

Steel for Flues, Fireboxes, and Rivets—Material used for these

purposes must be, above all things, thoroughly ductile. In the formation by forging of the difficult shapes into which the joints have to be constrained it is impossible to avoid permanently straining the material. The hard treatment to which the steel is subjected is not possible in any but one which is very ductile. While extremely ductile it must be strong to resist the heavy treatment it is likely to have to endure both cold and when under heat. It should be free from phosphorus and sulphur, so that it may not crack either when hot or cold. The desired ductility is tested by bending the material, whether plate or bar, through a large angle without any sign of cracking. This angle may be as great as 180 deg., or, in other words, the plate or bar should be capable of being closed up on itself (Plate XVII), this of course taking place when the steel is cold.

The ultimate tensile strength should be between 60,000 and 50,000 lb. per sq. in. (26·8 and 22·3 tons), yield point from 40,000 (17·8) to 30,000 (13·4), with elongation of 25 to 35 per cent., and reduction of 50 to 65 per cent.

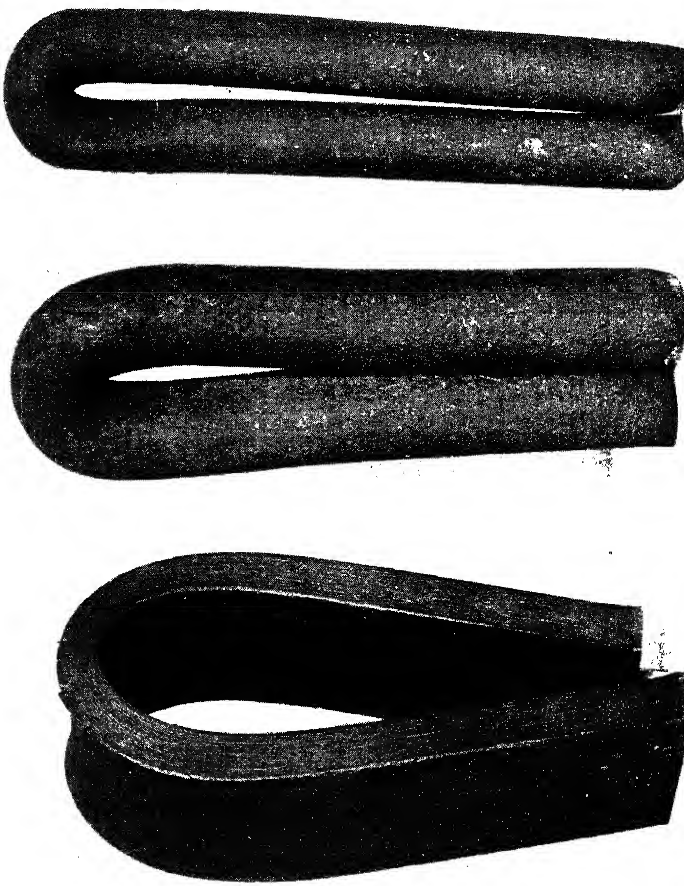
The Homogeneity of Open-hearth Steel—The fact that open-hearth steel can be produced in large quantities, in which the quality is very constant and departs little from the average, is well illustrated in the following results:

TABLE XVII

Elastic limit, per sq. in.		Ultimate strength, per sq. in.		Elongation, per cent. on 8 in.	
Lb.	Tons.	Lb.	Tons.		
Acid Open-hearth Steel. ¾-in. round bars.					
37,580	16·77	57,670	25·75	33·50	Highest of 20.
35,860	16·00	53,510	23·90	28·75	Lowest of 20.
36,510	16·300	56,538	25·24	31·15	Average of 20.
Acid Open-hearth Steel. 1¼-in. round bars.					
33,580	14·99	53,860	24·05	34·00	Highest of 16.
31,040	13·86	52,000	23·20	31·50	Lowest of 16.
31,809	14·20	52,853	23·59	32·75	Average of 16.
Acid Open-hearth Steel. ¾-in. round bars.					
37,480	16·72	58,400	26·07	32·50	Highest of 15.
35,700	15·94	56,700	25·31	28·00	Lowest of 15.
36,634	16·35	57,201	25·53	30·25	Average of 15.
Basic Open-hearth Steel. ⅞-in. round bars.					
33,880	15·125	49,440	22·07	37·00	Highest of 14.
31,530	14·07	47,380	21·15	33·75	Lowest of 14.
32,745	14·61	48,384	21·60	35·37	Average of 14.*

"Trans. Am. Inst. Min. Engrs.," vol. xxviii, Mr. H. H. Campbell. by kind permission of Mrs. Campbell.

PLATE XVII.



COLD BEND TESTS ON MILD STEEL

In the preceding figures there is a fairly large number in each batch of specimens, and in all four the difference between mean and extreme is relatively small.

NOTE AS TO TEMPER (JOHNSON)—TEMPERING (STEEL-MAKER'S MEANING):

TABLE XVIII.

Definition of Temper.	Steel-maker's meaning, per cent. of Carbon.	Steel-user's meaning.	
		Temperature.	Colour.
Very high . . .	1·50	About 400° F. (204° C.)	Light straw.
High	1·0 to 1·2	About 450° F. (232° C.)	Straw.
Medium	0·7 to 0·8	About 500° F. (260° C.)	Brown to pigeon wing.
Mild	0·4 to 0·6	About 550° F. (288° C.)	Light blue.
Low	0·2 to 0·3	About 600° F. (316° C.)	Dark blue.
Soft or dead soft .	Under 0·2	About 650° F. (343° C.)	Black.

The following figures are interesting as showing the limits imposed on the quantities of some of the constituents of different grades of steel by one of the largest producers of steel plates, namely, the Illinois Steel Company (1895).

TABLE XIX.

Quality.	Carbon, per cent.	Manganese, per cent.	Sulphur, per cent.	Phosphorus, per cent.
Firebox . .	0·16	0·35 to 0·50	Not over 0·04	Not over 0·02
Boiler . .	0·18	0·35 to 0·60	Not over 0·045	Not over 0·04
Flange . .	0·18	0·35 to 0·60	Not over 0·045	Not over 0·04
Ship . . .	0·15	0·35 to 0·65	Not over 0·06	Not over 0·08
Tank . . .	0·10	0·40	Not over 0·10	Not over 0·12

These figures show the quantity of carbon held to be desirable for the kind of plates mentioned, as well as the minimum quantities of the deleterious elements sulphur and phosphorus.

British Standard Tramway Rails and Fishplates (1922) *—

* Abstracted by permission of the British Engineering Standards Association from B.S. Specification (post free 1*s.* 3*d.*).

The steel may be made by acid Bessemer, basic Bessemer, or other approved process. Analysis to conform to the following limits:

	Bessemer Process.		Open-hearth Process.	
	Acid.	Basic.	Acid.	Basic.
	Per cent.	Per cent.	Per cent.	Per cent.
Carbon .	0·50 to 0·60	0·45 to 0·55	0·60 to 0·70	0·60 to 0·70
Manganese	Maximum 1·0	Maximum 1·0	Maximum 1·0	Maximum 1·0
Silicon .	,, 0·10	,, 0·30	,, 0·10	,, 0·30
Phosphorus	,, 0·075	,, 0·08	,, 0·06	,, 0·06
Sulphur .	,, 0·07	,, 0·07	,, 0·06	,, 0·06

For the impact or drop tests lengths of 5 ft. are cut from a rail selected by the engineer and shall be placed in a horizontal position with the head uppermost upon the two steel bearers of the British Standard Falling Weight Testing Machine, the bearers being 3 ft. 6 in. apart at their centres, and having their upper surfaces curved to a radius of 3 in. The height of the drop for all sections of rails shall be 10 ft. and the weight 20 cwt.

The steel for the fishplates is to be in all respects similar to that used for the rails.

High-silicon Steel Rails—In recent years much trouble has been caused to railway and tramway engineers by the formation of grooves on the treads of the rails used, now well known as "corrugation," and also by the general wear and reduction in weight caused by the heavier loads carried and by the greater frequency of the passing trains. The precise cause of rail corrugation has not yet been clearly defined. It seems to be partly caused by abrasion, but mainly by oscillatory movement of the wheels on the rails. The action is a very complex one, involving many parts of the travelling vehicle, and in spite of many ingenious theories put forward, it is most difficult to locate the exact cause. It seems likely, however, that the origin of the trouble will eventually be traced to the torsional spring of one wheel relatively to its fellow at the other end of the same axle, which relative movement causes a periodic chattering blow on the rail, accompanied by abrasive action of wheel on rail. However this may be, it has been found absolutely necessary to provide a remedy. Corrugation occurs in rails of every kind, steam railway, electric railway, and electric tramway. Also it occurs at all parts of a system, straight, large-radius curve, and sharp curve, the most frequent cases being, apparently, on rails with slight curvature.

In some systems it has been found necessary to clean off the corrugations with a portable milling machine, in the doing of which many thousands of pounds have been expended and much metal wasted. The most promising remedy for corrugation and abnormal wear is the more radical one of using harder rails in

the first instance. By doing this both corrugation and undue wear are prevented, and the life of the rail thereby increased.

In this connexion * Mr. Mattinson (of the Manchester Tramways Department) gives the analyses of three rail steels used with different degrees of success in the Manchester district. These are:

	Carbon.	Manganese.	Silicon.	Phosphorus.	Sulphur.
(A)	0·49	0·82	0·05	0·06	0·052
(B)	0·50	0·78	0·34	0·06	0·057
(C)	0·54	1·06	0·26	0·06	0·060

Of these three (A) is a basic Bessemer steel laid in Market Street, Manchester, and containing, as will be seen, a low percentage of silicon. (B), as shown by the composition, is a high-silicon steel. Rails of this steel were laid in Oxford Street, Manchester, and as compared with (A) proved to be 25 per cent. more durable. As a result, steel having the composition (C) has been used since 1907 with marked success. This particular steel (C) was found to have the following strength properties:

Tensile strength: 54·8 tons per sq. in. =122,800 lb. per sq. in.

Elongation on 2 in.: 18 per cent.

Reduction in area: 23 per cent.

Drop test: Tup, 1 ton, drop 15 ft.: first blow $\frac{7}{8}$ in., second blow $1\frac{5}{8}$ in.; no sign of fracture.

Brinell test: 3·3 mm., with 50 tons on 19 mm. ball.

Mr. William Willox† gives the lengths of life of four kinds of rail steel tried on the very heavily loaded and hard-worked Metropolitan Line.

Of these,

No. 1 is a High-silicon Basic Open-hearth Steel (from 0·204 to 0·212 per cent. Si).

No. 2 is a High-silicon Acid Bessemer Steel (from 0·182 to 0·381 per cent. Si).

No. 3 is a Basic Open-hearth Titanium Steel (from 0·166 to 0·161 per cent. Si, and a trace of titanium).

No. 4 is an ordinary Basic Open-hearth Steel (from 0·166 to 0·171 per cent. Si).

The results of the working of these rails showed that their effective lives were:

No. 1, over 23 months.
No. 2, 11 and 15 months on two lengths.
No. 3, $9\frac{1}{3}$ months.
No. 4, $9\frac{1}{3}$ and 11 months on different lengths.

* Correspondence on the above paper.

† See papers by Mr. W. Willox and Mr. Sellon in "Min. Proc. Inst. C.E.," vol. cxcvii, 1913–14.

These high-silicon steel rails were made by the "Sandberg" process, in which the whole of the silicon is first eliminated and afterwards the precise amount desired re-added. Mr. Willox strongly favours the open-hearth process for rails on account of the facility it provides for controlling the composition.

163. Steel Castings—Castings in steel are made from either acid or basic open-hearth steel or from the steel made in small tipping converters. A small open-hearth furnace is found to be the most useful for the all-round work of a steel foundry of any considerable size. The open-hearth process permits of greater facilities in controlling both composition and temperature of the charge when ready for casting.

The most frequent defect in steel castings is the sponginess of the metal sometimes produced and the frequent occurrence of blowholes. Both these defects can be combated successfully, when proper care is taken. As a matter of fact the quality of steel castings has improved to an enormous extent of late years, a state of things brought about by improved detail in the methods employed as gained from experience. One essential thing necessary for sound castings is what is known technically as a "dead melt," by which is meant that the charge just before pouring into the moulds should have a slag which is both thick and clean and non-oxidizing. An oxide slag conduces to a spongy casting. It is found that some *fluorspar* powder supplied to the metal just before pouring helps to clear it.

In order to secure good castings it is essential that the metal be poured at the correct temperature. A high temperature causes it to be more fluid and likely to fill up the moulds in a satisfactory manner, but a too high temperature may result in unsoundness by reason of the largeness of the contraction, which, being possibly unequal, brings about unsoundness and piping.

A temperature which is too low may give defective castings and the want of fluidity may result in the moulds not being properly filled, so that experience is essential, not only with a given steel but in producing castings of any particular type. Such experience has shown that it is better to cast at too low than at too high a temperature, so long as the metal is sufficiently fluid. The addition of a quantity of aluminium before pouring greatly improves the castings. It is supposed to remove oxides and thus help the flow from the ladle and by doing this allow pouring to take place at a lower temperature than is possible without the aluminium, resulting in cleaner surfaces.

Annealing the Castings—In order to prevent cracking and "flying" of hard-steel castings and at the same time equalize any initial stresses brought about during the cooling period, it

*TABLE XX. COMPOSITION AND STRENGTH OF TYPICAL STEEL CASTINGS

Description of Casting.	C per cent.	Si per cent.	Mn per cent.	S & P per cent.	Max. Strength per sq. in.		Elong. on 2 in. per cent.	Bend Test. Degrees.
					Lb.	Tons.		
For rudder frame	0·16	0·49	0·576	below 0·06	66,300	29·6	37	a.b. 90
For shaft bracket	0·21	0·53	0·63	,,	75,800	33·8	23	a.b. 80
Eccentric rod .	0·20	0·361	—	,,	66,300	29·6	24	a.b. 120
Pivot plate . .	0·40	0·326	—	,,	87,980	39·2	24	85
Casting . . .	0·47	0·501	—	,,	94,100	42·0	14	40
Roller path . .	0·33	0·501	—	,,	92,000	41·0	18	64
Roller path . .	0·22	0·42	0·594	,,	67,900	30·3	29	a.b. 100

is desirable that they be annealed before use, especially if they are of large size.

Messrs. Harbord and Hall mention usual temperatures for the annealing as 900° C. for castings in low-carbon steel, and for castings containing more than 0·5 per cent. carbon a somewhat lower temperature, at from 800° C. to 850° C.

Steel castings which have not been annealed are brittle when subjected to shock, and the annealing removes much of the liability to be cracked by blows. The actual time required for satisfactory annealing depends largely on the size and shape of the casting, and may be as long as three or four days at the annealing temperature and somewhat longer for cooling.

It is found that annealing by removing internal strain increases the ductility, but somewhat lowers the tensile strength, thus changing the condition to one of greater "toughness."

The following examples (taken from "Metallurgy of Steel," by Harbord and Hall, are typical :

TABLE XXI.—EFFECT OF ANNEALING STEEL CASTINGS

Carbon, per cent.	Unannealed.			Annealed.		
	Maximum Strength per sq. in.		Elongation on 2 in. per cent.	Maximum Strength per sq. in.		Elongation on 2 in. per cent.
	Lb.	Tons.		Lb.	Tons.	
0·23	68,780	30·68	22·4	67,220	30·00	31·4
0·37	85,300	38·18	8·20	82,220	36·70	21·80
0·53	90,100	40·23	2·35	160,480 (rather abnormal)	47·5	9·8

* From "Metallurgy of Steel," by Harbord and Hall, 2nd Edition, with Messrs. C. Griffin & Co.'s kind permission.

The following is a good example of the effect of annealing, hardening, and tempering of samples taken from a forged gun jacket.

The two samples given had the following analyses :

	C.	Si.	Mn.	S.	P.	Iron.
(A)	0·35	0·023	0·252	0·019	0·038	99·35
(B)	0·43	0·028	0·216	0·023	trace	99·30

This steel was hardened by quenching in oil at about 820° C., and reheated to 650° C.

Table XXII.

	Annealed.					Hardened and Tempered.				
	Maximum Strength per sq. in.		Elastic Limit per sq. in.		Elong. on 2 in. per cent.	Maximum Strength per sq. in.		Elastic Limit per sq. in.		Elong. on 2 in. per cent.
	Lb.	Tons.	Lb.	Tons.		Lb.	Tons.	Lb.	Tons.	
(A)	77,080	34·4	39,500	17·6	27	109,560	48·9	65,090	29·06	16·5
(B)	78,180	34·9	43,100	19·26	25·5	111,100	49·6	69,700	31·1	17·0

The following results of tests on steel castings are taken from a paper by Mr. Abbott of Messrs. Fairbanks :

Table XXIII. Tests of Steel Castings.

Elastic Limit (Yield point ?) per sq. in.		Average Young's Modulus : 24,416,000 lb. per sq. in., or 10,900 tons.		
		Maximum Tensile per sq. in.		Elongation on 10 in. per cent.
Lb.	Tons.	Lb.	Tons.	
28,900	12·90	43,570	19·45	6·00
{25,630	11·44	36,890	16·47	0·40
{25,650	11·45	42,110	18·85	0·80
23,070	10·30	44,240	19·75	6·00
21,370	9·54	38,300	17·10	10·00
31,140	13·90	46,260	20·65	2·00
35,840	16·00	56,670	25·30	10·00
35,500	15·85	52,190	23·30	9·00
{38,300	17·10	62,940	28·10	29·00
{38,300	17·10	62,720	28·00	20·50
17,520	7·82	32,140	14·35	8·00

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164. The Influence of Treatment on Steel.—There are three influences which may be regarded as likely to have important effects on the qualities of a given steel which are essential to the engineer. These influences are:

(i) The chemical composition.

(ii) Mechanical treatment during the processes of manufacture, such as hammering or cogging, the final rolling, cold rolling, drawing of tubes, wire-drawing, and all the other processes by which manufactured steel in its various sections is prepared for the user.

(iii) Heat treatment before use.

Of these (i) is frequently referred to in these pages, and, although it is the steel-maker who can vary the constituents, the engineer who makes use of the steel-maker's products is no less interested in the composition of a steel whose mechanical qualities of strength, hardness, and ductility affect him closely.

The same may be said of (ii) with the difference that the processes of manufacture are entirely in the hands of the steel-maker and cannot be affected by the user in any but a secondary degree. The influences included in (iii) are wholly in the hands of, or may be controlled by the engineer who is using the steel. A large amount of the heat treatment to which steel is subjected is carried out by firms who confine themselves to this work and may act under the instructions of engineers.

The main tendencies of heat treatment and the general effects produced are referred to in the following paragraphs.

165. Heat Treatment of Steel.—By the "Heat Treatment" of steel is meant, in the strict sense, the variations of temperature to which it is subjected from the time it enters the furnace until it is put into actual use as a finished product: but the more common meaning attached to the expression refers to the effects produced by variations in temperature imposed on the material as finished and supplied by the maker.

The best-known result of heat treatment is that met with every day in the workshop, during the hardening and tempering of cutting tools. A piece of carbon tool steel, though in a sense initially hard, as compared with such materials as structural steel or wrought iron, is not so hard that it can be used for such a purpose, for example, as the cutting of ordinary cast iron. But, by being submitted to the proper heat treatment, it is capable of being reduced to the precise degree of hardness desired. It is well known that if a bar of carbon (above 0·2 to 0·25 per cent. C) tool steel is heated to a bright redness and plunged into water so as to be very rapidly cooled or "quenched," it is thus brought to a state of extreme hardness: so hard, in fact, as to be capable of scratching any but the very hardest metals, but at the same

time it is too brittle to be used as a cutting tool. If such use were attempted the cutting edge would most certainly be chipped by the work attempted and the tool rendered useless. To prevent this damage to the edge of the tool occurring it is before use submitted to the process of softening or "letting down." In the case of a cutting tool like a chisel, turning tool, or drill, this softening or reducing to the proper temper consists in first applying the full temperature to the steel near the cutting part, and then quenching the end of the bar, including the portion previously ground to give the cutting edge. This latter is now cleaned by rubbing and carefully watched: as the heat stored in the bulk of the bar is gradually conducted into that portion which has just been quenched a thin oxide film is formed on the polished surface, which film gradually changes colour as the temperature is raised in that part of the bar. By knowing which colour tint corresponds to the desired degree of hardness, the smith is able to know when the tool has reached the wished-for point, which he then fixes by sudden quenching. It should be understood that the colour changes follow or accompany a gradual rise in temperature with a corresponding reduction of hardness, so that the smith gets the right temper by first hardening far beyond what is required and afterwards softening or letting down to the desired point.

The same result could have been attained by heating up to the same temperature as the smith let the tool down to, and quenching there, but without suitable appliances it is impossible to gauge the temperature precisely.

The word "temper" when referring to a particular steel always has the above meaning of "letting down" or softening; but when used in reference to carbon tool steels in general it is taken as meaning the percentage of carbon: as, a 1·5 per cent. C steel is a "high-temper" steel.

The following are the tints given by Mr. Metcalf *:

Light straw to straw . .	For lathe tools, files, etc.
Light brown (gold) to dark brown	For taps, reamers, drills, etc.
Pigeon wing (purple) . .	For axes, hatchets, and drills.
Light blue	For springs.
Dark blue	For some springs (seldom used).

166. Carbon in Steel—It is well known that in the case of cast iron the carbon is abundant in quantity and exists in part as free carbon, spoken of often as "graphite," and partly combined with the iron supersaturated with carbon.

This is partly true for iron and steel generally, the carbon being either free or combined, or both. Thus the carbon may

* "Manual for Steel Users," p. 117.

be present in iron in one of two forms. One of these can again be divided, with the result that the carbon in the iron may exist in three conditions, which are:

(*a*) Uncombined, as free carbon or graphite.

(*b*) Combined chemically in proportions not exactly known and called *hardening carbon*; in this state forming a compound which is at the same time strong and hard.

(*c*) Combined chemically as an iron carbide (Fe_3C) as far as the saturation point of 0·9 per cent. C. This is spoken of as the *cement carbon*.

In the molten state all the carbon is more or less combined, but when cooling takes place there is a separation of the free from the combined carbon unless the total carbon is below the saturation point of 0·9 per cent.

In the case of steel the carbon is wholly combined as *hardening carbon* or *cement carbon* or both. The effect of heat treatment is to vary the ratio of one of these to the other with the result of causing changes in the strength properties. According to Brinell, who made a long series of experiments on a 0·5 per cent. C steel, there is a particular temperature at which the carbon is changed from the *cement carbon* to *hardening carbon*, and this change is accompanied by an alteration from a relatively coarse structure to one which is so fine as to be approaching the amorphous condition. The temperature at which the above change was brought about has come to be denoted by the letter W. When hardening a given sample of steel it was found that by first heating to W and then quenching, the best results were obtained. If the temperature previous to quenching is much below W the steel will be left too soft, and if W is exceeded the structure becomes more crystalline and the metal more brittle.

Therefore, W is the temperature most suitable for the hardening of the steel.

Conversely, there is a temperature to be employed for obtaining the most satisfactory results when softening or annealing a steel previously hardened: here the carbon is changed from the *hardening* to the *cement* state. The temperature giving the best results has been called V. According to Brinell, if steel is heated to anything above V and below W, the carbon changes from the cement to the hardening state, which it still retains if raised above W. On heating to W or above the structure is the same independently of what it was before heating. If the heating is to below W the structure will depend on what it was before heating.

The two temperatures V and W are evidently of great and essential importance in the hardening and annealing of steel.

167. Annealing—The intention in annealing is by heating for a considerable time and cooling slowly to remove internal stresses caused during rolling and other processes of manufacture, to render the material more uniform, to raise the yield and maximum stresses and to improve the ductility. As a rule the effect of annealing is to increase the ductility of a steel, increase the elastic-limit stress, and generally to a less degree to increase the ultimate tensile strength. Some valuable results of annealing a number of samples of steel, having different percentages of carbon, and annealed at a number of different temperatures, along with figures representing their strength properties, have been given by Brinell and are quoted here in the form of the tables below. The results were for steels having carbon varying from 0·09 per cent. to 1·17 per cent. These steel samples were subjected to the following stages of treatment previous to the tests.

1. Hot-rolled with no further treatment (normal or untreated).
2. Annealed at 350°, 750°, 1,000°, 1,100°, and 1,200° C. with subsequent slow cooling.
3. Quenched from 750°, 850°, and 1,000° C. in water.
4. Quenched from 750°, 850°, and 1,000° C. in oil.
5. Quenched from 750°, 850°, and 1,000° C. in lead.
6. The test bars quenched in water were subsequently reheated to 350°, 550°, 650° C.

The oil-quenched bars were reheated to 350° and 500° C. The temperature of the water used for quenching was 20° C., of the oil 80° C., and of the lead 550° C.

The tensile tests were carried out on bars 0·708 in. diameter, and elongations were measured on 7·08 in.

Brinell's figures* embodied in the following tables, should prove useful as showing the effects on strength properties of nine grades of steel, each with a different percentage of carbon, of annealing them at four different temperatures, and quenching in water, oil, and lead, with suitable after-treatment.

It is further to be noted that our knowledge is continually being added to as regards the effects of the special heat treatment required by high-speed cutting tools and the very numerous family of alloy steels generally.

In the following tables are given figures showing the strength properties of a number of steels containing different quantities of carbon after being subjected to various kinds of treatment as specified. The different treatments or conditions referred to in the tables have the following index letters:

(A) The normal steel, untreated.
(B) The same steel after being annealed at 350° C.

* Brinell, in Iron and Steel Institute, published in "Metallurgy of Steel," Harbord & Hall. Published by C. Griffin & Co., Ltd.

(C) The same steel after being annealed at 750° C.
(D) ,, ,, ,, ,, ,, ,, ,, 850° C.
(E) ,, ,, ,, ,, ,, ,, ,, 1,000° C.
(F) The same steel after being water-quenched at 750° C.
(G) ,, ,, ,, ,, ,, now reheated to 550° C.
(H) ,, ,, ,, ,, ,, water-quenched at 850° C.
(I) ,, ,, ,, ,, ,, now reheated to 550° C.
(J) ,, ,, ,, ,, ,, next reheated to 650° C.
(K) ,, ,, ,, oil-quenched at 750° C., and reheated to 550° C.
(L) The same steel oil-quenched at 850° C., and reheated to 550° C.

TABLE XXIV. CARBON 0·09 PER CENT.

	Elastic Limit per sq. in.		Maximum per sq. in.		Elongation per cent. on 7·08 in.
	Lb.	Tons.	Lb.	Tons.	
A . .	21,053	9·4	46,700	20·7	26·1
B . .	25,080	11·2	45,900	20·5	36·0
C . .	23,700	10·6	46,800	20·9	34·3
D . .	20,600	9·2	48,160	21·5	33·6
E . .	31,360	14·0	47,260	21·1	36·3
F . .	19,500	8·7	80,200	35·8	15·6
G . .	25,100	11·2	46,100	20·6	34·9
H . .	25,300	11·3	87,800	39·2	15·8
I . .	30,900	13·8	50,800	22·7	30·9
J . .	25,300	11·3	49,060	21·9	33·1
K . .	28,000	12·5	45,900	20·5	35·1
L . .	28,000	12·5	50,400	22·5	34·9

CARBON 0·16 PER CENT.

	Elastic Limit per sq. in.		Maximum per sq. in.		Elongation per cent. on 7·08 in.
	Lb.	Tons.	Lb.	Tons.	
A . .	29,100	13·0	65,400	29·2	27·2
B . .	36,300	16·2	63,400	28·3	28·8
C . .	35,600	15·9	65,860	29·4	30·5
D . .	30,760	13·7	61,400	27·4	29·0
E . .	40,090	17·9	67,000	29·9	30·9
F . .	18,144	8·1	81,180	36·2	14·6
G . .	33,600	15·0	62,700	28·0	28·8
H . .	24,000	10·7	109,800	49·0	7·5
I . .	30,900	13·8	78,800	35·2	22·1
J . .	23,500	10·5	73,200	32·7	22·9
K . .	30,700	13·7	62,000	27·7	28·8
L . .	25,300	11·3	71,900	32·1	27·8

CARBON 0·25 PER CENT.

	Elastic limit per sq. in.		Maximum per sq. in.		Elongation per cent. on 7·08 in.
	Lb.	Tons.	Lb.	Tons.	
A . .	29,300	13·1	73,900	33·0	24·6
B . .	36,300	16·2	70,560	31·5	26·2
C . .	33,600	15·0	67,900	30·3	28·3
D . .	29,600	13·2	70,560	31·5	27·1
E . .	42,800	19·1	75,700	33·8	26·1
F . .	22,200	9·9	83,500	37·3	19·6
G . .	33,600	15·0	69,000	30·8	27·5
H . .	21,700	9·7	104,400	46·6	15·8
I . .	70,300	31·4	112,200	50·1	10·0
J . .	50,600	22·6	92,700	41·4	15·3
K . .	50,900	13·8	71,000	31·7	26·9
L . .	44,600	19·9	90,900	40·6	17·1

CARBON 0·34 PER CENT.

	Elastic Limit per sq. in.		Maximum per sq. in.		Elongation per cent. on 7·08 in.
	Lb.	Tons.	Lb.	Tons.	
A . .	26,650	11·9	78,400	35·0	23·6
B . .	36,300	16·2	78,200	34·9	25·6
C . .	34,900	15·6	74,800	33·4	25·3
D . .	30,900	13·8	76,800	34·3	26·2
E . .	42,800	19·1	81,500	36·4	25·9
F . .	22,200	9·9	87,800	39·2	19·2
G . .	30,700	13·7	76,400	34·1	24·4
H . .	25,100	11·2	129,000	57·6	1·3
I . .	59,100	26·4	127,200	56·8	10·8
J . .	47,500	21·2	99,900	44·6	12·7
K . .	36,300	16·2	75,700	33·8	25·0
L . .	61,600	27·5	102,800	45·9	17·8

Carbon 0·44 per cent.

	Elastic Limit per sq. in.		Maximum per sq. in.		Elongation per cent. on 7·08 in.
	Lb.	Tons.	Lb.	Tons.	
A . .	36,300	16·2	92,500	41·3	18·9
B . .	41,900	18·7	98,100	43·8	19·5
C . .	37,850	16·9	91,600	40·9	21·2
D . .	47,900	21·4	91,600	40·9	23·8
E . .	48,400	21·6	93,400	41·7	21·6
F . .	28,000	12·5	78,800	35·2	17·5
G . .	36,300	16·2	91,200	40·7	19·6
H . .	19,500	8·7	125,000	55·8	0·3
I . .	92,700	41·4	155,900	69·6	9·8
J . .	70,300	31·4	116,900	52·2	13·3
K . .	36,300	16·2	88,500	39·5	21·7
L . .	72,600	32·4	125,900	56·2	13·1

Carbon 0·65 per cent.

	Elastic Limit per sq. in.		Maximum per sq. in.		Elongation per cent. on 7·08 in.
	Lb.	Tons.	Lb.	Tons.	
A . .	40,300	18·0	121,850	54·4	10·6
B . .	51,500	23·0	125,200	55·9	11·6
C . .	41,900	18·7	114,700	51·2	15·0
D . .	56,000	25·0	126,100	56·3	13·8
E . .	54,200	24·2	131,000	58·5	12·3
F . .	33,400	14·9	121,850	54·4	13·2
G . .	44,600	19·9	116,500	52·0	15·6
H . .					
I . .	115,100	51·4	169,600	75·7	7·4
J . .	78,600	35·1	129,700	57·9	11·6
K . .	44,600	19·9	113,100	50·5	13·3
L . .	92,300	41·2	167,550	74·8	10·6

CARBON 0·79 PER CENT.

	Elastic Limit per sq. in.		Maximum per sq. in.		Elongation per cent. on 7·08 in.
	Lb.	Tons.	Lb.	Tons.	
A . .	44,800	20·0	127,450	56·9	10·0
B . .	54,650	24·4	134,600	60·1	10·8
C . .	49,050	21·9	124,100	55·4	12·1
D . .	56,200	25·1	130,600	58·2	12·5
E . .	62,700	28·0	134,200	59·8	12·2
F . .	25,300	11·3	130,100	58·1	11·5
G . .	47,500	21·2	125,000	55·8	13·1
H . .					
I . .	140,400	62·7	178,100	79·5	7·2
J . .	98,100	43·8	136,000	60·7	10·6
K . .	48,800	21·8	127,000	56·7	12·6
L . .	109,500	48·9	179,000	79·9	8·0

CARBON 0·94 PER CENT.

	Elastic Limit per sq. in.		Maximum per sq. in.		Elongation per cent. on 7·08 in.
	Lb.	Tons.	Lb.	Tons.	
A . .	49,050	21·9	139,550	62·3	6·7
B . .	61,600	27·5	145,400	64·9	8·5
C . .	44,800	20·0	133,900	59·8	10·6
D . .	59,600	25·7	130,600	58·3	13·4
E . .	68,300	30·5	153,000	68·3	8·4
F . .	44,800	20·0	141,300	63·1	6·2
G . .	47,700	21·3	143,100	63·9	10·4
H . .	50,400	22·5	136,600	61·0	
I . .	89,400	39·9	188,200	84·0	6·4
J . .	108,860	48·6	142,460	63·6	9·7
K . .	55,800	24·9	131,900	58·9	10·2
L . .	73,500	32·8	192,800	86·1	5·8

CARBON 1·17 PER CENT.

	Elastic Limit per sq. in.		Maximum per sq. in.		Elongation per cent. on 7·08 in.
	Lb.	Tons.	Lb.	Tons.	
A . .	50,400	22·5	125,900	56·2	2·6
B . .	69,900	31·2	144,000	64·3	3·6
C . .	65,850	29·4	135,960	60·7	9·8
D . .	57,400	25·6	125,000	55·8	15·1
E . .	79,300	35·4	147,680	65·9	4·1
F . .	36,500	16·3	142,680	63·7	4·9
G . .	58,900	26·3	140,000	62·5	5·6
H . .					
I . .	92,060	41·1	200,900	89·7	5·8
J . .	110,000	49·1	147,400	65·8	8·3
K . .	61,600	27·5	139,500	62·3	8·6
L . .	77,300	34·5	186,400	83·2	4·0

168. Modern Tool Steel—It has been pointed out that during the heating of carbon steel, changes in the arrangement of the contained carbon take place at certain temperatures. These changes result in the hardening of the steel when the increase in temperature changes the carbon in the steel from the cement form to the hardening form. On the other hand, during cooling the carbon changes back again and the steel becomes soft. These changes have been investigated by Professor H. C. H. Carpenter, who, in one of his papers dealing with the subject, gives several curves which illustrate what happens during the heating followed by cooling. One of these is shown in Fig. 212. The steel in this case was an ordinary carbon tool steel with 0·9 per cent. of carbon. Of the two curves shown, that on the left is the heating curve, and the one on the right the cooling curve.

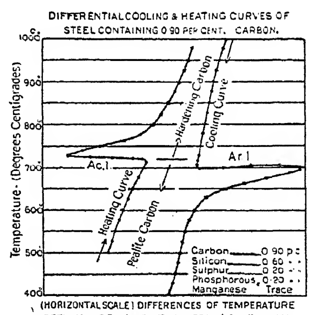


FIG. 212.

Starting from the top of the curve the steel is supposed to be in the hard state at a temperature of 1,000° C. and allowed to cool slowly. From 1,000° C. to 700° C. the cooling curve is represented by a smooth line. At 700° C. there is a swing out

of the galvanometer attached to the differential junction, indicating a large evolution of heat. This causes the break in the line marked Ar1, which break is represented by the horizontal step shown and marks a sudden change in condition. This sudden jump is followed by a more gradual return to the old line at 600° C.

Similarly on heating the curve is smooth up to about 720° C., after which there is a sudden break to the left, followed by a gradual return to the old line at about 800° C. The sudden break is marked Ac 1. The two points Ac 1 and Ar 1 correspond to the W and V previously mentioned and represent the limits of the critical period. If the carbon in the steel is in the pearlite and cement state the metal will be soft and show a coarse fracture. When Ac 1 is reached during heating, the cement carbon is changed to hardening carbon, with the evolution of much heat. The temperature is raised somewhat higher (to 1,000° C.) and the cooling allowed to follow. As the carbon has been brought to the hardening state the metal is in the desired condition and must not be allowed to resume the cement carbon state: hence quenching takes the place of gradual cooling.

This is for ordinary carbon tool steel, in connexion with which the changes brought about by temperature variations are by this time pretty well known: but with self-hardening steel and high-speed steel the conditions have not yet been so well established, although a great deal of experimental work has been completed and much is still being carried out.*

169. Heat Treatment of Self-hardening and High-speed Steel—In steels of this kind the heat treatment is of extreme importance, and, although much has yet to be discovered in this direction, sufficient has been established by the experience of the last fifteen years to enable the following short statement to be made.

This steel is made in crucibles which are placed in apertures called "melting holes," for the melting, after they have been filled with the charge of irons and alloys, which have been carefully weighed out in the proper percentages. The actual melting of the constituents which go to form the desired steel must be carried out with extreme care and a large amount of experience is needful in this work.

The molten steel is poured from the crucibles into ingot moulds and allowed to set. The ingots are reheated to temperatures dependent on their constituents. This reheating is the

* "Evolution of Modern Tool Steel" ("Engineering," May 3 and 17, 1907).

preliminary to hammering alone or hammering followed by rolling to the final sections.

A large proportion of the cutting tools made from the steel thus produced have to be machined in the process of finishing, in such cases as twist drills and milling cutters. It is therefore necessary that the steel be as soft and uniform as possible and for this purpose requires to be thoroughly and carefully annealed previous to being machined. The annealing itself consists in exposing the steel to a steady temperature in a " muffle furnace " where the metal does not feel the direct flame from the fuel but only the radiant heat from the inner surface of the muffle. This annealing removes internal stresses and has the effect of reducing the whole to a more uniform structure than it would have without annealing. In the case of tools of complicated forms, such as milling cutters, it is often advisable to anneal more than once.

Where steel has to be forged, Mr. Gledhill,* of Messrs. Armstrong, Whitworth & Co., recommends that before forging the steel should be raised to a very bright red heat, verging on a yellow heat (about 1,010° C.), at which temperature forging may be easy and rapid. Forging should not continue after the temperature has been allowed to fall to say a medium red (815° C.), when it ought to be reheated. For hardening the cutting part is to be brought gradually to a bright yellow heat, then rapidly to a white melting heat, on the completion of which the tool is cooled rapidly by means of an air blast. Competent authorities recommend the strongest blast possible. In order that the result may be quite satisfactory it is essential that the correct temperature be attained when hardening is aimed at. It has been found, for instance, that temperatures of 1,290° and 1,340° C. have resulted in very hard, crystalline steel, whereas 1,090° to 1,200° C. gave extremely good results.

170. Further Results—In addition to the many experimental results already quoted, the following, relative to tests carried out in the authors' laboratory, are of interest. In most cases the figures given are taken from groups of actual results, the number in the group varying from ten to fifty.

* See paper by Mr. J. M. Gledhill, Iron and Steel Inst., 1904.

Yield Stress per sq. in.		Maximum Stress per sq. in.		Per cent. elongation.
Forged Cast Steel (1) :				
118,700 to 132,000 lb.	or, 53 to 59 tons	130,000 to 152,320 lb.	or, 58 to 68 tons	18 to 19·5 on 2 in.
Forged Cast Steel (2) :				
49,280 to 71,680 lb.	or, 22 to 32 tons	96,320 to 123,200 lb.	or, 43 to 55 tons	9 to 22 on 2 in.
Forged Steel (*Medium*) :				
42,560 to 56,000 lb.	or, 19 to 25 tons	67,200 to 89,600 lb.	or, 30 to 40 tons	19 to 34 on 2 in.
Forged Steel (*Soft*) :				
31,360 to 42,560 lb.	or, 14 to 19 tons	56,000 to 62,700 lb.	or, 25 to 28 tons	23 to 35 on 8 in.
Steel Tyres for Tram Wheels :				
44,800 to 67,200 lb.	or, 20 to 30 tons	94,000 to 116,400 lb.	or, 42 to 52 tons	14 to 25 on 2 in.
Axles for Tram Cars :				
35,940 to 44,800 lb.	or, 16 to 20 tons	65,000 to 87,360 lb.	or, 29 to 39 tons	
Tram-rail Steel :				
58,240 to 76,160 lb.	or, 26 to 34 tons	107,500 to 116,500 lb. (From tread of rail.)	or, 48 to 52 tons	18 to 24 on 2 in.
40,320 to 53,760 lb.	or, 18 to 24 tons	78,400 to 89,600 lb. (From web and flange.)	or, 35 to 40 tons	24 to 32 on 2 in.
Tie Bars for Tram Rails :				
Screwed Ends ($\frac{7}{8}$ in. Whitworth thread) :				
44,800 to 67,200 lb.	or, 20 to 30 tons	67,200 to 87,360 lb.	or, 30 to 39 tons	
Body of Bar :				
39,200 to 45,920 lb.	or, 17·5 to 20·5 tons	56,000 to 76,160 lb.	or, 25 to 34 tons	29 to 32 on 8 in.
Fishplate Bolts (1 in. Whitworth thread) :				
47,000 to 73,900 lb.	or, 21 to 33 tons	85,100 to 107,500 lb.	or, 38 to 48 tons	15 to 32 on 2 in.
High-Compressive Steel :				
156,800 to 224,000 lb. (Proportional limit.)	or, 70 to 100 tons	336,000 to 358,400 lb. (Crushing.)	or, 150 to 160 tons	
Steel Castings :				
29,100 to 42,560 lb.	or, 13 to 19 tons	51,500 to 76,160 lb.	or, 23 to 34 tons	18 to 28 on 2 in.
Boiler-plate Steel :				
35,840 to 42,560 lb.	or, 16 to 19 tons	61,600 to 66,080 lb.	or, 27·5 to 29·5 tons	20 to 30 on 8 in.

(This boiler steel is evidently very uniform in quality.)

Yield Stress per sq. in.		Maximum Stress per sq. in.		Per cent. elongation.
Plate Steel for Tanks :				
38,080 to 42,560 lb.	or, 17 to 19 tons	64,960 to 69,440 lb.	or, 29 to 31 tons	22 to 27 on 10 in.
Steel for Plate Rivets :				
40,320 to 44,800 lb.	or, 18 to 20 tons	58,240 to 60,480 lb.	or, 26 to 27 tons	18 to 28 on 10 in.
Steel for Superheater Tubes :				
Higher tensile.		64,960 to 80,640 lb.	or, 29 to 36 tons	
Softer metal.		49,280 to 64,960 lb.	or, 22 to 29 tons	
Structural Steel : Girders, Channels, Angles, T, L, and other Structural Sections :				
38,080 to 44,800 lb.	or, 17 to 20 tons	56,000 to 71,680 lb.	or, 25 to 32 tons	20 to 29 on 8 in.
Steel for the Reinforcement of Concrete :				
31,360 to 48,160 lb.	or, 14 to 21·5 tons	49,280 to 70,560 lb.	or, 22 to 31·5 tons	17 to 30 on 8 in.
Steel Wire for Reinforcement :				
62,720 to 78,400 lb.	or, 28 to 35 tons	71,680 to 85,120 lb.	or, 32 to 38 tons	4 to 12 per cent.

The following abstracts from the British Standard Specification will be found useful :

* FLAT-BOTTOMED RAILWAY RAILS—These may be made by the Bessemer, Siemens-Martin or other process approved by the engineer. If a full specification is not available, the following extracts may be useful. Their chemical analysis must conform to the following limits :

STEEL MADE BY THE OPEN-HEARTH PROCESS

Element.	Acid.		Basic.	
	Ordinary Carbon.	Higher Carbon.	Ordinary Carbon.	Higher Carbon.
	Per cent.	Per cent.	Per cent.	Per cent.
Carbon	0·45 to 0·55	0·50 to 0·60	0·45 to 0·60	0·55 to 0·65
Manganese	0·90 (max.)	0·80 (max.)	0·90 (max.)	0·80 (max.)
Silicon	0·15 (max.)	0·10 to 0·30	0·15 (max.)	0·10 to 0·30
Phosphorus	0·06 (max.)	0·05 (max.)	0·05 (max.)	0·04 (max.)
Sulphur	0·06 (max.)	0·05 (max.)	0·06 (max.)	0·05 (max.)

* Abstracted by permission of the British Engineering Standards Association from B.S. Specification, "Flat-bottomed Railway Rails," official copies of which can be obtained from the Secretary of the Association, 28 Victoria Street, Westminster, S.W.1, price 1*s.* 2*d.*, post free.

STEEL MADE BY THE BESSEMER PROCESS

Element.	Acid.		Basic.
	Ordinary Carbon.	Higher Carbon.	Ordinary Carbon.
	Per cent.	Per cent.	Per cent.
Carbon	0·40 to 0·50	0·45 to 0·55	0·40 to 0·50
Manganese	0·70 to 1·00	0·90 (max.)	0·70 to 1·00
Silicon	0·15 (max.)	0·10 to 0·30	0·15 (max.)
Phosphorus	0·075 (max.)	0·06 (max.)	0·07 (max.)
Sulphur	0·07 (max.)	0·06 (max.)	0·07 (max.)

*STRUCTURAL STEEL FOR BRIDGES AND GENERAL CONSTRUCTION—Plates and rivet bars to be made by open-hearth process, acid or basic, and must not contain more than 0·06 per cent. sulphur or phosphorus.

Sectional material for bridges—same as above.

Sectional material for general building construction—ditto, except 0·07 per cent. S and Ph.

Tensile test pieces to be cut lengthwise and crosswise from plates and lengthwise for sectional material and bars.

One tensile test should be made from every cast or every 25 tons.

For plates, angles, etc., test pieces having gauge lengths of 8 in., and for bars with gauge lengths not less than 8 times the diameter. Tensile strength to be not less than from 28 to 32 tons per sq. in. with elongation of not less than 20 per cent. For material under $\frac{9}{16}$ in. cold bend tests only are required.

Rivet bars, where test pieces have gauge lengths of not less than eight times the diameter, should show tensile strength of from 26 to 30 tons per sq. in., and an elongation of not less than 25 per cent.

"Cold bend" and "temper bend" tests are the same as for ship material.

* STRUCTURAL STEEL FOR SHIPBUILDING—The steel is to be made by the open-hearth process, either acid or basic, as may be specified.

The material is to be free from cracks, surface flaws, and lamination. Standard test pieces are to be used, cut lengthwise and crosswise from the rolled steel. If the material is annealed before dispatch the test pieces must be annealed also.

In plates the original rolled surfaces are to be kept on the test piece.

* Abstracted by permission of the British Engineering Standards Association from B.S. Specification, "Structural Steel for Shipbuilding, etc.," official copies of which can be obtained from the Secretary of the Association, 28 Victoria St., Westminster, S.W.1, price 1*s.* 2*d.*, post free.

Elongation to be measured on a standard test piece on a gauge length of 8 in.

For plates more than $\frac{7}{8}$ in. thick the width not to exceed $1\frac{1}{2}$ in. on gauge length.

For plates between $\frac{7}{8}$ in. and $\frac{3}{8}$ in. thick the width not to exceed 2 in. on gauge length.

For plates less than $\frac{3}{8}$ in. thick the width not to exceed $2\frac{1}{2}$ in. on gauge length.

Necessary straightening of test pieces to be done cold.

Tensile strength of plates to be between 28 and 32 tons per sq. in.

Tensile strength of plates for cold flanging between 26 and 32 tons per sq. in.

Elongation not less than 20 per cent. on 8 in. for plates $\frac{3}{8}$ in. and over, 16 per cent.

Tensile strength of angles, bulb angles, channels, etc., to be between 28 and 33 tons per sq. in. Elongation as above.

Rivet bars to have tensile strength between 25 and 30 tons per sq. in. Elongation not less than 25 per cent. on Standard bar. These bars may be tested full size as rolled.

If a tensile test piece should break outside the middle half of its gauge length, the test may be discarded and repeated.

For "cold bend" and "temper bend" tests specimens to be sheared lengthwise or crosswise from the plates: for small bars the whole section to be used: for rivet steel "cold bend" tests are not needed. The specimens are not to be less than $1\frac{1}{2}$ in. wide. They are not to be annealed. Specimens $\frac{1}{2}$ in. thick, and only the rough edges due to shearing may be smoothed by file or grindstone: those 1 in. thick or over may be machined. In "temper bend" tests specimens to be brought to blood-red temperature and quenched in water not hotter than 80° F.

In both the above tests the specimens shall withstand, without cracking, being doubled over until the internal radius equals $1\frac{1}{2}$ times the thickness of the plate, and the sides are parallel. Bend tests may be made with pressure or by blows: for small sections these tests may be made upon the flattened bar.

For rivets the tests for ductility are:

(1) The shanks to be bent double upon themselves without crack.

(2) The heads to be flattened until $2\frac{1}{2}$ times the diameter of the shank.

*Structural Steel for Marine Boilers (1905)—Shall be made by the open-hearth process, acid or basic.

Round test pieces to have length not less than eight times the diameter, and sectional area of not less than $\frac{1}{4}$ sq. in.

* *Ibid.*

Plates for shells and girders, tensile strength from 28 to 32 tons per sq. in.: plates for flanging and welding and combustion chambers and furnaces, from 26 to 30 tons per sq. in.

Elongation on 8 in. not less than 20 per cent. in material $\frac{3}{8}$ in. or upwards in thickness and having the former of the above strengths: and not less than 23 per cent. where material has second of these strength limits.

For stay, angle, and tee bars tests as above, where used for same purposes.

"Cold" and "temper bend" tests same as for ship-building.

171. The Elasticity of Plastically Strained Steel—The following experiments were carried out on a turned bar of mild steel, 0·798 in. diameter, giving an area of section of $\frac{1}{2}$ sq. in. Five sets of readings were taken with a Martens extensometer, under the following conditions: (1) In the original state as supplied by maker; (2) when the bar had been stretched $\frac{1}{4}$ in. on the gauge length of 5 in.; (3) after another $\frac{1}{4}$ in. stretch; (4) after a third stretch of $\frac{1}{4}$ in.; (5) following a fourth and last stretch of $\frac{1}{4}$ in. Previous to the last set of readings the bar had thus been permanently elongated $4 \times \frac{1}{4}$ in., or 1 in. in 5 in., or 20 per cent. In the testing machine the material was apparently quite plastic when the stretches took place, but on the removal of the load in each case a hardening must have taken place, as shown by the following figures. Each set of elastic readings followed immediately on the one preceding.

The results obtained from the elastic experiments were the following:

TABLE XXV.

Experiment.	Young's Modulus E. lb. per sq. in.	Limit of Proportionality, per sq. in.		Diameter and	
		Tons.	Lb.	In.	S
1 . .	$26{\cdot}3 \times 10^6$	15·0	33,600	0·798	0
2 . .	$23{\cdot}3 \times 10^6$	16·8	37,600	0·779	0
3 . .	$23{\cdot}9 \times 10^6$	18·6	41,700	0·763	0
4 . .	$23{\cdot}6 \times 10^6$	20·8	46,600	0·744	0
5 . .	$23{\cdot}9 \times 10^6$	22·8	51,000	0·729	0

After each stretch the bar was treated as a new specimen, remeasured, and all calculations made with the new dimensions.

To find the ordinary commercial-strength properties two bars were broken, the original bar above and a second which had not been previously stretched. The first of these gave:

Primitive elastic limit . . .	33,600 lb.	15 tons per sq. in.
Yield point	42,600 lb.	19 ,, ,, ,,
Maximum stress . . .	58,700 lb.	26·2 ,, ,, ,,

Elongations: 46 per cent. on 2 in., 33·6 per cent. on 5 in., and 30·3 per cent. on 7 in.

The second bar of the same material, uninfluenced by the treatment to which the above had been submitted, was found to yield results which were practically the same.

The stresses and corresponding elastic strains are shown plotted on the accompanying diagram, Fig. 213. These, taken

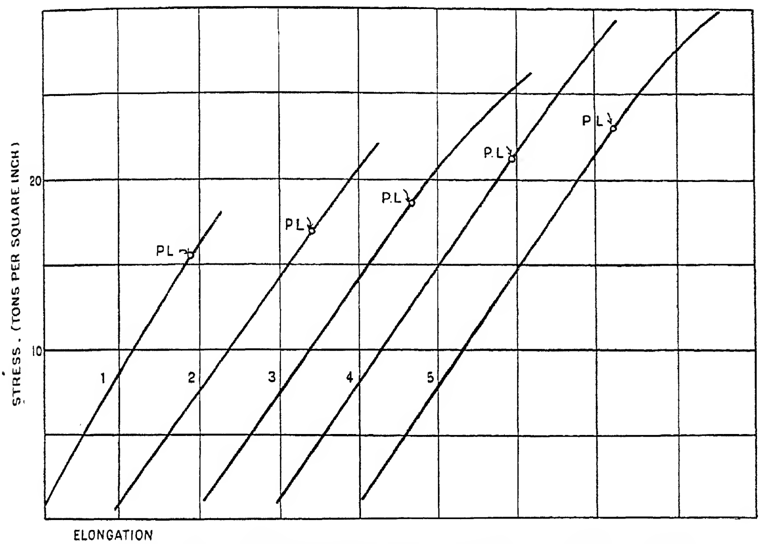


FIG. 213.—Plastically Strained Steel Plotting of Elastic Strains.

with the above numerical results, serve to emphasize one or two salient facts. The first of these is that after the first plastic stretch the value of the modulus drops from 26·3 to 23·3, or rather more than 12 per cent. Beyond this the modulus remains practically constant until the bar has been stretched 20 per cent. In other words the steel, even after sustaining a plastic stretch of 20 per cent., is nearly, but not quite, as elastic as in its primitive state.

The second point worth noting is that following the succeeding stretches there is a gradual exaltation of the P. Limit, and what is virtually a new material is created.

The third point is that, as the successive stretches reduce the cross-section, the ultimate strengths worked out on the new areas must have gradually increasing values. That is the

ultimate strengths will be 26·2, 27·5, 28·6, 30·0, and 31·2 tons, or 58,700, 61,700, 64,000, 67,100, and 69,800 lb. per sq. in. As the calculated breaking stress appears to increase after the successive stretches and their corresponding contractions, so the ductility as shown by the percentage elongation after fracture grows less. Thus, the percentage elongation at fracture of a bar whose gauge length as marked on the new bar comes out at 33·6 per cent., or 1·68 in. actual stretch. If, however, the 5 in. had been lengthened to 6 in., and 5 in. of this 6 in. had been taken as the new gauge length, the percentage would work out at $=\frac{5.0{\cdot}68}{6{\cdot}5} = 11{\cdot}33$ per cent. instead of 33·6 per cent. So that by subjecting steel to a certain amount of strain beyond the yield point a new material is formed whose elastic properties are only slightly impaired in comparison with those of the original material, whose tensile properties are increased, but whose ductility is reduced.

This process is often used in the steel for reinforced concrete, generally by twisting, and possesses obvious advantages. Whether subsequent heat treatment would partially restore the steel to its primitive condition the present experiments do not indicate.

172. The Relation between Stress and Plastic Strain— TENSION—The relation between tensile stress and the plastic strain caused by it was investigated as long ago as 1893 by Dr. T. E. Stanton. His work related more particularly to wrought iron and steel. Briefly, he found that by plotting logarithms of the stresses and logarithms of the corresponding strains the mutual relation was expressed by the following: $p = C.e^k$, p being the stress on the bar and e the ratio of the elongation to the original length. The values for the above symbols were found from the experiments. The value for k was 0·25.*

COMPRESSION—A similar relation to the above where a cylindrical bar is under a compressive stress has been found more recently. It was found when carrying out the tests that the specimen, originally cylindrical from end to end, only retained the cylindrical form for about two-thirds of its length near the middle, and that it terminated in two truncated cones. The strain measurements were taken between two centre dots marked on the surface of the portion which remained parallel. The results obtained are expressed in the following formula: $p = C.e^k$ where

wrought iron $p = 55{\cdot}30.e^{0{\cdot}178}$
annealed mild steel $p = 65{\cdot}31.e^{0{\cdot}251}$ †

* Manchester Lit. and Phil., 1893, Dr. Stanton.
† Manchester Lit. and Phil., 1905, W. C. Popplewell.

Torsion—A similar relation to the two mentioned above can be found between shear stress and angular distortion. This is best done by making a torsion experiment and carrying the twisting moments far beyond the yielding point up to fracture. The bars may be solid or hollow. On Fig. 214 is shown the view of a short length of a cylindrical shaft under a distorting twisting couple. When a shaft is twisted the point A moves towards C and the axial line BA takes the position BC by moving through an angle α which is the angle of distortion. The angle of twist measured is θ and from it and the radius the arc AC may be calculated. If the length of the shaft is l, AC thus found divided by l is the tangent of the angle α. This gives the tangent of the angle of distortion, from which can be obtained the angle itself. In an ordinary torsion-testing machine, such as that of Avery already described, the angles of twist during the semi-plastic and plastic stages are measured by the number of teeth or fractions of teeth turned through by the worm wheel. The shearing stress f_s is best obtained by assuming it to be uniform across the section and having a value equal to $3.T/2\pi(R_2^3-R_1^3)$. This is for a hollow shaft having an outer radius R_2 and an inner one R_1, T being the twisting moment. If a test is made and the twisting couples plotted as abscissæ and the angles of distortion as ordinates a curve of the form shown on Fig. 215 is obtained. If a second diagram be plotted with abscissæ representing logarithms of twisting moments and ordinates as logs of angles of distortion a curve as shown on Fig. 216 will be the result. The relation between twisting moment and distortion is given by $T = A.\alpha^m$ for the lower part of the line, and $T = B.\alpha^n$ for the upper part. Where these two parts of the line meet there is evidently a change in the law. Introducing the shearing stress and assuming it to be uniform at all points across the section,

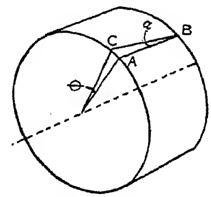


Fig. 214.

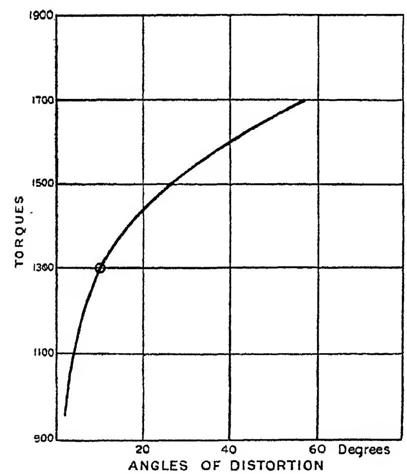


Fig. 215.—Curve for Torques and Angles of Distortion.

the above relations are represented by $f_s = B.a^n \times \frac{3}{2}/R_2^3 - R_1^3$.

In the case of a certain hollow mild-steel shaft the relationship was found to be $f_s = 20{,}200.a^{0\cdot231}$ where $R_1 = 0\cdot157$ and $R_2 = 0\cdot25$.

The following figures are interesting as showing the limits imposed on steel for gas bottles (from the "Report of the Committee appointed to inquire into the Causes of the Explosion and the Precautions required to ensure the Safety of Cylinders of Compressed Gas." 1903. Price 1*s.* 4½*d*).

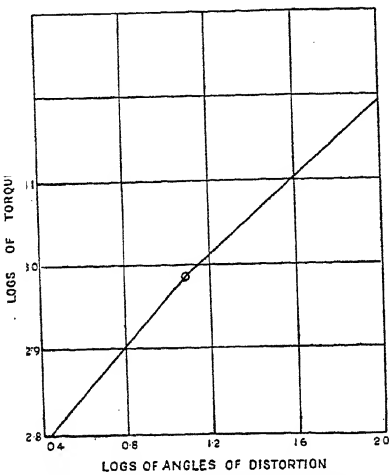


FIG. 216.—Logarithms of Torques and Angles of Distortion.

173. Summary: Cylinders of Compressed Gas (Oxygen, Hydrogen, or Coal Gas).

(*a*) LAP-WELDED WROUGHT IRON—Greatest working pressure, 120 atmospheres, or 1,800 lb. per sq. in.

Stress due to working pressure not to exceed 6½ tons per sq. in.

Proof pressure in hydraulic test, after annealing, 224 atmospheres, or 3,360 lb. per sq. in.

Permanent stretch in hydraulic test not to exceed 10 per cent. of the elastic stretch.

One cylinder in fifty to be subjected to a statical bending test, and to stand crushing nearly flat between two rounded knife-edges without cracking.

(*b*) LAP-WELDED OR SEAMLESS STEEL—Greatest working pressure 120 atmospheres, or 1,800 lb. per sq. in.

Stress due to working pressure not to exceed 7½ tons per sq. in. in lap-welded, or 8 tons per sq. in. in seamless cylinders.

Carbon in steel not to exceed 0·25 per cent. or iron to be less than 99 per cent.

Tenacity of steel not to be less than 26 or more than 33 tons per sq. in. Ultimate elongation not less than 1·2 in. in 8 in. Test bar to be cut from finished annealed cylinder.

Proof pressure in hydraulic test, after annealing, 224 atmoatmospheres, or 3,360 lb. per sq. in.

Permanent stretch shown by water jacket not to exceed 10 per cent. of elastic stretch.

One cylinder in fifty to be subjected to a statical bending test, and to stand crushing nearly flat between rounded knife-edges without cracking.

* The following bending tests were carried out in the classes of the Department of Engineering upon samples of tramway rails supplied by the Manchester Corporation Tramways Department.

The object of the tests was to find how the strength of jointed tram-rails compared with that of the solid rail.

The samples were classified as follows :—

A. New rail, solid throughout (no joints).

B. New rail with ordinary fishplate joint in middle.

C. New rail with ordinary Weldite fishplate joint in middle.

TABLE XXVI. TEST ON SPECIMEN A.—NEW SOLID RAIL

Load. lb.	Reading on Micrometer. In.	Total Deflection under Load. In.	Deflection under last increment of Load. In.	Permanent Set. In.
4,000	7·055	—	—	—
8,000	6·989	·066	·066	—
4,000	7·052	—	—	·003
12,000	6·935	·120	·117	—
4,000	7·054	—	—	·001
16,000	6·889	·166	·165	—
4,000	7·052	—	—	·003
20,000	6·863	·192	·189	—
4,000	7·051	—	—	·004
24,000	6·773	·282	·278	—
4,000	7·054	—	—	·001
28,000	6·717	·338	·337	—
4,000	7·051	—	—	·004
32,000	6·659	·396	·392	—
4,000	7·046	—	—	·009
36,000	6·603	·452	·443	—
4,000	7·045	—	—	·010
40,000	6·533	·522	·512	—
4,000	7·029	—	—	·026
44,000	6·458	·597	·571	—
4,000	7·007	—	—	·048
48,000	6·297	·758	·710	—
4,000	6·896	—	—	·159
52,000	5·890	1·165	1·006	—
4,000	6·561	—	—	·494
56,000	5·121	1·934	1·440	—
4,000	5·895	—	—	1·160

From Report published at the Manchester School of Technology.

TABLE XXVII. TEST ON SPECIMEN B.—NEW STEEL RAIL WITH FISHPLATE JOINT IN MIDDLE

Load. Lb.	Reading on Micrometer. In.	Total Deflection under Load. In.	Deflection under last increment of Load. In.	Permanent Set. In.
4,000	7·936	—	—	—
8,000	7·838	·098	·098	—
4,000	7·910	—	—	·026
12,000	7·753	·183	·157	—
4,000	7·902	—	—	·034
16,000	7·666	·270	·236	—
4,000	7·871	—	—	·065
20,000	7·633	·303	·238	—
4,000	7·898	—	—	·038
24,000	7·328	·608	·570	—
4,000	7·663	—	—	·273
28,000	6·448	1·488	1·215	—
4,000	6·878	—	—	1·058
32,000	5·611	2·325	1·267	—
4,000	6·137	—	—	1·799
36,000	3·976	3·960	2·161	—
4,000	4·682	—	—	—
44,000	Maximum load	Deflection several inches		

TABLE XXVIII. TEST ON SPECIMEN C.—NEW STEEL RAIL WITH WELDITE JOINT IN MIDDLE

Load. Lb.	Reading on Micrometer. In.	Total Deflection under Load. In.	Deflection under last increment of Load. In.	Permanent Set. In.
4,000	7·131	—	—	—
8,000	7·019	·112	·112	—
4,000	7·086	—	—	·045
12,000	6·935	·196	·151	—
4,000	7·015 ?	—	—	·116
16,000	6·853	·278	·162	—
4,000	7·029	—	—	·102
20,000	6·819	·312	·210	—
4,000	7·027	—	—	·104
24,000	6·713	·418	·314	—
4,000	7·004	—	—	·127
28,000	6·640	·491	·364	—
Crack occurred here 4,000	6·982	—	—	·149
32,000	6·562	·569	·420	—
4,000	6·959	—	—	·172
36,000	6·468	·663	·491	—
Several cracks here { 4,000	6·927	—	—	·204
36,000	Specimen broke with many loud cracks.			

All were of standard weight (103 lb. per yard) and pattern, and had an overall length of 10 ft.

The bending test was made on each sample separately, and under the same conditions of test. They were placed tread side down, and the load was applied upwards in the centre by hydraulic pressure. The span in each case was 9 ft. clear from centre to centre of rollers forming the end supports.

Method of Test—An initial load of 4,000 lb. was applied and increased by steps of 4,000 lb. After each deflection for the last load had been noted, the whole of the load (with the exception of the initial 4,000 lb.) was removed and the permanent set (if any) noted. Measurements of the deflection in the centre of the span were made by micrometers reading to ·001 in.

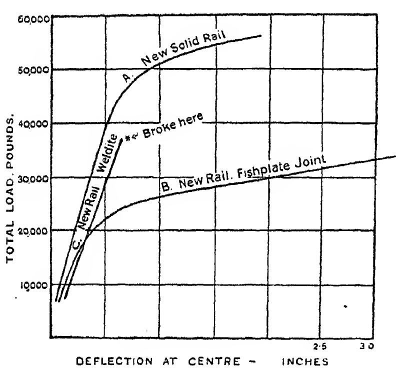


Fig. 217.

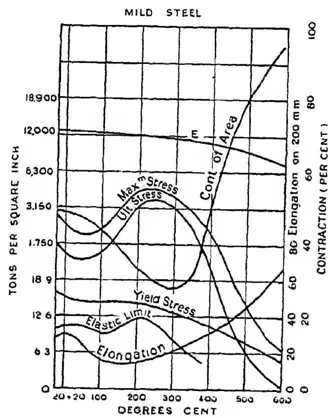


Fig. 218.

174. Metals at High Temperatures—All metals used in engineering can exist in three states, i.e. liquid, solid, and gaseous, and it may be looked upon as a coincidence that most metals are solid at atmospheric temperature and can be used in structures. The mechanical properties of metals at ordinary temperatures are well known but vary significantly as the temperature is increased, and in the case, for instance, of boiler, turbine, and internal-combustion engine work some knowledge of the relation between the properties and the temperature is desirable.

Probably the most complete investigation on the mechanical properties of metals at high temperatures was conducted in the Berlin Testing Laboratories by Professor Martens about 1890. The curves* from his results shown in Fig. 218 show the relation

* "Proc. Inst. C.E.," vol. civ. (1891). (Trans. by G. R. Bodmer.)

between the properties and the temperature for mild steel with a tensile strength when cold of about 23 tons per sq. in. It will be noted that the ultimate strength falls about 5 per cent. at from 20° C. to 70° C. and then gradually rises to about 25 per cent. above the value at 20° C. from 70° C. to 250° C. After this it falls, passing through the same value as at 20° C. at a little less than 400° C. The elastic limit and yield point vary in a somewhat similar manner.

The elongation falls to about one-half to one-third at 150° C., afterwards rising steadily as the temperature increases and passing through the same value as at 20° C. at about 350° C.

Thus, in the neighbourhood of 150° C. the steel is much more brittle than at ordinary temperature.

The value of Young's Modulus falls about 3 per cent. to 4 per cent. for every 100° C. up to 300° C. and then a little more rapidly, being at 600° C. about two-thirds of its value at 20° C.

The curves for wrought iron obtained in the above-mentioned

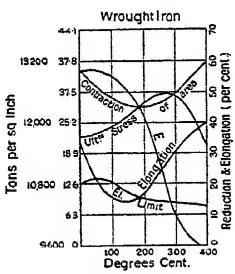


Fig. 219.

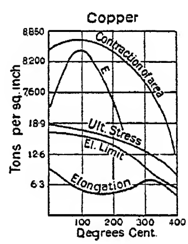


Fig. 220.

laboratories about 1893 are shown in Fig. 219 and the curves for copper in Fig. 220.

The bars in the above experiments were heated through a bath of oil or alloy which was fitted with a stirring arrangement. The temperatures were taken by a mercurial thermometer up to 400° C. and by an air thermometer at higher temperatures.

The mechanical properties of copper alloys at high temperature are sometimes required, and from experiments conducted by Professor Martens the change in the properties from 0° C. to 100° C. for cast and rolled delta metal and manganese bronze does not exceed 5 per cent.

Further experiments on copper alloys were conducted by W. C. Unwin * and Huntington,† who found that generally the tenacity elongation and contraction decreased with increase of

* Report of Brit. Assoc., 1889.

† "Inst. of Metals," vol. ii, 1912.

temperature. Unwin used an oil bath to heat the bars and Huntington tested his bars horizontally and applied gas jets to each enlarged end, the middle portion being heated by conduction. Huntington also gives the chemical analysis of the alloys he tested.

It may be noted that the power of a metal to resist shock depends to a large extent upon the temperature. In some tests on railway axles performed by T. Andrews* he found that they could only withstand one-third the number of blows at 300° C. that they did at 20° C. In the case of impact tests, such as the Izod, it is important that the temperature be noted at which the test is conducted.

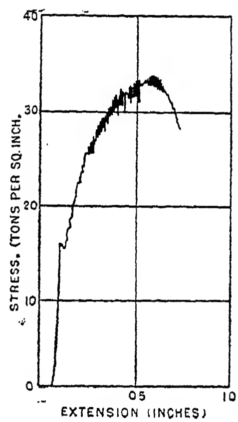


Fig. 221.

In Fig. 221 is shown an autographic record of a mild-steel bar tested at a temperature of about 150° C., and it will be seen that irregularities appear in the plastic portion of the curve which are absent for a cold specimen of the same material. An irregularity appears to be caused by a cessation in the elongation while the load continues to increase. The elongation then takes place suddenly, which causes the beam of the machine to oscillate and so obscure the proper shape of the irregularity. The sudden slight elongation is accompanied by a sound issuing from the bar suggestive of the crystals sliding over one another. In the above case the bar was heated by passing an alternating electric current through it, the temperature being taken by a copper-eureka thermo couple.

* Soc. Engineers, London, 1896.

CHAPTER XIV

ALLOY STEELS

THE extra constituents used for giving special properties to the steel of cutting tools are the metals tungsten, chromium and vanadium. Manganese in certain quantities also is capable of imparting to the steel some of the desired qualities, but the most important are the three metals above mentioned with the addition of another metal, molybdenum. Of these constituents the most useful is probably tungsten, after which these special steels are often named. Cutting tool steel may be divided into two distinct classes, called respectively carbon steel and high-speed or self-hardening or tungsten steel. Of these two the carbon steel had been used almost to the exclusion of any other kind until about sixteen years ago, when a great advance was made in the employment of what has now come to be recognized as the steel most suitable for the tools used in the cutting of metals. The old carbon steel is a carbon, manganese, silicon, sulphur, phosphorus steel, with a high percentage of carbon. The carbon content is somewhere about 1·25 per cent. High-carbon steel of this kind has the useful property of becoming extremely hard after being heated to redness and suddenly cooled by being plunged into water. After such treatment it is too hard and brittle for use in turning, planing, and drilling tools and has therefore to be tempered by slow heating up to some known temperature. This raising of the temperature reduces the hardness until the required degree has been reached, when the process can be stopped by quenching. The degree of this tempering or reduction of extreme hardness is carried to an extent fixed by the purpose for which the tool is about to be used. If the temperature be again raised to anything approaching redness, the steel on being allowed to cool slowly assumes its condition of comparative softness. The same result would be attained by hard cutting so as to overheat the tool.

Below are given tables showing respectively the percentages of carbon in several steels used for well-known purposes, and the temperatures at which a typical carbon steel attains certain definite degrees of hardness after tempering.

175. Self-hardening Tool Steel—Tool steel which is made

sufficiently hard for cutting purposes by the simple fact of being allowed to cool slowly after being forged into its required shape, was discovered by Robert Mushet between 1860 and 1870; but it is only recently that steel of this nature has come into anything like general use. Mushet found that a steel containing a moderate amount of tungsten in combination with a percentage of manganese somewhat higher than the amount usually contained in tool steel was as hard after slow cooling, following forging, as a high-carbon tool steel would be after being heated to redness and cooled very quickly by quenching in water. Steel of this kind was for many years—until about 1890—spoken of as Mushet steel and was used only to a very limited extent. At the date mentioned Mr. F. W. Taylor, in the United States, discovered that by reducing the excess manganese and using chromium in its place along with the tungsten he was able to produce a far more satisfactory self-hardening or air-hardening steel. Since that time the use of steel of this kind has rapidly increased and become more general.* The actual composition of a number of self-hardening steels is given below:

TABLE XXIX. COMPOSITION OF FOUR TYPICAL TOOL STEELS (PERCENTAGE)

Make.	Tungsten.	Chromium.	Carbon.	Manganese.	Vanadium.	Silicon.	Phosphorus.	Sulphur.	Speed in feet per minute when cutting Medium Steel.
Jessop . . .	—	0·207	1·047	0·189	—	0·206	0·017	0·017	16
Mushet self-hardening .	5·441	0·398	2·150	1·578	—	1·044	—	—	26
Original Taylor-White . .	8·000	3·800	1·85	0·300	—	0·150	0·025	0·030	58 to 61
Best high-speed, 1906 . . .	18·910	5·470	0·670	0·110	0·290	0·043	—	—	99

The above were given fourteen years ago, but more recent developments have not changed the composition of these high-speed steels to any great extent since that time. It is considered (by Mr. F. W. Taylor) that the carbon in high-speed steel should not be less than 0·50 per cent. and not more than 0·68 per cent.; that to obtain a steel that is tough, forgeable, and not liable to fire-cracks, the percentage of manganese should be kept low; the sulphur and phosphorus should be very low; and the best high-speed steels contain from 5·5 to 6·0 per cent. of chromium and 18 to 19 per cent. of tungsten. Durability is better when the silicon content is low.

* Quoted from a paper by Mr. F. W. Taylor.

The above figures will give the reader some idea of the marked influence exerted on tool steels by very small quantities of what might at first sight be considered almost extraneous substances. New discoveries are being made every day, and it is impossible to say in what direction the next notable development will take place.

By referring to the above table it will be seen that in the case of the first steel mentioned, that made by Jessop of Sheffield, the material is evidently that of a carbon tool steel containing about $\frac{1}{8}$ of 1 per cent. of chromium in addition to a little over 1 per cent. of carbon. When cutting a medium carbon steel the tool speed must not exceed 16 ft. per minute, or overheating of the tool may be expected, with consequent loss of cutting hardness. It will further be seen that, as in the case of the three remaining steels, the tungsten content grows to nearly 19 per cent. and the chromium to nearly $5\frac{1}{2}$ per cent. and the possible cutting speed has increased to approximately 100 ft. per minute. The speed thus attained would be quite impossible with a carbon steel, which would lose its temper almost at once. The net result of the peculiar properties given to tool steel by the addition of tungsten, chromium, and vanadium is that it is possible to run lathes and other machine tools at far higher speeds than formerly and consequently to remove greater amounts of the material in a given time, resulting in an increased output for the works in which these tools are in use. It was at the last Paris Exhibition that British engineers were first abundantly convinced of the marvellous efficacy of high-speed steel by seeing an American lathe at work, using Taylor-White steel—in which the tool was visibly red-hot and the shavings were coming off from the tool at a blue heat. This was the period of the revival of the Mushet principle, which had been dormant for nearly thirty years.

Such briefly are the peculiarities induced in carbon steels, for all the tool steels contain sufficient carbon to be referred to as carbon steels, although the tendency is in the direction of a reduction of the carbon percentage—by the addition of relatively small percentages of two or three of what may be called the subsidiary metals. Some of these rarer metals are also used for the purpose of giving added strength properties to steels other than tool steels.

They include those already mentioned, with the addition of nickel. Most steels in this category are placed in the general class of "alloy steels."

176. Manganese Steel—This can hardly be called an alloy steel in the ordinary sense, since all steels contain more or less manganese, but the kind of steel which goes by this name, often called "Hadfield's manganese steel," after the name of the man

who developed its manufacture, has some very remarkable properties.

This steel which contains the relatively high percentage of manganese, giving it the special properties to be mentioned, may contain as much as 12 per cent. of manganese in addition to about 1·5 per cent. of carbon. The peculiar property which an adequate percentage of manganese gives to the steel is entirely dependent on the precise quantity present. For instance, when the manganese is increased from below 1 per cent. to 1·5 per cent. it is found that its effect is to make the steel brittle, and this brittleness is further increased as the manganese is increased, until with a percentage of 4 to 5·5 per cent. the metal becomes so brittle that it can be easily broken with a hammer and even pulverized. When the manganese is increased beyond this the steel becomes more ductile until a point of maximum strength is reached with about 18 per cent. This strength is accompanied by great hardness, and it is this combination of hardness with ductility that adds to its value. The ductility is deeply affected by the rate of cooling. The effect of the rate of cooling is unlike that of carbon steel. In this latter rapid cooling conduces towards hardness and slow cooling gives increased ductility and softness, but in Hadfield's manganese steel it is found that the effect of sudden cooling is to make the steel very ductile and that slow cooling renders it more brittle. At the same time the hardness is not affected materially by the rate of cooling. Manganese steel is used in cases where hardness and ductility are desired at the same time. Such instances are found in railway crossings, car wheels, machinery for crushing rock, and safes: in the last case an application of the burglar's blow-pipe, in which case the heating is slow, has no appreciable effect in attaining the desired result of softening the metal. This last property, known as that of self-hardening, is found to be at its best with about 14 per cent. of manganese. During a tensile test of manganese steel it is noticed that the elastic limit is high and that there is no well-marked yield point. On the accompanying figure (222) is shown a diagram in which is represented the extent to which

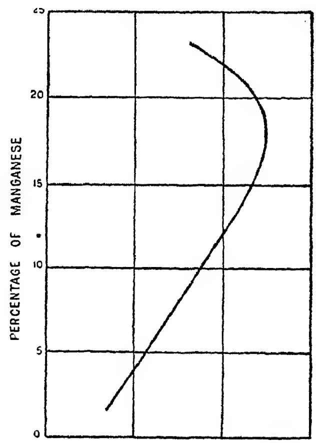


FIG. 222.—Effect of Manganese on (ultimate) Strength of Manganese Steel.

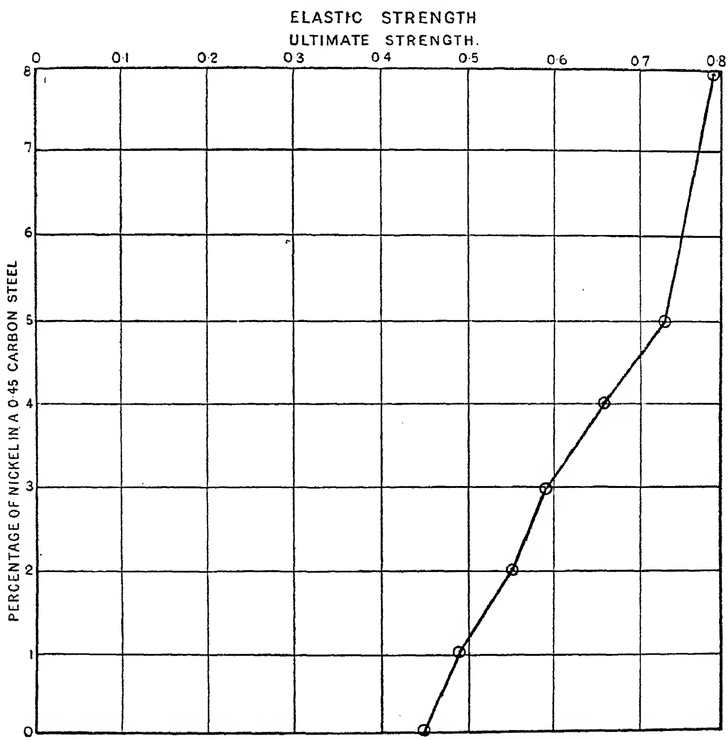
the tensile strength of manganese steel in an untreated state is dependent upon the quantity present.

177. Nickel Steel—The most important effects of adding a reasonable quantity of nickel to a carbon steel are increase of tensile strength, increased ductility, and, most important, a raising of the elastic limit or increase of the ratio of elastic limit to ultimate strength. The following table contains some figures which show the effect of adding nickel to a carbon steel containing 0·45 per cent. carbon.

EFFECT OF NICKEL ON ELASTIC RATIO

Per cent. Nickel .	0	1	2	3	4	5	
Ratio $\frac{\text{Elastic}}{\text{Ultimate}}$ . .	0·45	0·49	0·55	0·59	0·66	0·73	0·79

This is shown graphically in Fig. 223.



INFLUENCE OF NICKEL ON $\frac{\text{ELASTIC STRENGTH.}}{\text{ULTIMATE STRENGTH.}}$

FIG. 223.

In the more usual cases of nickel steel about 3·5 per cent. of nickel is added to a low-carbon steel containing say 0·25 per cent. carbon—in fact an ordinary structural steel. The result is a steel of high elastic and ultimate strength, of which the former is especially important, combined with great hardness and ductility.

This alloy, largely by reason of its combination of ductility with hardness as well as strength, has been very largely used in the construction of armaments, notably for the armour of warships and for shell cases. The armour of warships, both the main armour and the lesser barbette armouring, is now made with a steel containing 0·25 per cent. carbon, 3·25 per cent. nickel, and 1·5 per cent. chromium. In the case of side armour plates there is a deeply carburized layer on the outside surface which is likely to be exposed to the impact of enemy projectiles, the ductility preventing cracking, though some penetration is possible.

Another important use of nickel steel is in revolving shafting, especially marine shafting, in which a high elastic strength and non-liability to develop cracks when overstrained are important. Apart from its initial strength properties, elastic and ultimate, a steel containing nickel in a suitable quantity is better able to withstand repetitions and reversals of stress than a simple carbon steel. When a length of shafting is running, bending stresses may be added to the more or less constant torsional stress, either by the fact that the shaft is constrained to run out of line by movements of its bearings, or, in the case of shop shafting, by the belt tension on the pulleys. In such cases as these the stress in the material is not only added to, but the extra stress is applied over and over again, and it has been known for something like fifty years that if a stress of the same kind is repeated many times the material exposed to it will fracture sooner than if the same stress had been steadily applied for the same length of time. Further, if the stress is not simply repeated but completely reversed every time, fracture will occur even sooner.

178. Vanadium Steel—Although it is only comparatively recently that the manifold virtues of vanadium have been fully recognized by the engineer, it was actually discovered as long ago as 1831, to be thoroughly examined by Sir Henry Roscoe thirty-six years after. Vanadium itself appears to possess extreme hardness and a very high melting-point. Its great usefulness lies in its power of affecting the properties of the carbon steel to which it may be added in small quantities. Briefly, the effects produced on carbon steel by the addition of small percentages of vanadium are :

(1) A raising of the elastic limit when under steady load ;

(2) An increase of power to resist rapid alternations of stress and stresses due to shock;

(3) Raising the ultimate stress under steady load; and

(4) Giving steel the property of retaining its cutting hardness even when raised to a very high temperature in the process of heavy work.

These added properties in steel are all extremely important to engineers in different ways. The first three are important to the designer of munitions of war, where great strength is needed, and in high-speed machinery generally and motor-car and aeroplane engines in particular. Quality No. (4) becomes important when the steel is in the hands of the users of high-speed lathes.

It is important to observe that as the melting-point of vanadium is higher than that of platinum and therefore very much higher than the temperature of molten steel, the vanadium cannot be conveniently added to the steel in its pure state or commercially pure state. Therefore an alloy of iron and vanadium is produced in the process of the electrical smelting which melts at a very much lower temperature, and it is in this form that the vanadium is added to the steel as desired. This alloy is called "ferro-vanadium."

The following figures will show what kind of influence is gained by adding vanadium to steels of different compositions and qualities.

TABLE XXX. VANADIUM AND CARBON STEEL (Carbon 1·1 per cent.)

Vanadium, per cent.	Elastic Limit, per sq. in.		Percentage Reduction in Area.
	Lb.	Tons.	
0·00	67,000	29·9	7·0
0·14	96,120	43·0	6·9
0·29	99,320	44·7	10·0
0·58	144,800	64·7	7·6
0·77	131,300	58·8	9·3
1·11	120,600	54·0	17·6

From these figures it appears that the addition of small quantities to this particular carbon steel has the effect of increasing the tensile strength as represented by the elastic limit without any loss in ductility. The maximum effect is attained when the quantity of vanadium added amounts to about 0·6 per cent.,

after which there is a tendency for the strength to fall off again. These figures are plotted on Fig. 224. The following table* is interesting :

TABLE XXXI. EFFECT OF VANADIUM ON SOME ALLOY STEELS

Kind of Steel.	C per cent.	Mn per cent.	Chr per cent.	Ni per cent.	Van per cent.	Elastic Limit per sq. in.		Reduction in Area per cent
						Lb.	Tons.	
Nickel forging . . .	0·26	0·50	1·00	—	0·16	61,920	27·6	57·3
Chr-Van spring . . .	0·40	0·77	1·22	—	0·19	67,520	30·5	61·7
Chr-Van-Ni	0·30	0·27	1·51	3·45	0·08	69,140	30·8	68·5
Ni-Van	0·24	0·72	—	3·40	0·15	79,260	35·0	64·0
Chr-Van-Ni	0·57	0·27	0·93	2·04	0·07	65,150	29·2	49·8
Van casting . . .	0·19	0·60	—	—	0·76	44,340	19·8	44·9
Chr-Van spring . .	0·40	0·77	1·22	—	0·19	183,400	50·6	50·6
Carbon spring . . .	1·00	0·30	—	—	—	63,800	28·4	15·2
Carbon casting . . .	0·18	0·65	—	—	—	34,690	15·5	44·9
Carbon spring . . .	1·00	0·30	—	—	—	101,000	45·0	16·1

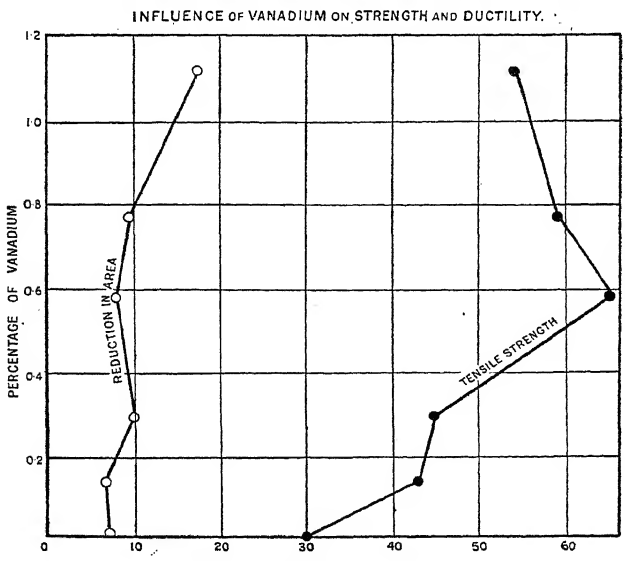


FIG. 224.

Of these, the first seven were annealed, while the rest were oil-tempered at 900° C. and afterwards reheated to 550° C.

* Capt. H. R. Sankey, Proc. Inst.M.E., 1904.

It is possible from these figures to compare the strength properties of several typical steels with and without vanadium, and to note how both the tensile strength and the ductility are raised by its addition.

Experiments carried out at University College on the endurance of steel in the form of rotating beams showed that annealed vanadium steel gave the following results among others.

TABLE XXXII

Material.	Tenacity.	Elastic Strength.	Maximum Stress endured.	Range of Stress endured.
Annealed van. steel . .	38·4	26·8	20·5	41·0
Oil-tempered van. steel .	57·2	46·5	24·5	49·0
These results may be compared with some obtained by Mr. Popplewell on carbon steel, the experiments being carried out in a similar manner.*				
Carbon tool steel	55·7	22·2	22·4	44·8
Forging steel	28·5	12·5	12·2	24·4

The elastic limit given in the results is the primitive limit of proportionality. All stresses are in tons on the sq. in.

The evidence contained in the above figures should be sufficient to make it clear that the presence of vanadium in steel conduces to high elastic and maximum strengths and to endurance under repetition and reversal. More cogent reasons why vanadium steels are especially suitable where reversals and shocks have to be sustained have been found from actual practical experience in the use of vanadium steel under these trying conditions.

One of the most valuable of the alloy steels containing vanadium is chrome-vanadium steel, which, in the opinion of some authorities, is the best material for machine parts and springs. Besides having high elastic and ultimate strength, it is extremely ductile, and at the same time can be rendered so extremely hard by quenching as to be capable of scratching glass.

179. Molybdenum Steels—The effect of the addition of a certain quantity of the metal molybdenum to a steel already comparatively high in carbon is to improve some of its strength properties. The yield point is raised as well as the ultimate strength. It is the opinion of some metallurgists that molybdenum has a power about $2\frac{1}{4}$ times that of tungsten in hardening the steel. The

* W. C. Popplewell, "Inst. C.E.," vol. cxcvii.

following are a few figures which serve to illustrate the above effects of molybdenum in steel. These figures have been taken from a paper by Professors Arnold and Reed read before the Institute of Mechanical Engineers. Here are analyses of five samples of steel in which the carbon is nearly constant, and the percentage of molybdenum gradually increases.

TABLE XXXIII. TABLE SHOWING THE EFFECT OF MOLYBDENUM ON CARBON STEEL

Carbon per cent.	Molyb-denum per cent.	Yield Point per sq. in.		Maximum Strength per sq. in.		Per cent. Elonga-tion on 2 in.	Per cent. Reduction in Area.
		Lb.	Tons.	Tons.	Lb.		
0·78	2·430	56,540	25·24	45·80	102,600	15·6	29·1
0·75	4·95	85,750	38·28	49·32	110,480	11·6	19·9
0·71	10·15	87,000	38·84	49·16	110,120	11·6	23·5
0·79	15·46	91,840	41·0	55·26	123,780	14·5	24·9
0·82	20·70	88,800	39·64	53·72	120,330	13·6	19·3

The appearances of the fractures were respectively (when taken in the same order):

(1) Very fine crystalline.
(2) Fine crystalline.
(3) Rather coarse crystalline.
(4) Somewhat crystalline (radial).
(5) Fine crystalline.

The above figures are plotted on Fig. 225.

180. Nickel-Chromium-Carbon Steels—Apart from the uses which have been mentioned, there are certain kinds of alloy steels (in addition to vanadium steel) whose qualities render them especially useful for resisting shocks and rapid reversals of stress such as are likely to be met with in high-speed engines and machinery and in motor cars and wagons. Among the most important steels used for this purpose are the now well-known "chromium" and "nickel-chromium" steels. These, besides being strong to resist steady loads, are especially endowed with shock-resisting qualities which render them valuable for the purposes above mentioned. The following figures, taken from a paper read before the Institution of Automobile Engineers, will show the effect of varying the analyses of the steels.

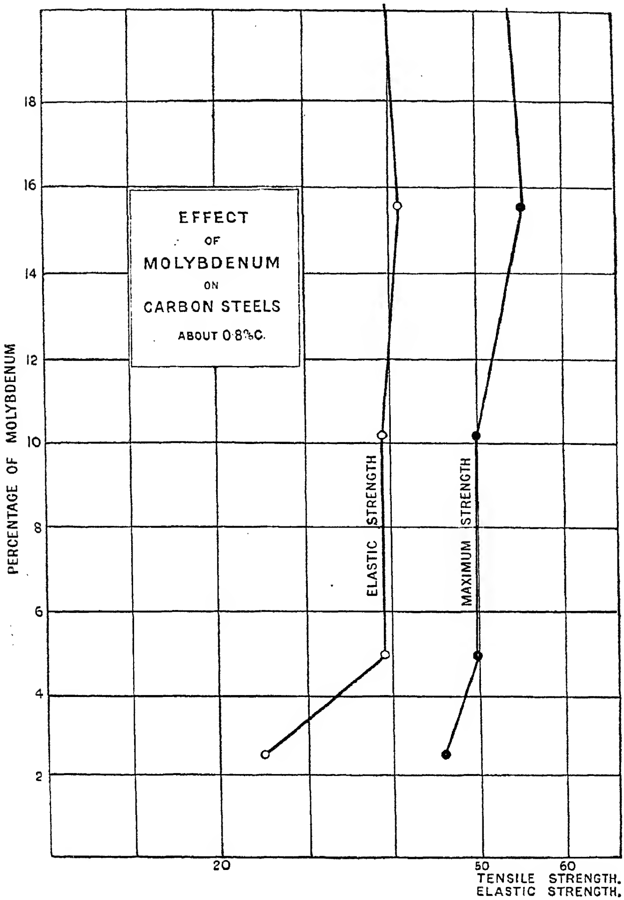


Fig. 225.

TABLE XXXIV. AUTOMOBILE CRANK-SHAFT STEELS

Analysis. Kind of Steel.	Carbon per cent.	Nickel per cent.	Chromium per cent.	Yield per sq. in. Tons	Yield per sq. in. Lb.	Maximum per sq. in. Tons	Maximum per sq. in. Lb.	Elongation per cent.	Reduction per. cent.
Carbon steel . . .	0·40	nil	nil	30	67,200	50	132,000	24	52
Nickel steel . . .	0·35	3·5	nil	35	78,400	50	112,000	25	64
Chromium steel . .	0·35	nil	1·5	35	78,400	50	112,000	24	60
Nickel-chrom. steel .	0·3	3·5	0·5	45	100,800	55	123,200	23	66
Nickel-chrom. steel .	0·3	3·5	0 5	55	123,200	65	145,600	18	58

These test results were obtained from specimens 0·564 in. diameter with a measured gauge length of 2 in. They show the strength properties to be expected in a carbon steel, in a nickel steel without chromium, in a chrome steel without nickel, and in two steels containing all three constituents : the results emphasize the advantage which the last has over the first three.

The above nickel-chromium steels are considered specially suitable for resisting shocks and rapid reversals of stress. There are certain parts of motor vehicles, such as the front axle parts of the steering gear, in which it is more important to have a ductile material to resist violent shocks and jars than to have a steel with a high elastic limit. With this in view, the following are given:

TABLE XXXV. AUTOMOBILE STEEL POSSESSING GREAT DUCTILITY

Kind of Steel.	Carbon per cent.	Nickel per cent.	Chromium per cent.	Yield per sq. in.		Maximum per sq. in.		Elongation per cent.	Reduction per cent.
				Lb.	Tons	Lb.	Tons		
Carbon steel . . .	0·3	nil	nil	53,760	24	89,600	40	30	66
Nickel steel . . .	0·3	3·5	nil	73,920	33	100,800	45	27	66
Nickel-chrom. steel .	0·25	3·5	0·5	89,600	40	112,000	50	25	66

The figures in this table make it clear that, though the nickel steel appears better than the carbon steel, the addition of chromium to the nickel and carbon still further advances the strength without lowering the ductility.

181. Automobile Steels—The requirements and qualities of steels suitable for use in automobile work are discussed very thoroughly in a paper read by Mr. J. H. S. Dickenson before the Institution of Automobile Engineers, in December, 1915.

The author begins by classing the steel parts of a motor into three main groups, somewhat as follows :

"(1) Those which are subjected to considerable alternating stresses and which are principally liable to fail by fatigue.

"(2) Those which are less liable to this type of failure but which, while being strong, must resist fracture by excessive sudden blows or shocks.

"(3) Parts, of which gear wheels are by far the most important, in which some special difficulties arise in connection with crushing stresses and abrasion."

This is the division of steel parts according to the probable manner of their failure, but in addition to these, which are mostly stress conditions, there are others to be taken account of, in

selecting steels, such as the readiness and certainty with which any necessary heat treatment can be performed; the behaviour of the steel in working; and the liability to develop flaws.

Failures of steel parts are classified in three groups:

"(1) Failure by deformation without fracture. Such cases are easily rectified by strengthening the section, or by substituting a steel of a high elastic limit.

"(2) Failure by surface deformation or wear.

"(3) (*a*) Sudden fracture of more or less hard or brittle parts by the application of shock. This is often to be attributed to the steel being in a rotten condition owing to overheating in forging or stamping.

"(*b*) Fracture of non-brittle parts by direct over-stressing. This type of failure is seldom encountered, and such fractures are, of course, a completion of the type mentioned in No. 1.

"(*c*) Fracture by fatigue, that is, by the development of a growing flaw, induced by repeated alternations of stress. This class of fracture is by far the most common, and the most difficult to guard against."

182. Typical Steels suitable for Crank-shafts—The following figures are given in the above paper:

TABLE XXXVI

Type of Steel.	Carbon Steel.	Nickel Steel.	Chromium Steel.	Ni-Cr Steel.	Ni-Cr Steel.
Analysis:					
Carbon, per cent.	0·40	0·35	0·35	0·30	0·30
Nickel, per cent.	nil	3·50	nil	3·50	3·50
Chromium, per cent.	nil	nil	1·50	0·50	0·50
Tensile Strength:					
Yield point (tons per sq. in.)	30	35	35	45	55
Maximum (tons per sq. in.)	50	50	50	55	65
Elongation, per cent.	24	25	24	23	18
Reduction in area, per cent.	52	64	60	66	58

(The above results were obtained on specimens 0·564 in. diam. and 2 in. gauge length.)

These results show the advantage of nickel-chrome steel.

In parts subjected to variable stresses, such as the front axle parts of the steering gear and so forth, ductility is of more importance than a high elastic limit. The following table gives figures for three steels suitable where toughness is of importance. The steel in this and the last table is first suitably heat-treated:

TABLE XXXVII

Type of Steel.	Carbon Steel.	Nickel Steel.	Nickel-Chromium Steel.
Analysis :			
Carbon, per cent.	0·30	0·30	0·25
Nickel, per cent.	nil	3·50	3·5
Chromium, per cent.	nil	nil	0·50
Tensile Strength :			
Yield point (tons per sq. in.)	24	33	40
Maximum (tons per sq. in.)	40	45	50
Elongation, per cent.	30	27	25
Reduction in area, per cent.	66	66	66

The following load-strain diagram was obtained from a sample of nickel-chrome steel tested in the authors' laboratory, January 26, 1916.

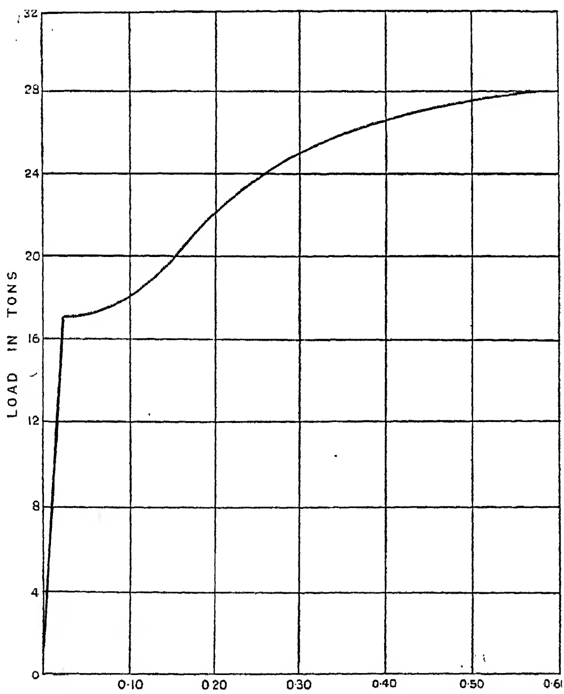


FIG. 226.—Load-strain Diagram for Nickel-Chromium Steel. Tested in Authors' Laboratory, January 26, 1916.

The ultimate strength of the above sample was 63·6 tons per sq. in. and the ductility on a length of 2 in. was 22·5 per cent.

183. Nickel-Chromium Steel in Short Tubular Struts*— The following notes form a brief account of recent experiments on the collapse of short tubular struts of a steel containing carbon, nickel, and chromium. The importance of the results obtained is largely influenced by the increased employment of such tubes in aeroplane construction. It is further emphasized by the fact that the use of this high-tensile steel enables thinner and therefore lighter tubes to be used, and this renders the likelihood of local failure more evident than where the tubes are thicker. The precise manner of failure may be by general flexure or by local flexure in the form of crinkling or corrugation. The specimens here tested failed by crinkling. Whether a tubular strut fails by general or local flexure depends upon its length, the manner in which its ends are held, and the extent to which it is supported by lateral stays.

In the present experiments the tubes were turned on their faces and placed between the platens of a 30-ton Amsler-Laffon testing machine, and the load increased until collapse took place by the crinkling of the tube. For the purpose of avoiding crinkling close to the ends and bending due to the load not being quite axial, several special methods of holding were adopted, especially in the cases of the very thin tubes. These were:

(P) Squared ends simply resting against the platens.

(Q) The ends very slightly turned to fit plates which rested against the platens.

(R) (P) Condition with tube ends plugged with discs.

(S) (Q) Condition with similar disc plugs.

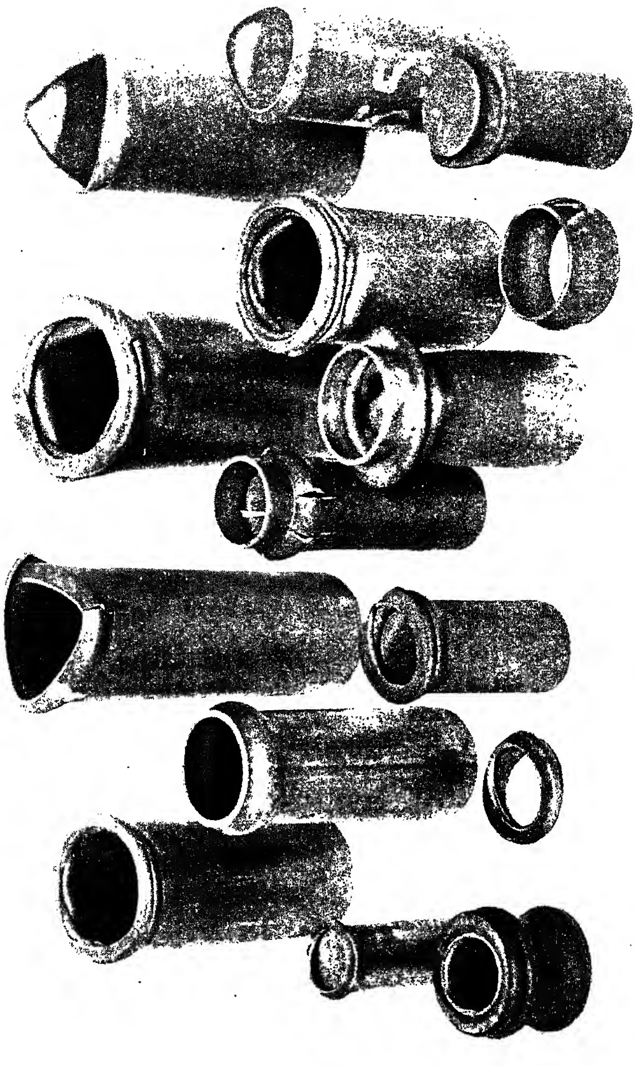
The above machine works on an oil-pressure principle, in which friction is reduced to a minimum by the avoidance of the use of packings and by a partial rotation given to the intensifier ram on which the pressure is dependent. The low-pressure of the intensifier is derived from a mercury column on the top of which rests a float, in turn connected so as to give axial movement to the drum of the autographic recorder. Rotation of the latter was obtained by the use of a scissors magnification apparatus connected with the platens.

The main results were given by the autographic records taken during the test of each tube, by diagrams plotted from extensometer readings taken during some of the experiments, and by notes made during the experiments.

The tubes tested were formed from material having a compo-

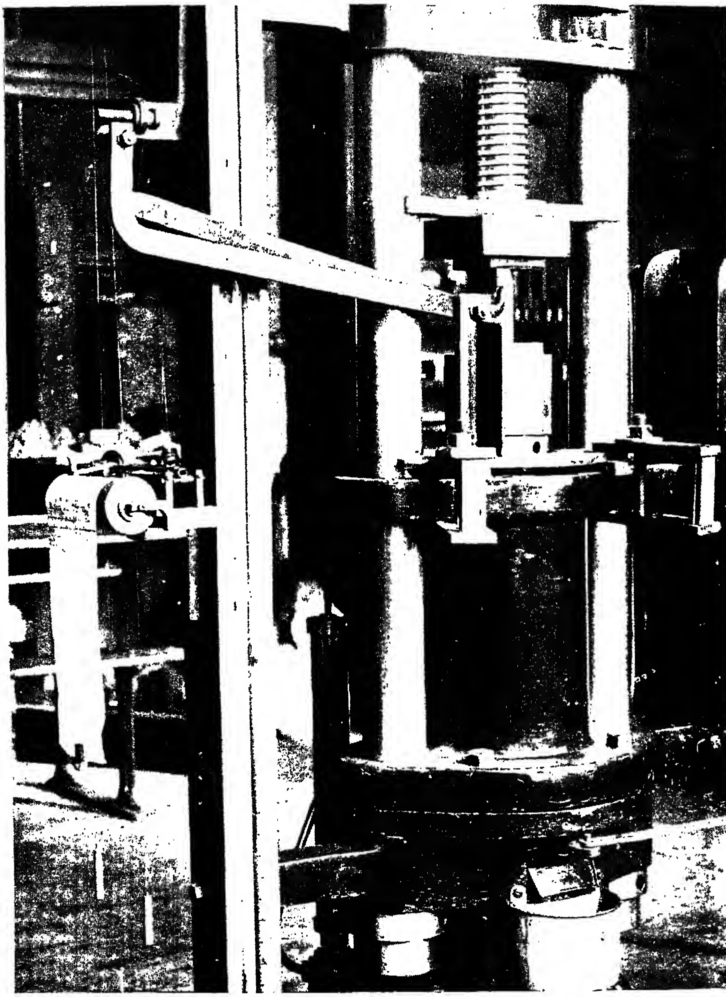
* Popplewell and Carrington, Min. Proc. Inst.C.E., 1917, vol. cciii.

PLATE XVIII.



RESULTS OF CRINKLING TESTS ON TUBES OF STEEL

PLATE XIX.



DEVICE FOR DRAWING AUTOGRAPHIC DIAGRAMS IN CRINKLING TESTS OF STEEL TUBES ON AMSLER LAFFON COMPRESSION MACHINE

sition within the following ranges, as regards its chief constituents :

Carbon	0·2 to 0·3 per cent.
Nickel	4 to 5 ,, ,,
Chromium	1 to 1·5 ,, ,,

The specimens were either annealed, of which there were thirty-three, or unannealed or hard, of which there were twenty-one. Their dimensions are given in the table below:

The strength properties as ascertained by tensile tests showed:

Annealed material	Yield point, from 36·4 to 38·5 ton per sq. in. Maximum, 41·3 to 49·0 tons per sq. in. Elongation on 2 in. and 8 in. 19 to 28·5 and 11·5 to 17 per cent.
Unannealed or hard	Yield point, practically same as maximum. Maximum, 75·1 to 104·5 tons per sq. in. Elongation on 2 in. and 8 in. 5·1 to 14 and 4 to 6 per cent.

Photographic views of some of the tubular specimens which were tested are shown on Plate XVIII and the machine in which the tests were made showing the autographic recording device on Plate XIX.

Autographic Records—One of these was taken for each specimen tested. As stated above, the amount of the load was indicated by the axial motion of the pencil as it marked the diagram on the rotating drum, and the corresponding strain was taken from the shortening of the specimen as magnified from the movement towards one another of the platens. Fig. 24 shows a record taken during an experiment. The letter E indicates the point where the first crinkle began to form as fixed by micrometer measurement. C marks the termination of an experiment, while M marks the point where in some of the experiments irregularities in the curves were caused by the formation of cracks. The crinkling stress or "elastic breakdown" point was found by noting the load at which the line ceased to be straight.

By following the curve in Fig. 24 it is seen that the line is straight until E is being approached :

184. Valve Steels for Internal-combustion Engines—The effect of the high temperatures in internal-combustion engines on the condition of the surfaces of the valves as well as on the strength is such that it is found that certain steels are more suitable than others. In this connexion Mr. Leslie Aitchinson has carefully considered the suitability of all available steels from every point of view, and has embodied his investigations and conclusions in a paper,* recently published. In the following table, taken from the above paper, the various steels are given

* "The Engineer," December 26, 1919.

numbers which indicate their relative suitability, the most suitable steel being marked 1 and the worst 5.

TABLE XXXVIII. GENERAL PROPERTIES OF VARIOUS VALVE STEELS

Property.	High-tungsten, high-carbon Steel.	High-chromium low-carbon Steel.	High-chromium, high-carbon Steel.	3 per cent. nickel Steel.	Nickel chromium Steel.
Tensile strength at high temperatures	1	3	2	4	4
Ease of forging	4	3	4	1	2
Ease of sound manufacture	4	3	4	1	2
Ease of heat treatment	2	1	2	4	3
Scaling	3	1	2	4	4
Retention of physical properties	1	2	2	3	4
Self-hardening in running	2	3	4	1	5
Freedom from distortion	1	1	1	2	2
Wear in stem	1	3	1	2	2
Hardening in the foot	2	1	1	3	3
Ease of machining	3	1	5	2	4

Finally, he allocates to the different valves steels having the following compositions:

All inlet valves	3 per cent. nickel steel.
Exhaust valves with a temperature not exceeding 600° C.	3 per cent. nickel steel.
Exhaust valves with a temperature between 600° and 700° C.	High-chromium steel.
Exhaust valves at temperatures greater than 700° C.	Tungsten steel.

185. Wrought Iron—In the complete range of the irons and steels cast iron stands at one end and wrought iron at the other. Whereas cast iron contains moderate percentages of a number of foreign ingredients, including carbon, silicon, manganese, phosphorus, and sulphur, besides traces of others, the material known to engineers as wrought iron is the purest form of iron known to commerce, containing as it does percentages of these other elements which are very low and often little more than traces. In a broad sense the process of obtaining wrought iron from cast iron, in the form of pig, consists in getting rid of as much as possible of the carbon and the other constituents. This is most often effected by what is known as the "puddling" process, in which the carbon, etc., are burnt out of the molten

bath of pig iron by an oxidizing flame which is allowed to play upon its surface. The bath of molten slag and pig iron is exposed to the flame, and continually stirred meanwhile until the purer iron, having got rid of much of its foreign content, gradually forms into an approximate ball. This is removed from the furnace, hammered, and rolled into its final shapes. This is briefly the manner of getting wrought iron from pig. The material so obtained has the following characteristic qualities.

The appearance of its fracture is distinctly fibrous, this being caused by minute parts of the iron oxide slag being mixed with the ball and pressed in the rolling until thoroughly assimilated as "part of the finished material." The melting temperature is very high as compared with cast iron and steel. Wrought iron does not possess the property of being rendered hard by heating followed by rapid cooling, as in some kinds of steel. But it does possess the quality of being easily welded at a white heat: this is a very important and valuable property, rendering the use of wrought iron imperative for many purposes. Also, wrought iron is a soft, ductile material, capable of being easily shaped when either hot or cold. As compared with average mild steel, wrought iron is neither quite so strong nor so ductile, but its welding is easier and more certain, and it is not so liable to be damaged by overheating. The fracture of a wrought-iron bar, broken by a pull in a tensile testing machine, appears to be distinctly fibrous, while mild steel, broken in the same way, has a "silky" or "velvety" appearance.

PUDDLING—The essential parts of the puddling process may be thus briefly described:—The process is carried out in a simple "reverberatory" furnace. In this a fire of relatively large size is allowed to burn on the fire-bars of the grate, and the flame from this fire, being caused by the roof of the furnace to impinge on the pig iron on the hearth, first melts this and afterwards gradually absorbs the carbon, silicon, etc.

The whole process is generally considered as being divided into the following four stages:

(*a*) *Melting Down*—This lasts probably for thirty minutes from the time the furnace flame begins to play on the pig. During the previous "heat" the furnace has become very hot and some of the heat still retained is utilized in starting the present heat. The hearth is prepared by placing on it an abundant quantity of "fluxing cinder" or "hammer slag," which is the well-known black iron oxide, squeezed out of previous balls by the hammer. To this is added some 5 cwt. of grey forge iron. When the hearth has been prepared in this way, the furnace door is closed and

combustion allowed to proceed until all the materials on the hearth are melted. During this part of the process, as soon as the iron is so far heated that it is beginning to melt, the pigs should be moved so that they all melt more or less together and quite uniformly. During the "melting" stage most of the manganese and silicon are removed, as well as some of the phosphorus.

(*b*) *Clearing Stage*—At the end of the melting stage, a thorough stirring of the bath is effected by a workman who plies a long iron bar bent near the end at right angles. This mixes intimately the melted pig with the oxide "cinder," and at the same time exposes the fluid mass to the action of the hot flame, high in oxygen from the air. While this is proceeding the high temperature is maintained. This stage of quiet fusion lasts about ten minutes. During this clearing stage all the manganese and silicon yet remaining are eliminated, as well as some further phosphorus.

(*c*) *The "Boil"*—In this stage the puddler closes his damper, cutting off the draught and lowering the temperature. This has the effect of thickening the metal on the hearth, and a vigorous stirring of the mass is begun. This brings about a more intimate mixing of the molten iron and the oxide cinder, and causes oxidation of the carbon with the formation of carbon monoxide. This gas, as it is formed, bubbles through the molten mass, thus giving the name to the stage. This "boiling" stage occupies about thirty minutes, and is held to be complete when violent ebullition has ceased and the carbon monoxide no longer burns with its blue flame. When this point is reached the molten mass rises up on the hearth to as much as six inches and a quantity of slag is allowed to flow away from the furnace. The violent boiling has settled down and the iron is now more or less a sticky, porous mass, and is said to have "come to nature." In this third stage, besides elimination of the greater part of the carbon, a further quantity of phosphorus is cleared away.

(*d*) *"Balling Up"*—This concluding stage occupies about twenty minutes and is mainly devoted to rake-work by the puddler, during which he works up the crude wrought iron resulting from the first three stages into a soft mass or ball, which he divides up into a number of balls of smaller size and more convenient for handling. This is often about six, each weighing something like 80 lb.

The next operation consists in treating the puddled ball by hammering under a steam-hammer, or squeezing in some form of mechanical squeezer, or pressing by hydraulic means. The intention of this hammering or squeezing is to get rid of most of

the black iron oxide, which has become intimately mixed with the iron.

The crude iron thus resulting from hammering or pressing, during which processes it has been reduced to a roughly prismatic form, is next passed through rolls which are worked in a progressive manner. The resulting product is a bar of the iron referred to technically as a "puddled bar."

The puddled bar thus produced by rolling the puddled ball, after this has been pressed, is cut up into shorter lengths: these are tied together in bundles, raised to a white heat, and again hammered and put through the rolls. It has been found, however, that there is a limit to the number of pilings, so that up to about six repetitions of the process there is a distinct improvement in quality with repetition, but that as the number is still increased beyond ten there is observed a falling off in strength. This limit to the number of advantageous repetitions of piling after the first one or two becomes largely a matter of cost, and the manufacturer has to decide whether the cost of repetition can be recovered in the price obtained for his commodity.

186. Finished Wrought Iron—For a great many important uses the place of wrought iron, so long held, has been taken by mild steel. The latter material possesses several qualities denied to wrought iron, which have compelled its use for the greater part of all that comes under the head of structural work. Formerly all steam boilers, metal bridges, roof trusses, rails, wheels and metal ships were constructed of wrought iron, but now all are built of steel of appropriate grades and qualities.

Wrought iron has, however, inherent qualities which make its use certain in some kinds of work. One great advantage which wrought iron has over mild steel lies in the ease and freedom with which it can be welded, coupled with its freedom from the danger of overheating when being worked.

The following test results, by various authorities, give some idea of the usual strength properties of commercial wrought iron.

TABLE XXXIX. TENSILE STRENGTH OF WROUGHT IRON

Kind of Iron.	Yield Stress per sq. in.	Maximum Stress per sq. in.	Elongation per cent.	Reduction per cent.	Authority.
Swedish Hammered Bars	24,750 lb. 11·05 tons	42,100 lb. 18·80 tons	24·6 on 15 in.	—	Kirkaldy.
Iron Ship Plates .	—	49,300 lb. 22·01 tons	5·7 on 10 in.	5·5	Board of Trade Report, Unwin.
Iron Boiler Plates* .	—	47,380 lb. 21·15 tons	9·6 on 10 in.	13·07	Ditto.
Iron Boiler Plates† .	—	41,400 lb. 18·48 tons	3·2 on 10 in.	4·41	Ditto.
S.C. Crown . . .	38,370 lb. 17·13 tons	55,000 lb. 24·56 tons	—	—	Platt & Hayward.
Netherton Crown Rivet	37,680 lb. 16·82 tons	56,000 lb. 25·01 tons	—	—	Ditto.
Netherton Crown Best	33,700 lb. 15·04 tons	48,380 lb. 21·60 tons	—	—	Ditto.
Best W.I.. . . .	35,100 lb. 15·66 tons	52,600 lb. 23·44 tons	—	—	Popplewell and Coker.
Lowmoor (*a*) . .	31,360 lb. 14·00 tons	64,850 lb. 28·95 tons	7·01	5·9	The Steel Committee.
Lowmoor (*b*) . .	26,430 lb. 11·80 tons	55,530 lb. 24·79 tons	12·65	48·8	Ditto.
Yorkshire . . .	29,500 lb. 13·16 tons	53,070 lb. 23·69 tons	17·87	51·4	Ditto.

The following results may be added to those already given. The important point to be borne in mind in connexion with tensile tests of good wrought iron is the remarkably small difference between average and extreme results. These figures refer to a large number of tests made at widely different times.

* With fibre. † Across fibre.

TABLE XL

Yield Stress per sq. in.		Maximum Stress per sq. in.		Elongation per cent.	Reduction per cent.
Lb.	Tons.	Lb.	Tons.		
27,000 to 40,000	12 to 18	47,000 to 56,000	21 to 25	20 to 30 on 8 in.	35 to 50
Wrought Iron for Bolts:					
33,600 to 44,800	15 to 20	49,300 to 58,200	22 to 26	18 to 25 on 8 in.	—
Specimens turned out of Bolts:					
26,900 to 35,600	12 to 15	49,300 to 58,200	22 to 26	25 to 35 on 2 in.	—

Interesting test results have at different times been obtained from strips of wrought-iron boiler plate, taken from boilers which had been in constant use for many years previously, sometimes forty years.

27,000 to 40,000	12 to 18	35,800 to 44,800	16 to 20	1·5 to 8·0 on 8 in.	—

The results are remarkably good considering the exposure of the iron to intense and continuous heat for more than a generation. The direction in which signs of fatigue are most apparent is in the reduction of ductility. Unfortunately all wrought-iron boilers were built so very many years ago that no data exist regarding the original properties, with which comparisons might be made.

CHAPTER XV

WIRE

ANY ductile metal may be drawn into wire by the use of suitable appliances and the exercise of a reasonable amount of care. The metals from which wire is most generally drawn include:

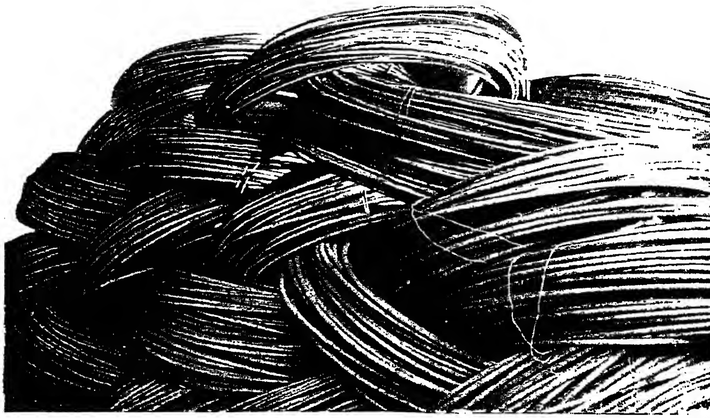
Most kinds of steel; wrought iron; copper; aluminium; the bronzes; platinum; tin; lead; silver; gold. Materials having little ductility, such as cast iron, steel castings, and the harder of the ductile steels, cannot be drawn into wire.

The form of the wire is in most cases cylindrical, though not always so. In order to produce it, a thin bar of the material is forcibly drawn through a series of taper-holes cut in what are called "draw-plates." On each "draw-bench" there are several of these plates, each of which is pierced by a number of holes, arranged in concentric circles, the holes varying in size.

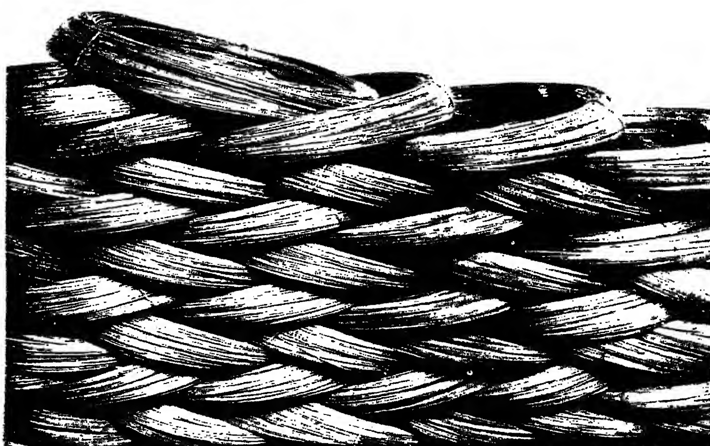
The process of wire-drawing is not unlike what takes place in a progressive series of elongations such as have been described in Chapter XIII (p. 374). In this case, however, there was no surface planishing by drawing through dies and there was no attempt to anneal at any stage. Wire-drawing proper is a combination of elongation caused by tension and forcible passage through rigid holes.

Special methods are employed where extremely fine wires are to be produced. For example, where it is desired to draw the very fine platinum wire used for the cross-webs of some surveying instruments, the wire is first drawn down as fine as possible by the ordinary means, the holes being cut in rubies or other suitable precious stones. This fine wire is then surrounded by a layer of silver, making a silver wire with a platinum core. This silvered wire is now drawn down as fine as possible, when the platinum will be reduced proportionately. When the combined wire has been drawn as fine as possible it is immersed in strong acid, which dissolves the silver and leaves an extremely fine platinum wire. Dr. Wollaston was, after the further drawing to $\frac{1}{3,000}$th of an inch and dissolving, able to produce a platinum wire $\frac{1}{30,000}$th in. in diameter.

PLATE XXI.*



(*a*) WIRE RODS



(*b*) FINISHED WIRE

In the case of steel or iron wire the metal is first formed from the original ingots (Plate XX) into billets of smaller size; and these, after being heated, are rolled out into long rods of relatively small diameter, say No. 5 gauge to No. 8 gauge, equal to 0·212 to 0·160 in. diameter (Plate XXI (*a*)). These are next drawn cold in successive stages until the required final diameter is reached (Plate XXI (*b*)). A rolled rod is pointed at one end and the point pushed through the first draw-plate of the series. The pointed end, which thus projects beyond the back of the plate, is taken hold of by a jaw or vice attached to the end of a strong chain, whose other end passes round a small pulley on the main shaft of the bench. This is started with a clutch and the wire pulled some distance through the die. The jaw is now released and allowed to take hold at a point close to the plate, and again pulled out. This is repeated until there is sufficient wire formed to pass round the main drawing pulley or block, to which it can be secured by a permanent catch. Drawing can now definitely commence. This is repeated in stages, the diameter being reduced each time. After five or six such reductions it is found that the steel has become very hard, and in order to restore the ductility the wire is at this stage annealed. This is effected by placing the coil in a muffle furnace to which very little air is admitted, bringing it to a red heat and allowing it to cool slowly. In doing this it is impossible to avoid a certain amount of oxidation and it becomes necessary to remove the black scale thus formed before submitting the wire to any further drawing. The presence of scale cuts and damages the holes of the draw-plates. The scale is removed by pickling in an acid bath.

The drawing can now be resumed for another set of reductions when a second annealing and pickling becomes necessary. The extent to which these drawings are continued depends on the final diameter desired.

For all metals what is known as "continuous wire drawing" is now the rule. This means that instead of carrying out a set of say five drawings before annealing, these five are effected at one time. To do this the draw-bench is provided with a "block" as before, for the final coil, but in addition there are four other draw pulleys. The result is that the wire passes through the first plate, round the first pulley, through the second plate, round the second pulley, through the third plate, and so on —to the final coil on the block. The pulleys and the block all work together, and the five operations are completed in one stage. In this way much time is saved.

Below are given a few results of tests of wires:

TABLE XLI. TEST RESULTS FOR WIRES

Kind.	Diam., in.	Area, sq. in.	Breaking Load, lb.	Breaking Stress, Tons per sq. in.	Breaking Stress, Lb. per sq. in.	Yield Point. Tons per sq. in.	Yield Point. Lb. per sq. in.	Elongation per cent.	Authority.
Silicium bronze . .	0·080 to 0·436	—	—	27·6 to 50·0	61,820 to 112,000	—	—	— 2·0	Preece.
Copper	0·081	—	—	28·4 to 30·3	63,620 to 57,870	—	—	—	,,
Copper trolley wire,.	0·404	0·1283	2·9	22·6	50,620	—	—	5·7	Popplewell
Copper trolley wire annealed . . .	0·403	0·1274	1·8	14·1	31,580	—	—	45·8	,,
Steel	0·160	0·0199	—	48·0	107,520	42·40	94,976	2·5	,,
,, ...	0·175	0·0241	—	46·0	103,000	40·0	89,600	3·5	,,
,, ...	0·190	0·0286	—	41·0	91,840	40·0	89,600	3·8	,,
,, ...	0·213	0·0356	—	43·5	97,440	37·5	83,000	5·0	,,
,, ...	0·298	0·0697	—	34·5	77,280	32·0	71,680	4·7	,,
,, ...	0·325	0·0829	—	37·0	82·880	33·0	73,920	4·0	,,
,, ...	0·274	0·0590	—	36·5	81,760	34·0	76,160	6·6	,,
,, ...	0·348	0·0951	—	34·0	76,160	31·5	70,560	6·4	,,
,, ...	0·250	0·0491	—	38·0	85,120	35·5	79,520	5·0	,,
,, ...	0·372	0·0109	—	38·5	86,240	35·0	78,400	5·5	,,
,, ...	0·232	0·0422	—	34·5	77,280	34·3	76,800	6·0	,,
,, ...	0·400	0·0126	—	38·0	85,120	35·5	79,520	6·5	,,
,, ...	0·064	—	1,100	—	—	—	—	—	,,
Aluminium . . .	0·192	0·0289	690	10·6	23,800	—	—	0·25	,,
,, . . .	0·146	0·0167	400	10·6	23,800	—	—	0·24	,,
,, . . .	0·146	0·0167	405	11·0	24,700	—	—	0·200	,,

187. Steel Wire Rope—Steel used in the form of rope is an exceedingly important material in many kinds of engineering work. The uses of steel wire rope include hauling and traction in mines; all kinds of lifts and cranes, both for passengers and material; ropeways for the transmission of goods and material across large open spaces; cable tramways; suspension bridges; and the standing rigging of ships. In all these directions different forms of ropes are needed to suit the particular uses in question and certain materials may be more suitable than others.

The makers of wire ropes employ the qualities of steel named below:

TABLE XLII

	Tensile Strength. Tons per sq. in.	Tensile Strength. Lb. per sq. in.
1. Extra plough steel, special high grade	140 to 160	313,600 to 358,400
	120 ,, 140	268,800 ,, 313,600
	110 ,, 120	246,400 ,, 268,800
2. "Improved" plough steel . . .	100 ,, 110	224,000 ,, 246,400
3. Mild plough steel, or special crucible	90 ,, 100	201,600 ,, 224,000
4. Crucible steel	80 ,, 90	179,200 ,, 206,600
5. Siemens-Martin, mild steel . . .	45 ,, 50	103,300 ,, 112,000

These figures, indicating the breaking strength of the various steels employed in rope manufacture, show a very wide range in the ultimate strengths.

The classification given above is that used by the wire-rope makers. The metallurgists and steel-makers employ a classification which differs from the last more in appearance than reality. Mr. J. Bucknall Smith gives it as :

TABLE XLIII

	Breaking Stress.	
	Tons per sq. in.	Lb. per sq. in.
Special qualities of tempered and improved cast-steel wire	150 to 170	336,000 to 380,800
"Improved" cast "plough steel"	Up „ 120	Up „ 268,800
Crucible cast steel "improved"	„ „ 100	„ „ 224,000
High-carbon	„ „ 80	„ „ 179,200
Siemens wire	„ „ 60	„ „ 134,400
Common Bessemer wire	„ „ 40	„ „ 89,600

What is called by the wire-maker "Improved" or "Patented" steel refers to wire which passes through a muffle furnace and thence into an oil bath. The furnace heats the wire to the necessary temperature and immersion in the oil bath hardens it. The wire so treated finally passes through a bath of molten lead which has a tempering effect. This "improving" process is generally employed immediately before the last drawing, and is such that mild-steel wires having an ultimate strength of say 50 tons per sq. in. may have this raised to as much as 80 tons per sq. in.

Plough steel wire, so called because it was first employed in the ropes used in steam-ploughing operations, has a composition as follows :

C	0·828 per cent
Mn	0·587 „ „
Si	0·143 „ „
S	0·009 „ „
P	nil.
Cu	0·030 „ „

Dr. Percy has shown that in the plough steel with the above analysis the ultimate strength is raised by successive drawings in the following way :

TABLE XLIV

Diameter, in.	Ultimate Strength.	
	Tons per sq. in.	Lb. per sq. in.
0·191	90	201,600
0·159	100	224,000
0·132	115	257,600
0·093	154	344,960

188. Wire Ropes—First, a number of wires are twisted together to form a strand. Then, several strands, generally six or less frequently seven, are twisted together to make up the rope. The strands are twisted or "laid" round a central core, which in all but the very stiffest ropes is of hemp or other soft fibre. Sometimes the strands themselves have hemp cores to make them more flexible.

The strands are made in "stranding" machines. In these the bobbins of wire are set in rotating "flyers" which themselves turn in bearings formed in "rings" attached to a central shaft. The main shaft is driven, and as it revolves carries the rings round and with them the flyers, which also have a separate rotation of their own, provided by a kind of modified "sun-and-planet" motion. By this the bobbins are caused to retain always the same position relative to the ground. This is done to prevent any torsion being given to the wires. The core, whether fibrous or of wire, passes through the main shaft, which is hollow, and the machine as it rotates lays the wires spirally around it. The laying machine is of similar construction to the stranding machine but larger, and has the same provision against torsion. In both these machines the finished product, whether strand or rope, is finally wound on to a separate bobbin or reel.

SECTIONS OF WIRE ROPES AND DIRECTIONS OF LAY—A wire rope has to perform functions for which it requires to possess certain definite qualities. They are referred to in the following paragraphs.

LAY—The wires in the strand may be given either a right-hand or left-hand twist, and the same is true for the strands as they lie in the rope. In what is known as "ordinary" lay, the twist of the rope is opposite to that of a strand, that is, if the strand twist is right-hand then the strands are laid in the rope with a left-hand twist, and vice versa. In "Lang's lay" the twist of the rope strands is the same hand as that of the wires in the strands. The advantage of Lang's over the ordinary lay is that a rope so constructed is more durable (see Plate XXII (*a*)).

PLATE XXII.*

(*a*) WIRE ROPE: ORDINARY LAY

(*b*) WIRE ROPE: LANG'S LAY (WORN)

With the ordinary lay it is found that the wires at the surface wear through and break, whereas the Lang's lay wires (Plate XXII (*b*)) lie together more closely and gradually wear into a roughly cylindrical form with few exposed and prominent points.

FLEXIBILITY—It is found that greater flexibility is arrived at by the use of many small wires, and by using hemp cores in both strands and rope. Ropes of this class, intended to possess extreme flexibility, are used for passenger lifts, and other uses where they have to pass round many pulleys of relatively small diameter. It is a general rule that the diameter of the pulley around which a wire rope has to pass must be not less than from 150 to 200 times the diameter of the rope. The use of too small a pulley very soon causes fracture of the outer wires, not so much by straightforward wear as by repeated bendings over the small guide pulleys. This point requires to be carefully watched in ropes used for power transmission.

STIFF ROPES—There are many of these in use for such purposes as suspension ropes for bridges, guide-ropes for mine cages, standing rigging on ships, and others where the rope is not intended to move in the course of its work.

WINDING ROPES FOR COLLIERIES AND OTHER MINES—These are among the most important of the various wire ropes in use. They are subjected to very heavy wear, the loads which have to

TABLE XLV

Diameter of Pulley.	Working Load, to	Size of Rope.
3 feet	4 tons	Up to 1 in. diameter
4 ,,	5 ,,	,, ,, ,,
5 ,,	5 ,,	,, ,, ,,
6 ,,	5 ,,	,, ,, ,,
7 ,,	5 ,,	,, ,, ,,
8 ,,	6 ,,	,, ,, ,,
9 ,,	6 ,,	,, ,, ,,
10 ,,	7 ,,	,, 1⅛ in. ,,
11 ,,	8 ,,	,, ,, ,,
12 ,,	8 ,,	,, ,, ,,
13 ,,	9 ,,	,, 1¼ in. ,,
14 ,,	9 ,,	,, ,, ,,
15 ,,	10 ,,	,, ,, ,,
16 ,,	10 ,,	,, ,, ,,
17 ,,	10½ ,,	,, 1½ in. ,,
18 ,,	12 ,,	,, ,, ,,
20 ,,	12 ,,	,, ,, ,,

* All the above illustrations (Plates XX–XXIII) relative to steel-wire ropes are from photographs kindly provided by Messrs. Allan, Whyte & Co., the well-known wire-rope manufacturers, of Rutherglen, Glasgow.

be lifted by them are large, their own weight adds considerably to the tension, and they are exposed to damp atmospheres. Consequently, a great amount of care has been devoted to the manufacture of ropes for this purpose, and therefore very good results have been attained. Colliery ropes are only moderately flexible but they are supplied with pulleys of the relatively large diameters given in Table XLV on previous page.

STRENGTH—The ultimate strength of a steel-wire rope may be found directly. But when it is desired to find the safe load that may be applied to an existing rope, the usual plan is to calculate the theoretical strength of all the wires and make a reduction to get the strength of the complete rope. The amount of this allowance is derived from the accumulated results of breaking tests and is of course only approximately correct. It would be possible to carry out breaking tests on short lengths of every rope put into actual use and take this as the real strength of the rope, but, apart from the cost and loss of time involved, it would be necessary to test a considerable number of lengths; and the test of a short length does not necessarily give quite the same result as would be obtained from a long piece. Therefore it is better to calculate on the wires and reduce according to experience. It is usually safe to take the breaking strength of a rope as about 85 per cent. of the combined strength of all the wires, or 15 per cent. less. The customary allowances as used by the makers are as follow:

	Per cent.
For 6 strands of 7 wires and 6 strands of 12 wires . . .	5 to 10
„ 6 „ „ 17 „ „ 6 „ „ 19 „ . . .	10 to 15
„ 6 „ „ 27 „ „ 6 „ „ 37 „ . . .	20 to 25

Thus, where there is a greater number of wires the reduction allowance increases.

EXAMPLE OF CALCULATION OF BREAKING STRENGTH—The following illustration will serve to show how the above estimate of strength is made:

Number of strands . . .	6
Wires in strand . . .	7
Diameter of wire . .	0·150 in.
Sectional area of wire	0·0177 sq. in.
Aggregate sectional area of rope	0·743 sq. in.
Material of wires	Improved plough steel
Breaking strength of wire	100 tons per sq. in.
Estimated theoretical strength of rope = 0·743 × 100 = 74·3 tons	

Making 10 per cent. allowance for "actual strength," this becomes 74·3 − 7·43 = 66·87 tons or = 66 tons approx.

In making this estimate the sizes and number of wires are known exactly, and the strength of the wire can easily be found,

but the reduction allowance is a matter of judgment based on the results of previous tests to destruction of similar ropes.

SAFE LOADS—For colliery winding where the loads caused by rapid acceleration are great, the safe load should be taken as one-tenth of the breaking load. For hauling ropes the factor is one-sixth to one-eighth.

LUBRICATION—In order to prolong the life of a rope by diminishing loss from abrasion and rust, it should be frequently subjected to the effect of a hot lubricant.

FOR USE IN DAMP SITUATIONS—Where they are liable to rust, ropes should be galvanized.

189. Wire-rope Grips—When a rope is in use or is being tested for strength it is necessary that both ends shall be held so firmly that the working tension on the one hand and the test load on the other cannot disturb the grip. Several ways have been used to effect this end. There is little possibility of obtaining a positive hold on the wires, and in nearly all cases the strength of the grip is made to depend on the friction between the wires and some part of the holding shackle.

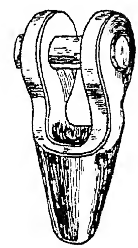


FIG. 227.

There are four main types of these rope grips or capples or sockets, namely:

(*a*) Thimble in spliced eye.
(*b*) Conical socket (Fig. 227).
(*c*) Capple with internal and external rings (Fig. 228).
(*d*) Capple fitted with rivets.

(*a*) In this, which is suitable for the lighter ropes and for those in use on shipboard, the rope is looped round an ordinary iron thimble and secured by means of a splice. When the splice is

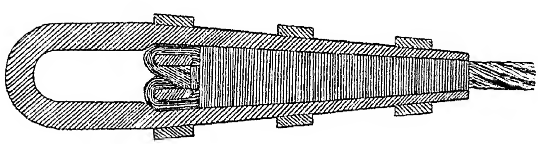


FIG. 228.

well made this form of socket is extremely efficient, and if a rope so fitted at each of its ends is loaded in a testing machine, the loading should result in a clean break of the rope without loosening of the splice. This is probably the most efficient of the holdings which are dependent on friction.

(*b*) The conical socket may be as shown or it may be part of a testing machine. The best method of fixing the rope is the following:—At a distance of say 9 in. from its end, the rope

is wrapped with fine steel wire for a distance of 2 in., and for some 2 or 3 ft. beyond the wire wrapping is tightly wound with cord or rope of small diameter. This is to prevent the rope from becoming loosened. The end part beyond the wire wrapping is now opened with the wires spread out like a brush, and the core cut away. The next stage is to bend the end of each wire round so that it forms a hook whose length should be rather more than one inch. The writers think that the fairest pull upon the wires is obtained when the hooked wire ends are turned outwards. The opened-out rope-end should be of such a size as will fit complete into the socket or test shackle. It is next thoroughly cleaned by immersion in caustic soda and dilute acid. Some operators afterwards "tin" the wire ends.

The complete end, prepared in the manner described, is placed in its socket and lead poured in. This results in a solid end being formed on the rope, which, resting in the socket into which it has been cast, very strongly resists all attempts to extract the rope by tension. This arrangement, applied to a single strand, is shown on Plate XXIII (*a*) (before pouring in the metal) and (*b*) afterwards. The fusible metal used in this process may be simple lead or one of the well-known antifriction metals. Where lead is used it is desirable to harden it by the addition of a small quantity of antimony.

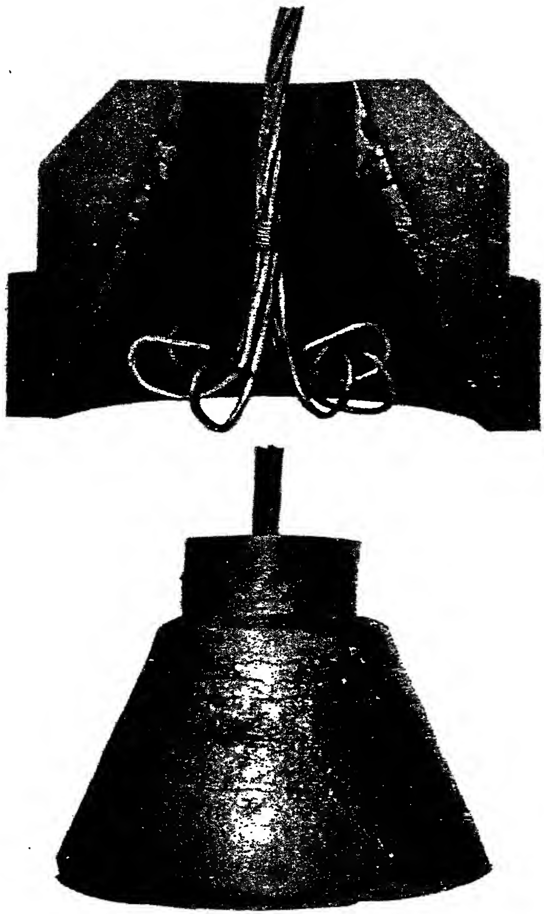
It may be thought that the effect of pouring fusible metal round the wires will be to soften them and render breaking more easy. The following table of melting-point of several alloys of tin and lead should be noted :

TABLE XLVI

*Tin. By weight.	Lead. By weight.	Melting-point.	
		Deg. Fahr.	Deg. Cent.
150	100	330	166
100	100	370	188
57	100	420	216
50	100	440	227
33	100	480	249
25	100	500	260
16	100	520	271
10	100	540	282
4	100	560	293

W. C. Popplewell, "Testing of Materials."

PLATE XXIII.*



(a)

(b)

(*a*) HOLDER FOR WIRE ROPE
(*b*) END CAST ON TO ROPE

so that when an alloy of this kind is used, the steel will not be subjected to a temperature much above 600° F. or 316° C. (which is the melting-point of pure lead). Assuming that the temperature of the oil used in quenching the steel has been 850° C. or 1,557° F., followed by a lead bath at 550° C. or 1,022° F., the temperature of the molten fusible metal (in the neighbourhood of 600° F.) cannot be expected to have any softening effect on the steel.

When the pull is applied to the rope it comes on the several wires in the gripping socket with the tendency to pull them out of the matrix in which they lie. The efficiency of the holding ultimately depends on the difficulty with which the wires can be pulled out of the metal, and for this reason the most suitable kind of metal should be used. The following are a few of the mixtures which have been found suitable. Tetmajer used a mixture of 9 parts lead, 2 antimony, and 1 part bismuth for hard steel wires; and 8 parts tin, 1 antimony, and 1 copper for iron and mild steel. Arnould used equal parts of lead and zinc.

The precautions necessary for success are thorough cleaning of the wires, the use of a fusible metal which is sufficiently hard to prevent the wire ends from being straightened and pulled out of their matrix, and the metal when poured should be only sufficiently fluid to fill up all interstices, as too great fluidity means a higher temperature with a longer time for cooling. With metal not too highly raised in temperature before pouring and run into a socket with thick cast-iron or steel walls, cooling will be rapid and no harmful softening of the wires may be expected.

It has been found that in the majority of cases where this method has been used for holding test lengths, the rope breaks without any pulling out of the wires. The same plan has also been employed for securing the ends of wire ropes to sockets.

(c) This is typical of many of the ways of holding the ends of metal ropes. The strands are loosened and wrapped with fine wire close to the ends. These loose strand ends are now turned backwards and inwards and passed through a steel ring so as to lie against the body of the rope. The complete end so prepared is covered by a wrapping of rope yarn and this conical end is placed in a socket, and a number of rings driven tight on the conical outside. This closes up the socket tightly. When the pull comes on the rope the conical head formed on the rope end is drawn tightly into the socket and a considerable load is required to disturb the fixing. When it does begin to give way the loose strands pull through the ring against all friction. When once movement begins it continues until the ends are pulled clean out. This plan of forming a conical head on the rope end is often used in the conical sockets, with or without lead.

(*d*) In this case the rope end is placed in a conical socket and a number of rivets are placed clean through socket and rope. The end may or may not be thickened into a conical form by turning back the wires.

From observation of test results on sockets of the above types—which are only four out of many—(*a*) appears to be quite satisfactory; (*b*) is probably the most completely efficient, and, when properly made, can be relied upon to break the rope every time; (*c*) may or may not pull out, but the chances are that it will eventually do so if sufficient pulling force is applied. Of course the force necessary to do this will be very near the breaking load of the rope, so that in the fastenings of winding ropes in mines there is a considerable margin for safety. (*d*) This fastening is essentially weaker than any of the others. Although (*a*) and (*c*) may fail by pulling loose, the loosening is in any case very gradual and may not be expected under the working loads adopted.

In testing the strength of a length of wire rope, specially prepared ends, as in (*b*), are often dispensed with and the ends of the rope are held by wedges like a solid bar. The writers have had little experience of testing ropes in this manner, but recognize that much may be done with wedge grips of this kind, provided the wedges are cut with sufficiently fine teeth, are long enough to cover completely at least two pitches of the rope coil, and that the rope itself is of fairly heavy construction and not of very small wires laid round hemp cores.

The following examples illustrate the manner of calculation:

Examples of Breaking Tests

(1) *Pit Winding Rope:*

Circumference of rope	3 in.
Weight per fathom	8·5 lb.
Number of strands	6.
Number of wires in a strand	19 (12 outside 7).
Total number of wires in rope	114.
Size of wire	0·072 in. (15 B.S.W.G., 13 Am.W.G.)
Sectional area of wire	0·0041 sq. in.
Sectional area of rope	0·4674 ,, ,,
Average breaking load of one wire	813 lb.
Total breaking load of all wires	41·43 tons.
Breaking load of rope	33·72 ,,
Efficiency $= \frac{33 \cdot 72}{41 \cdot 43}$	= 81·7 per cent.
Stress in rope on fracture	70·7 tons per sq. in.

For the purposes of the above breaking test the specimen was 3 ft. long between the grips, and was provided with fusible metal conical ends as described.

(2) *Flexible Wire Rope for Lift* :

Circumference	1·74 in.
Weight per fathom . . .	2·78 lb.
Number of strands . . .	6.
Number of wires in strand . .	19 (12 outside 7).
Number of wires in rope . .	114.
Size of wire	0·036 in. (20 B.S.W.G., 19 Am.W.G.)
Sectional area of wire . .	0·00130 sq. in.
Sectional area of rope . . .	0·1482 ,, ,,
Average breaking load of one wire	203 lb.
Breaking load of total wires .	10·32 tons.
Breaking load of rope . . .	9·5 tons.
Efficiency	92·0 per cent.
Stress in rope at fracture . .	64·1 tons per sq. ft.

On Fig. 229 are autographic diagrams of tests of two wire ropes.

INDEX	Circumference of ROPE	Number of Strands	Number of wires in each Strand	Diameter of WIRE	Maximum LOAD in tons	Maximum Extension in inches	Original Length of rope	MAKER
1	9"	6	61	0·108	279·64	11·4	14 ft.	*BULLIVANT*
2	9"	6	37	0·138	273·20	10·8	14 ft.	*BULLIVANT*

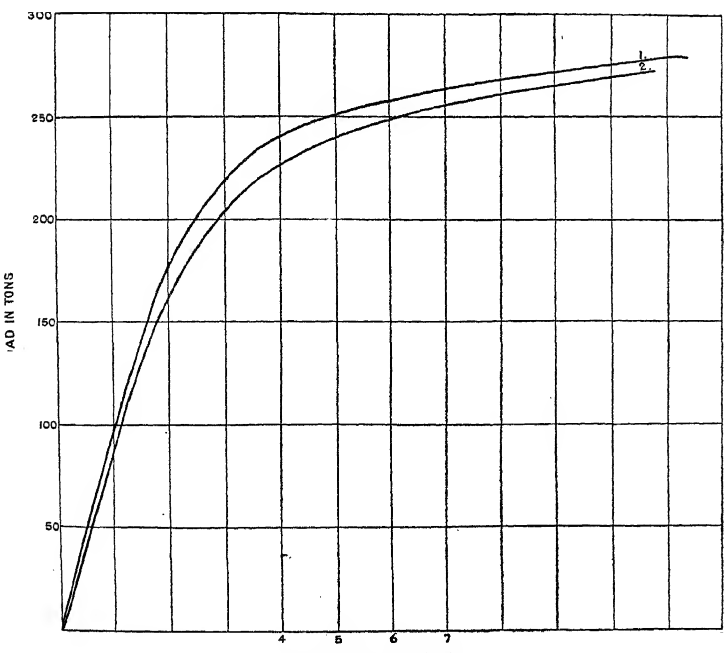


Fig. 229.

190. Copper—Copper is one of the most useful metals employed by the engineer, both on account of its own valuable qualities and because of the many brasses and other alloys of which it forms one of the chief constituents. In commerce it exists in one of two states, namely as hard-rolled or hard-drawn plates, bars and wire; and the same when in the annealed or softened condition. In its unhardened state copper is a soft metal with a strength which is not very high. It is easily worked by rolling or drawing into wire: both these have a hardening effect. Hardened copper can be softened by heating to a dull redness and quenching in water.

The specific gravity of worked copper is about 8·9, which means a weight per cubic ft. of 550 lb. Its melting-point is 2,000° F. and it volatilizes at a white heat. As its electrical conductivity is very high its chief use is for electrical conductors, both when forming parts of electrical machinery and as wire in transmission lines.

Below are given figures relating to the strength properties of both classes:

INVESTIGATIONS MADE UPON COPPER TROLLEY-WIRE SUPPLIED BY THE MANCHESTER CORPORATION TRAMWAYS DEPARTMENT

The object of the tests was to find how the ductility and strength of copper trolley-wire is affected by the following processes:

i. Annealing.
ii. Wear and tear on the straight run.
iii. Soldering and ditto under ear.

The samples tested were classified as follows:

(A) New wire annealed.
(B) New wire.
(C) Old wire under ear.
(D) Old wire on straight.

Test on Specimen A.—New Annealed Trolley-wire

Measurement of Cross-section of Wire.—The diameter of the wire was measured by accurate micrometers in the middle of each alternate inch for a length of 12 in. near the middle of the specimen. In each case two measurements were made along diameters at right-angles to each other.

Position.	In Plane of Centre Dots.	Perpendicular to Plane of Dots.
(*a*) . . .	0·402 in.	0·404 in.
(*b*) . . .	·402 „	·404 „
(*c*) . . .	·402 „	·403 „
(*d*) . . .	·403 „	·404 „
(*e*) . . .	·403 „	·404 „
(*f*) . . .	·402 „	·404 „

From these the diameter was taken as 0·403 in. and the area of cross-section 0·1274 sq. in.

Loads and Extensions

Readings were taken over a length of 8 in. and extensions were measured by means of Ewing's Extensometer, and dividers and scale.

Load in Tons.	Extension in 8 in.		Load in Tons.	Extension in 8 in.	
0·05	0·0002	Taken with Ewing's Extensometer	0·375	0·021	Taken with dividers and scale
·075	·0004		·40	·030	
·10	·0006		·50	·081	
·125	·00092		·60	·127	
·15	·00116		·70	·191	
·175	·0016		·80	·258	
·20	·0020		·90	·338	
·225	·0026		1·0	·421	
·25	·0039		1·1	·515	
·275	·0048		1·2	·631	
·30	·00882		1·3	·796	
·325	·01004		1·4	·931	
·35	·0158		1·5	1·110	
·375	·02058		1·6	1·411	
			1·7	1·891	
			1·8	2·81	

Under the maximum load of 1·8 tons the specimen stretched quickly, and finally broke in the clips.

The bar was then replaced in the testing machine and re-loaded, when it broke clear of the clips within the marked portion, giving the following results :—

Extension over 8 in., including fracture . . .	3·66 in.
Diameter of fracture	0·21 „
Extension over 2 in. portion, including fracture .	1·3 „
Diameter of bar at maximum load (i.e., before local contraction began)	0·35 „

Test on Specimen B.—New Hard-drawn Trolley-wire

Measurement of Cross-section of Wire—The measurements of diameter were made at inch intervals, and over diameters at right-angles to each other.

Position.	In Plane of Centre Dots.	Perpendicular to Plane of Dots.
(*a*) . . .	0·403 in.	0·404 in.
(*b*) . . .	·403 „	·404 „
(*c*) . . .	·403 „	·403 „
(*d*) . . .	·403 „	·404 „
(*e*) . . .	·403 „	·405 „
(*f*) . . .	·403 „	·404 „
(*g*) . . .	·403 „	·403 „

From these the diameter was taken as 0·404 in., and the area of cross-section 0·1283 sq. in.

Loads and Extensions

Conditions as in test on Specimen A

Load in Tons.	Extension in 8 in.		Load in Tons.	Extension in 8 in.	
0·1	0·0002	Taken with Ewing's Extensometer	1·6	0·01448	Ewing's Extensometer
·2	·0010		1·7	·0158	
·3	·0018		1·8	·01704	
·4	·0027		1·9	·01852	
·5	·00352		2·0	·02006	
·6	·0044		2·1	·02188	
7	·00528		2·2	·02394	
·8	·00636		2·3	·02654	
·9	·0072		2·4	·02778	
1·0	·00816		2·5	·03196	
1·1	·00914		2·5	·0320	
1·2	·0108				
1·3	·0112		2·6	·0370	Dividers and scale
1·4	·01224		2·7	·042	
1·5	·0134		2·8	·063	
			2·9	·225	

The stretch at the end was very sudden, the specimen breaking in the clips with a load of 2·9 tons.

Diameter clear of fracture, ·396 in.

This wire after breaking as above was further stretched, when the 8-in. length became 8·457 in., including the fracture. The diameter at fracture was ·285 in.

The 2-in. length containing the fracture was found on measurement to elongated to 2·28 in.

Test on Specimen C.—Old Hard-drawn Trolley-wire from under Ear

Measurement of Cross-section of Wire—The measurements of diameter of the wire were made at inch intervals over a length of 8 in. in the middle of the specimen.

Readings *a*, *b*, and *c* were under the ear.
Readings *d* and *e* were just at end of ear.
Readings *f*, *g*, *h*, and *i* were clear of ear.

Position.	Plane of Centre Dots.	Perpendicular to Plane of Dots.
(*a*) . . .	0·405 in.	0·403 in.
(*b*) . . .	·400 ,,	·402 ,,
(*c*) . . .	·401 ,,	·402 ,,
(*d*) . . .	·400 ,,	·401 ,,
(*e*) . . .	·390 ,,	·398 ,,
(*f*) . . .	·367 ,,	·405 ,,
(*g*) . . .	·365 ,,	·396 ,,
(*h*) . . .	·365 ,,	·398 ,,
(*i*) . . .	·365 ,,	·400 ,,

From these the area was taken to be 0·114 sq. in.

Loads and Extensions

Conditions of test similar to those on Specimen A.

Load in Tons.	Extension in 8 in.	Load in Tons.	Extension in 8 in.
0·1	—	1·3	0·0118
·2	0·0009	1·4	·01292
·3	·0018	1·5	·0140
·4	·0027	1·6	·0155
·5	·0036	1·7	·0172
·6	·0045	1·8	·0189
·7	·0055	1·9	·0209
·8	·0064	2·0	·0238
·9	·0074	2·1	·0274
1·0	·0084	2·3	·028 ?
1·1	·0094	2·5	·123
1·2	·01094		

The specimen fractured at a point about 1½ in. clear of the ear.
Under the maximum load of 2·5 tons the specimen stretched rapidly to breaking-point.

Extension over 8 in., including fracture . . . 0·325 in.
" " 2 in., " " . . . 0·254 in.
The diameter of the fracture was approximately 0·254 in.

Test on Specimen D.—Old Wire on the Straight

Measurement of Cross-section of Wire as for Specimen B.

Position.	In Plane of Centre Dots.	Perpendicular to Plane of Dots.
(*a*)	0·374	0·398
(*b*)	·373	·398
(*c*)	·373	·399
(*d*)	·374	·399
(*e*)	·373	·399
(*f*)	·374	·399
(*g*)	·374	·399
(*h*)	·374	·399

From these the area of cross-section was taken as 0·117 sq. in.

Loads and Extensions

Conditions as in test on Specimen A.

Load in Tons.	Extension in 8 in.	Load in Tons.	Extension in 8 in.
0·1	0·0008	1·3	0·0122
·2	·0016	1·4	·0133
·3	·00244	1·5	·0148
·4	·00338	1·7	·0159
·5	·0041	1·8	·0174
·6	·005	1·9	·0191
·7	·006	2·0	·0208
·8	·0069	2·1	·023
·9	·00794	2·2	·0255
1·0	·0088	2·3	·0288
1·1	·0098	2·6	·119
1·2	·011		

A well-defined maximum load was obtained on this specimen at 2·60 tons, and on relieving the stress the bar finally broke at 2·31 tons.

Extension over 8 in., including fracture . . . 0·28 in.
" " 2 in., " " . . . 0·24 in.
Diameter at fracture . . . . . . 0·24 in.

Diameters at maximum load (clear of fracture)—Vertical, 0·374 in.; Horizontal, 0·396 in.

The graphs for the tests are shown on Fig. 230.

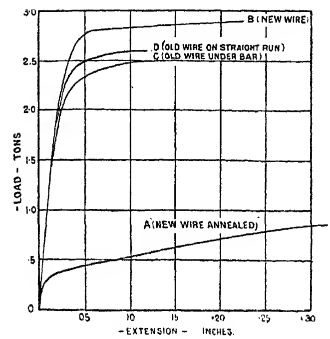


FIG. 230.

SUMMARY OF TESTS ON TROLLEY-WIRE

Specimen.	A.	B.	C.	D.
Date of test . . .	17/5/06	24/5/06	24/5/06	24/5/06
Initial diameter—in. .	·403	·404	·365–·396	·374–·399
Initial area of section —sq. in. . . .	·1274	·1283	·114	·117
Maximum load—tons .	1·8	2·9	2·5	2·6
Diam. of fracture—in.	·21	·285	·254	·24
Area of fracture—sq. in.	·0346	·0638	·0506	·0453
Stress at max. load—tons sq. in. . . .	14·1	22·6	21·9	22·2
Elongation at max. load —per cent. on 8 in. .	35·2	2·8	1·54	1·49
Elongation after rupture —per cent. on 8 in. .	45·8	5·7	4·06	3·5
Contraction of section at fracture—per cent. .	72·9	50·3	55·6	61·3

191. The Alloys of Copper—There are many of these, of which the simpler are the brasses, made up chiefly of copper and zinc, and the bronzes, made up largely of copper and tin. Of the two white metals thus alloyed with copper, zinc has a specific gravity of nearly 7. It melts at 800° F. and vaporizes at 1,900° F. In producing brass great care must be observed to control this vaporization and consequent waste of zinc.

Tin is a silvery white metal more stable than zinc. It is perhaps best known as a coating for thin sheet iron called tinplate. Its specific gravity is 7·3.

The most remarkable alloy of copper and tin is speculum metal. This consists of 2 parts of copper to 1 of tin. It is white in appearance, both hard and brittle, and may be ground and polished to a mirror-like surface, in which condition it is used for the reflectors of astronomical instruments.

192. Aluminium—Aluminium is a uniform white metal, soft, and extremely light, having a specific gravity from 2·56 to 2·75. It melts at 1,150° F. It does not corrode or rust in air. Its strength when rolled is from 10 to 12 tons per sq. in. Its modulus of elasticity is 12 million lb. per sq. in.

The following are results of strength tests of samples of the less common metals—aluminium, copper, brass, bronze, and gunmetal.

	Yield Stress, per sq. in.		Maximum Stress, per sq. in.		Elong. per cent. on 2 in.	Authority.
	Lb.	Tons.	Lb.	Tons.		
ALUMINIUM. Castings .	6,720 to 24,640	3 to 11	20,160 to 35,840	9 to 16	0·5 to 3·5	Popplewell.
COPPER	22,400	10	29,120	13	12 to 24 35 to 45	Popplewell.
Rolled bars for stays. .	—	—	31,360 to 33,600	14 to 15	35 to 45	Popplewell.
BRASS. Copper plus Zinc plus small amount of Tin in some cases . .	6,720 to 28,000	3 to 12·5	15,680 to 51,520	7 to 23	6 to 40	Popplewell.
BRASS (C 60, Zn 40 p. c.)	29,100	13	57,100	25·5	20	Popplewell.
BRASS (C 80, Zn 15, Tin 5 per cent.) . .	17,900	8	40,300	18	30	Popplewell.
BRASS (C. 85, Zn 15 per cent.)	12,310	5·5	29,120	13	30	Popplewell.
NAVAL BRASS (C 62, Zn 37, Tin 1 per cent.) .	22,400	10	56,000	25	35	Popplewell.
Usual specification, 22 to 26 tons per sq. in. and 30 per cent. elongation.						
NAVAL GUNMETAL (C 88, Zn 2, Tin 10 per cent.) (2 samples) . . .	—	—	36,960 35,800	16·5 16	15 14	Popplewell.
Usual specification, 14 tons per sq. in and 7½ per cent. elongation.						
GUNMETAL	13,440 to 26,880	6 to 12	17,920 to 49,300	8 to 22	4 to 18	Popplewell.
MANGANESE BRONZE .	17,900 to 31,360	8 to 14	40,300 to 71,700	18 to 32	3 to 10	Popplewell.
PHOSPHOR BRONZE . .	17,900 to 24,640	8 to 11	22,400 to 35,800	10 to 16	0 to 4	Popplewell.

*Tensile Tests of Aluminium-Tin-Copper Alloys (Guillet)

Composition, per cent.			Yield Point.	Max. Stress, tons per sq. in.	Elongation, per cent.	Reduction, per cent.	Relative hardness.
Copper.	Tin (by diff.).	Alumin. (by diff.).					
90·93	9·00	0·07	7·4	14·7	26·2	34·0	50
88·66	9·87	0·47	5·7	9·9	2·0	1·4	57
89·54	9·70	0·76	4·8	8·1	2·0	0·0	61
89·05	8·74	2·21	4·8	8·1	2·0	2·8	63
90·71	5·18	4·11	4·1	6·2	5·0	12·5	65
89·10	3·59	7·31	7·8	10·7	3·0	13·0	63
Tensile Tests. Chill [illegible]ngs							
88·47	1·18	10·35	16·4	[illegible]	[illegible]	5·6	—
91·24	1·06	7·70	9·6	[illegible]	[illegible]	24·2	—
89·66	2·36	7·98	[illegible]	[illegible]	[illegible]	27·0	—
93·58	0·94	5·48	[illegible]	[illegible]	[illegible]	16·97	—
91·64	2·60	5·76	[illegible]	[illegible]	[illegible]	[illegible]	—
89·51	4·93	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	—
90·20	7·41	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	—
[illegible]ngs							
88·47	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	15·2	—
91·24	[illegible]	[illegible]	[illegible]	[illegible]	68·1	46·4	—
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	35·8	31·8	—
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	43·5	41·2	—
[illegible]	[illegible]	[illegible]	[illegible]	17·34	33·1	31·6	—
[illegible]	[illegible]	[illegible]	[illegible]	15·61	11·7	12·3	—
[illegible]	[illegible]	[illegible]	[illegible]	12·85	9·9	8·7	—

Effect of Replacing Tin by Aluminium
Tensile Tests

[illegible]n, per cent.			Chill Castings.		Annealed Castings,		
Copper.	Tin.	Aluminium.	Max. Stress, tons per sq. in.	Elongation, per cent. on 2 in.	Temper. per cent. on 2 in. of Annealing.	Max. Stress, tons per sq. in.	Elongation, per cent.
					Degs. C.		
90·10	—	9·90	36·93	30·5	800	26·4	5·0
89·66	2·36	7·98	29·88	25·0	750	27·32	35·8
89·51	4·93	5·56	14·98	2·6	700	15·61	11·7
90·20	7·41	2·39	14·61	3·8	700	12·85	9·9
Admiralty gunmetal			16·5	15·0	700	21·4	40·0

* From paper by Prof. A. A. Read, D.Met., and R. H. Greaves (Inst. Metals, March 1, 1916).

TABLE XLVIII. HARDNESS TESTS

Composition, per cent.			Hardness Number.				
			Schleroscope.		Brinell. Load, 3,000 kilos.		
Copper.	Tin.	Aluminium.	Cast.	Annealed.	Quenched.	Cast.	Annealed.
88·47	1·18	10·35	32	26	43	189	145
91·24	1·06	7·70	11	9	11	81	62
89·66	2·36	7·98	16	15	16	102	94
93·58	0·94	5·48	8	8	9	62	66
91·64	2·60	5·76	11	10	11	78	69
89·51	4·93	5·56	19	15	15	120	91
90·20	7·41	2·39	18	15	15	114	94

193. *Duralumin—This light alloy of aluminium has come into use in recent years and for many purposes has proved of great value. In addition to the aluminium it contains the following : Copper, 3·5 to 5·5 per cent. ; magnesium, 0·5 per cent. ; manganese, 0·5 to 0·8 per cent. It can be either rolled or forged. It is extremely light with a density of about 2·8. The melting-point is 650° C. Heating and quenching produce little immediate effect, but after ageing for several days both hardness and ductility are increased. For example, an alloy which after rolling showed a tensile strength of 37,000 lb. per sq. in. and 17 per cent. elongation, after quenching and ageing gave 58,000 lb. and 23 per cent. respectively. Its modulus of elasticity is about 12×10^6 lb. per sq. in.

* From Inst. C.E., Abstracts, New Series, No. 1, October, 1919.

CHAPTER XVI

TIMBER

TIMBER is almost indispensable to the engineer for certain purposes. As its strength properties largely depend upon varying conditions and circumstances, these should be noted. The dominant feature in timber of all kinds is the fact that it is made up of bundles of natural vegetable cells which have been formed during the period of its growth as part of a tree. The quality of a particular wood is to a great extent dependent on the size and arrangement of the ultimate cells. The main formation is in the direction of the trunk or branch from which the wood is taken. These lines of cell bundles go to create what is known as the grain of the wood. Forces applied in the direction of the grain are more strongly resisted, both in tension and in compression, than forces applied in directions normal to the grain. The former tend to pull apart the fibres or to crush them by folding up, while the latter tend to squeeze the fibres of the grain closer together.

All timbers used by the engineer, with the exception of some of the bamboos, are of the exogenous class, that is they grow in thickness by the yearly addition of new layers of cells between the body of the stem or trunk and its bark. The annual increments to the diameter are shown as the annual rings which can be seen in the case of most timbers on a cross-section of the trunk. In some timbers it is impossible to pick out the rings from one another because they have become merged into one general tint. On Plate XXIV (*a*) is seen a microscopic view of a small portion of an annual ring. The individual cells constituting the end views of the fibres are distinctly seen. The noticeable hole is a resin duct. (*b*) shows a section along the grain and normal to the annual layers and (*c*) is a section along the grain but tangential to the annual layers. The wood seen in these photographs is spruce used in aeroplane construction. That part of the stem immediately below the bark and including the most recently added annual rings, serves as a channel for the passage of the sap from the root to branches, and is spoken of as the "sapwood," as opposed to the "heartwood" nearer the centre of the stem.

The heartwood is matured but unseasoned timber, while the sapwood is only partially mature. For this reason it is weaker and less sound, and not so durable as the former (see Fig. 231 (A)). When a tree is fully grown—at an age of from 70 to 100 years—it is felled, the bark stripped and the sapwood cut away in the form of slabs. The remaining log consists wholly of heartwood, and after a certain period of drying and seasoning, is cut up into baulks, beams and planks, ready for use.

In the pines and firs maturity is attained at 70 or 80 years, while the harder timbers, like oak, may not be fully grown in much less than 100 years.

The engineer or the carpenter is chiefly concerned with those qualities of a timber which bear on the uses to which he intends

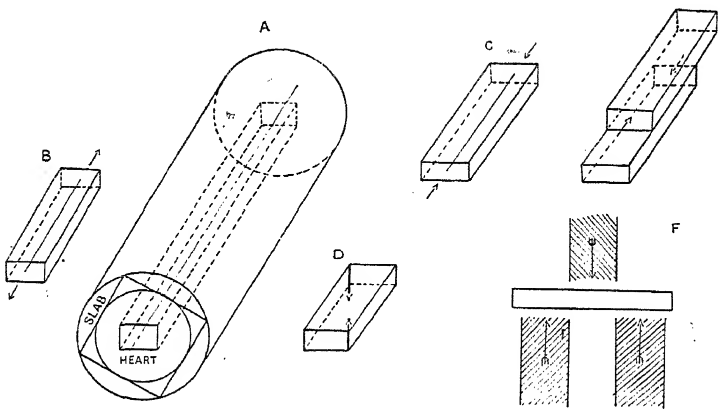


FIG. 231.

to devote it, and so long as these are satisfactory he is not greatly interested in its previous history. At the same time he does subdivide the timbers into "pinewood" and "leafwood," or "softwoods" and "hardwoods."

Seasoning may be natural or artificial. In natural seasoning the timber is cut up into planks and stacked up in a place where it is fully exposed to an abundant supply of dry air. For artificial seasoning the timber is stacked in a closed vessel or tank and exposed to a stream of hot air. The temperature employed is generally from 150° F. to 180° F. Seasoning of this kind is very useful when speed is desired and the timber is urgently required for use. In the final result timber which has been seasoned naturally is held to be better than that which has been dried artificially.

Creosoting—This consists in saturating the timber with the tar-product creosote. By many authorities it is considered to be by far the best preservative of timber, under all conditions. The actual creosoting is effected by stacking the timber in closed metal tanks. The timber should be as dry as possible and the temperature of the creosote should be not less than 120° F. The creosote is admitted under a pressure of not less than 180 lb. per sq. in. The rapidity of absorption depends, of course, on the sizes of the pieces of timber, that is, whether planks, sleepers, or boards. The actual quantity absorbed varies from 10 lb. for each cu. ft. of timber to 25 lb. per cu. ft. The higher figure applies to baulks of timber which are likely to be exposed to the attacks of sea worms.

In the pinewood class, which provides the greater part of the timber used for constructive purposes, are included pitch pine, Baltic pine, American red pine, larch, yellow pine, and spruce, this order being roughly that of their strength and hardness; but though some of these are harder and stronger than others, they all come under the general classification of "softwood." They are easier to cut and work than the "hardwoods," are lighter in weight, less durable, and relatively weaker. The "leafwoods" include a number of "hardwoods," as oak, ash, beech, birch, elm, teak, and greenheart, all of which are important for their strength and durability in varying degrees; and several softer timbers such as mahogany and baywood, which are especially useful because of their facility in being worked, though their strength is not great.

The uses to which timber is put by the engineer include the construction of wharves, piers, and marine work generally (built wholly of timber in the form of large baulks and beams), false-work for reinforced-concrete construction, coffer dams and temporary work in connection with foundations, foundation piles, poles for telegraph and telephone wires, and for carrying transmission lines, railway sleepers, signal poles, and those parts of rolling stock which are constructed in timber. Besides the above there are innumerable minor uses mostly in smaller parts of machinery, where special qualities are of more importance than strength and size. There are also the thousand and one ways in which wood is employed by the joiner in the fittings of buildings and ships.

In the United States, Canada, and in many of our Dominions and Colonies, as well as in other parts of the world, great quantities of timber are used in the construction of engineering works of major importance, such as highway and railway bridges, and viaducts and embankments which are formed entirely from timber. The employment of timber in these directions is very largely

dependent on the ease with which it can be procured and its cost. In some parts of Canada, for instance, pine wood is easily and cheaply obtained in large quantities, and is held to be quite suitable for the uses mentioned above.

When timber is seasoning it shrinks, and the shrinkage must be taken into account when arranging the details of structural parts, as the percentage of shrinkage is generally different in different directions. Thus it shrinks (obviously) more in a lateral direction than along the grain: also there is more shrinkage along the annual rings than at right-angles to them. In average seasoning the loss of weight is from one-fifth to one-seventh of the total weight: when the seasoning extends over a considerable time and is very perfect, the loss of original weight may be as much as one-third. The shrinkage varies with the wood: for example, while a plank of oak may shrink one-twelfth in width, a pine plank of the same size may shrink only one-fortieth.

Pine and Fir—The characteristic qualities of timbers belonging to this family are "regularity and evenness of grain," the possibility of obtaining it in pieces of considerable length, the relative cheapness, the lack of great strength, except in the cases of a few varieties, and, what is perhaps of most importance, the ease with which it can be cut and worked.

Timbers of this class are used for such purposes as the masts of ships and telegraph poles. They form the chief materials for great works such as temporary scaffolding and centerings, and in many more permanent works where questions of cost and strength are of supreme importance. Though the general characteristics of all the fir and pine woods are alike, there are a great many varieties with useful properties peculiar to themselves. The greatest durability is found in the larch and the pitch pine as well as the firs from Norway and the Baltic ports. Besides possessing the quality of durability the firs and pines of this class contain great quantities of resin. This no doubt contributes to their durability and adds greatly to their strength.

A class of the pines which is found to be useful to mechanical engineers is the Canadian white or yellow pine. This wood lacks strength but possesses grain of exceptional uniformity and is very easy to work. On these accounts it is largely used in the making of foundry patterns and similar purposes where it is desired to use a clean-grained timber which is easy to work. Additional valuable properties of Canadian pine are the smoothness of its surface when planed, its almost complete freedom from knots, and its non-liability to warp after being cut into form.

TELEGRAPH POLES—Telegraph poles, which are made of white pine, depend for their preservation on their seasoning and

their after-treatment. Although the heartwood contains a low percentage of water, it is found difficult to eliminate this in the process of seasoning. It is helped by a preliminary soaking which removes some of the gums and so accelerates the rate of air-drying in the case of natural seasoning. There are two objects in seasoning, one to remove the food supply of the fungus which causes decay and the other to render the timber more ready to receive preservative liquids. Where artificial seasoning is employed it is necessary to control the temperature with great care. When the timber has been dried it should be impregnated with the preservative at once. There are two kinds of preservative used, mineral salts absorbed by the wood and paints or varnishes applied to the outside. Oil paint is comparatively soon removed from the surface and so loses its effect, but tar is more permanent by reason of its elasticity when the pole bends and to some extent is absorbed into the wood and helps to preserve it. Sheathing of concrete is sometimes used, but though effective is costly. Charring is not good.

Oak—This timber, especially the British variety, has long been regarded as the synonym for strength and durability, and with the exception of a few of the foreign timbers is the strongest and most durable of timbers. Its actual strength to resist transverse loads is high, it is very tough, and its stiffness under flexure is great. For these reasons oak has been employed from time immemorial for shipbuilding purposes, in the making of carriage wheels, casks, trenails, and the frames of large and small roofs, as well as for household matters, especially furniture.

Against oak it may be said that it contains an acid which has a corroding effect on iron and steel, for which reason copper bolts were, and are still, largely used in wooden ships. It is also a difficult timber to work on account of hardness and want of evenness in its grain.

Ash—Though ash is a coarse-grained wood, it possesses much strength; but its distinguishing qualities are its toughness and flexibility. For these reasons ash is employed where shocks and wrenches are to be resisted in such cases as the felloes and spokes of wheels, the shafts of carriages, and the shafts of hammers. As the characteristic properties of ash are its flexibility and want of rigidity it is seldom used where these qualities are not desired. It has one or two undesirable properties which limit its use to purposes similar to those named. These include its lack of durability when exposed in damp situations, though very durable when kept dry.

Elm—Possesses important qualities which render its use almost imperative in many cases. Its grain is coarse, but it has considerable strength and toughness, and a non-liability to

split when nails or bolts are driven in. It is used for the naves of carriage wheels and for wooden pulley blocks. It will withstand constant immersion but fails when kept alternately wet and dry. Perhaps its greatest defect is its liability to become warped.

Beech—The qualities which distinguish this timber are its smooth and uniform grain. The grain is fine and in many cases beautiful in appearance. Its strength is great, and on this account it is used for the teeth in mortise wheels, where strength, toughness, and the power to resist wear are of great importance. It will withstand constant immersion in water, but when kept alternately dry and wet it rapidly shows signs of deterioration. Its stiffness is not so high as that of oak, but its toughness is greater.

Hornbeam—A most valuable wood, and, like beech, used in relatively small parts such as the teeth of wheels, for which purpose it has no superior. Its value for this use is due to its extreme toughness and ability to resist abrasion.

Mahogany—A close-grained wood having great uniformity of texture; used for small work requiring exactness of form rather than mere strength. The varieties used in engineering work, such as the "baywood" from Honduras and the African mahoganies, possess particular uniformity, and are sufficiently soft to be easily worked. Mahogany is used by the pattern-maker for such work as the teeth of wheels and by the joiner for small parts in light machinery. It has little real strength to resist force and is not a durable wood unless kept at all times quite dry.

Teak—A valuable timber to the constructive engineer. It comes chiefly from Burma, whence it is shipped in quantity to all parts of the world. Its characteristic qualities are its great strength, the small amount of its shrinkage even in very hot climates, and, what is important in many cases, its invulnerability to the attacks of certain small marine insects found in tropical climates and spoken of as "worms." Its grain is rough and it is very liable to split, but as regards tenacity and stiffness it is superior to oak but is less tough.

Greenheart is a hard timber of great value, obtained from Guiana, where the trees grow to a height of 70 ft. In the interior of the country it grows in very great abundance.

There is little in their appearance to distinguish the heartwood from the sapwood; both have a dark-green colour. It is the heartwood which forms the valuable timber so well known to and used by engineers for such purposes as lock-gates, dock-gates, piles, and other harbour works, and canal works. It is peculiarly well adapted for works of this nature because it is able to

resist the attacks of certain insects who drill small holes into the surfaces of most timbers, thus causing ultimate destruction. This effect was shown very strikingly in two pieces of ship planking exhibited in the Kelvingrove Museum, Glasgow. Both formed parts of a wreck which had been submerged on the west coast of Scotland for some eighteen years. Of these, one was greenheart, the other teak. The teak was almost entirely eaten away, while the interior of the greenheart was quite sound, with a slightly pitted appearance of the surface.

Greenheart is an exceedingly strong wood as shown by all the tests. Its use for trout-fly rods shows it as a flexible wood with great elasticity.

Its typical strength properties are as given below:

Tensile strength	3·3 tons per sq. in.
Compressive strength	5·4 ,, ,, ,, ,,
Transverse, 1 in. × 1 in. × 1 ft. span . .	835 lb. (Thurston).

194. Strength Properties of Timber from Experiment—As a general rule, it is always better to make any kinds of tests

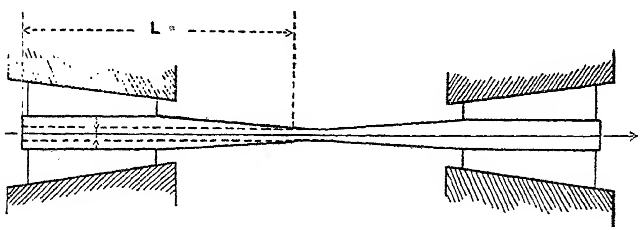


Fig. 232.

on timber on large rather than on small specimens. As timber is not in any sense a homogeneous material, the results obtained from large specimens give a better average than those taken from small pieces of the same material. Small pieces cut from a large plank or baulk may themselves happen to be good or bad samples and not averages of the whole mass. Of course, tests of large pieces are more costly than those of small samples. Consequently where expense is of supreme importance small samples may be used, and if sufficient numbers are tested the strength properties should be indicated well enough for design purposes.

For any one kind of timber the strength properties vary greatly in different samples, and for this reason a wide margin should always be allowed when selecting values for purposes of

design. The information available in this direction includes (B) The Tensile strength, Fig. 231.

A typical test specimen of the best form is shown in Fig. 232. The whole specimen is long, the actual breaking section is relatively small, and the proportions are so arranged that the load required to break the bar in tension is less than the load required to shear through one of the large ends.

Fig. 231 (C), Compressive strength along the grain.
,, 231 (D), Crushing across the grain.
,, 231 (E), Shearing along the grain.
,, 231 (F), Shearing across the grain.

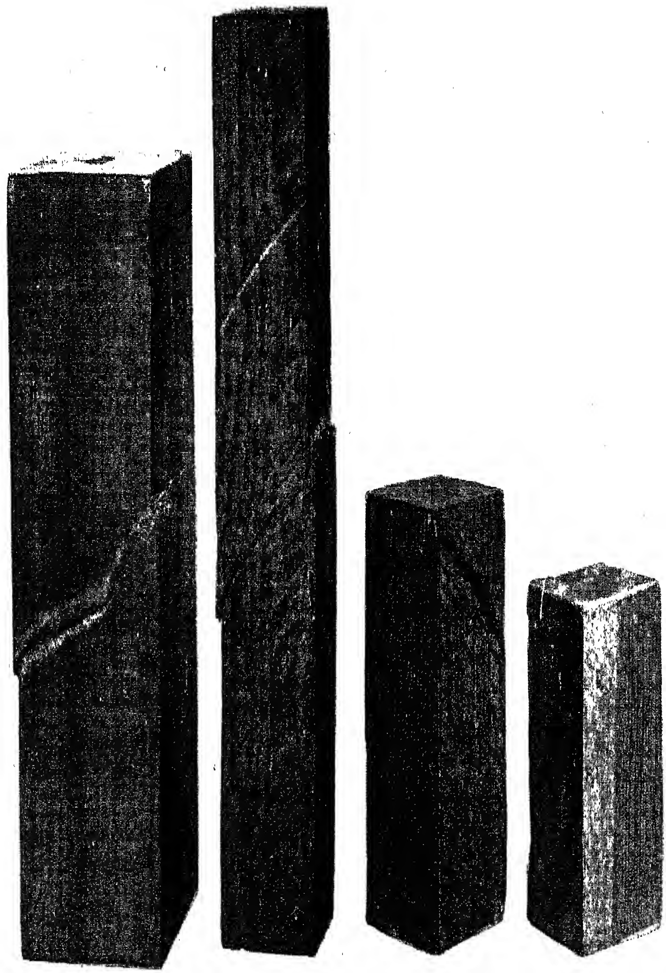
Several of these are given in the following tables:

TENSILE STRENGTHS OF VARIOUS TIMBERS

Timber.	Ultimate Tenacity per sq. in.		Dimensions.	Authority.
	Tons.	Lb.		
Pitch Pine	7·8	—	$1\frac{1}{2}$ in. × $\frac{3}{4}$ in	Popplewell.
Fir	—	7,000 to 18,100	—	Anderson.
,, American	—	12,000	—	,,
,, Memel	—	11,000	—	,,
,, Riga	—	12,600	—	,,
Larch	—	10,200	—	,,
Oak	—	9,000 to 19,800	—	,,
,, English	—	15,000	—	,,
,, Canadian	—	12,000	—	,,
,, African	—	14,000	—	,,
Ash	—	15,780 to 19,600	—	,,
Beech	—	11,500 to 22,200	—	,,
Elm	—	13,490 to 14,400	—	,,
Hornbeam	—	4,250 to 20,240	—	,,
Mahogany	—	8,000 to 21,800	—	,,
Teak	—	8,200 to 15,000	—	,,

195. Crushing Strengths of Timber—The kind of specimen used in crushing tests of timber is usually a prism of square or circular cross-section. Let the base be ABCD: the load may be applied in a direction parallel to AB, normal to AB, or normal to the base. The following are a few results:

PLATE XX



COMPRESSION TESTS OF TIMBER, SHOWING PLANES OF FAILURE

Teb.	Direction of Load.	Sizes.	Crushing Load per sq. in.		Authority.
			Lb.	Tons.	
Teak (A) .	Along grain	3·5″ × 3″ × 3″ high	—	4·13	Popplewell.
„ (B) .	Across grain	—	—	1·33	„
„ (C) . .	Across grain	3″ × 3″ × 3″ high	—	0·89	„
Various African mahoganies	Along grain	1″ × 1″ × 2″ high	6,000 to 10,000 (aver. 8,300)		„
	Across grain	2″ × 1″ × 1″ high	1,850 to 5,150 (aver. 3,670)		„

Plate XXV shows a photograph of a specimen of teak which has been subjected to a crushing load.

It is to be observed that the planes of fracture are inclined to the axis of the pieces and indicate a slide having taken place.

Transverse Tests of Timber

Timber.	Width.	Depth.	Span.	Central Breaking Load.	Equivalent on 1 in. × 1 in. × 1 in. Span.	Authority.
Various African mahoganies	1 in.	1 in.	24 in.	250 to 450 lb. (aver. 380)	760 lb. 1·3 in. Defl.	Popplewell.
Douglas Fir, Brit. Columbia (Green)	8 in.	16 in.	Modulus of rupture, 6,605 lb. sq. in.			Gov. Brit. Col.

Firs

	Douglas (Brit. Columb.)	Longleaf Pine.	Shortleaf Pine.	Redwood.	Norway Pine.
Cross-section . .	8 in. × 16 in	Various	Various	Various	6 in. × 12 in.
Moisture per cent.	31·8	29·2	48·4	90·2	52·1
Condition . . .	Green	Green	Green	Green	Green
Modulus of rupture, lb. per sq. in. .	6,605	6,437	5,948	5,327	3,767
Modulus of elasticity, lb. per sq in. . .	1,611,000	1,466,000	1,546,000	1,202,000	1,042,000
Weight per cu. ft., oven dry . . .	28·9	35·4	31·4	23·3	25·2
Cross-section . .	As above	As above	As above	As above	As above
Moisture per cent. .	20·9	21·6	16·3	17·3	17·0
Condition . . .	Air seasoned	Air seasoned	Air seasoned	Air seasoned	Air seasoned
Modulus of rupture	7,142	5,957	7,033	4,573	5,255
Modulus of elasticity	1,641,000	1,720,000	1,782,000	946,000	1,103,000
Weight per cu. ft. lb. (oven dry) . .	27·8	38·6	32·1	22·2	26·4

The last results, published by the Agent-General for British Columbia, are averages.

The following are results of transverse tests of samples of timber in the form of beams 1 in. wide, 1 in. deep, tested on a span of 24 in. Graphs from similar experiments, showing the relation between the central deflection and central load, are shown on Fig. 233.

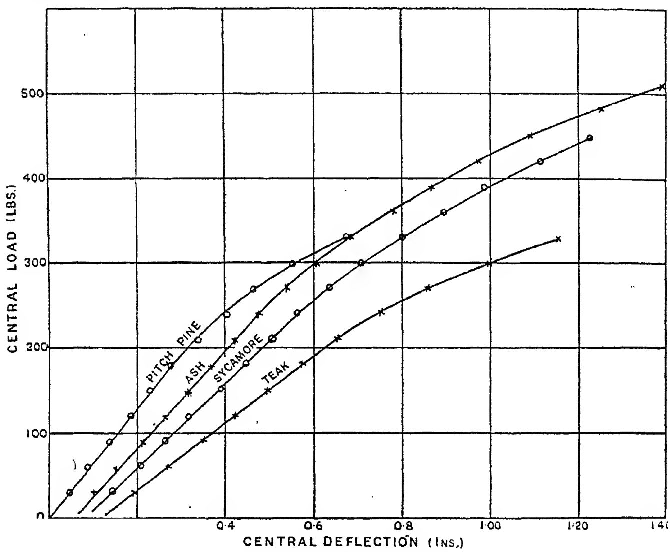


FIG. 233.—Transverse Tests on small Timber Beams 1 in. deep by 1 in. wide, on a span of 24 in.

Timber.	Central Breaking Load, lb.	Modulus of Rupture, per sq. in.		Modulus of Elasticity, lb. sq. in.
		Lb.	Tons.	
Pitch Pine . . .	500	—	—	—
Pitch Pine . . .	384	—	—	1,150,000
Oak	300	—	—	—
Oak	350	—	—	1,382,000
Beech.	300	—	—	1,110,000
Ash	250	—	—	1,016,000
Ash	320	—	—	—

The following data are taken from Mr. Record's book on "The Mechanical Properties of Wood." It should be noted that

these tests were made on samples of timber grown in the United States and tested at the Government Forests Laboratory, Madison, Wisconsin.

Tensile Strength

Name of Species.	Surface of Failure, Radial. Lb. per sq. in.	Surface of Failure, Tangential. Lb. per sq. in.
White Ash	645	671
Beech	633	969
Birch, Yellow	446	526
Elm, Slippery	765	832
Oak, Post	714	924
„ Red	639	874
„ Swamp White	757	909
„ White	622	749
„ Yellow	728	929
Sycamore	540	781
Conifers.		
Cypress, Bald	242	251
Pine, Red	179	205
„ Western Yellow	230	252
„ White	225	285

Endwise Crushing Tests on Small Pieces in Green Condition

Name.	Stress at Elastic Limit.	Crushing Strength.	Modulus of Elasticity.
White Ash	3,510	4,220	1,531,000
Beech	2,770	3,480	1,412,000
Birch, Yellow	2,570	3,400	1,915,000
Elm, Slippery	3,410	3,990	1,453,000
Oak	2,780	3,330	1,062,000
„ Red	2,290	3,210	1,295,000
„ Swamp White	3,470	4,360	1,489,000
„ White	2,400	3,520	946,000
„ Yellow	2,870	3,700	1,465,000
Sycamore	2,320	2,790	1,073,000
Conifers.			
Cypress, Bald	3,560	3,960	1,738,000
Fir, Douglas	2,390	2,920	1,440,000
„ Alpine	1,660	2,060	882,000
„ White	2,610	2,800	1,332,000
Pine, Red	2,470	3,080	1,646,000
„ Western Yellow	2,100	2,420	1,271,000
„ White	2,370	2,720	1,318,000
Spruce	1,880	2,170	1,021,000

COMPRESSION TESTS ACROSS THE GRAIN

Name.	Stress at Elastic Limit.	Fibre Stress in 853 lb. per sq. in.
White Ash	828	97·1
Beech	607	71·2
Birch, Yellow	454	53·2
Elm, Slippery	599	70·2
Oak, Post	1,148	134·6
,, Red	778	91·2
,, White	853	100·0
,, Yellow	857	100·5
Sycamore	433	50·8
Conifers.		
Cypress, Bald	548	64·3
Fir, Douglas	427	50·1
Pine, Red	358	42·0
,, Western Yellow	348	40·8
,, White	314	36·8
Spruce, White	262	30·7

SHEARING STRENGTH ALONG THE GRAIN (GREEN CONDITION)

Species.	Surface of Failure, Radial.	Surface of Failure, Tangential.
Ash, Black	876	832
,, White	1,360	1,312
Beech	1,154	1,375
Birch, Yellow	1,103	1,188
Elm, Slippery	1,197	1,174
,, White	778	872
Oak, Post	1,196	1,402
,, Red	1,132	1,195
,, Swamp White	1,198	1,394
,, White	1,096	1,292
,, Yellow	1,162	1,196
Sycamore	900	1,102
Conifers.		
Cypress, Bald	836	800
Fir, Alpine	573	654
,, Douglas	853	858
,, White	742	723
Pine, Red	812	741
,, Western Yellow	686	706
,, White	649	639
Spruce	607	624

SHEARING STRENGTH ACROSS THE GRAIN OF VARIOUS WOODS

Kind of Wood.	Lb. per sq. in.
Ash	6,280
Beech	5,223
Birch	5,595
Cedar, White	1,372
Chestnut	1,536
Ebony	7,750
Maple	6,355
Oak	4,425
Oak, Live	8,480
Pine, White	2,480
Pine, Northern Yellow	4,340
Pine, Southern Yellow	5,735
Pine, very resinous Yellow	5,053
Spruce	3,255
Walnut, Common	2,830

CHAPTER XVII

CEMENT, CONCRETE, BRICK, AND STONE

196. Cements.—Several important materials coming under this head have been employed in engineering work for many years. Their sphere of usefulness is increasing, and in all probability will continue to do so. The peculiar property common to all cements is that of solidifying and hardening after being mixed into a paste with water. Most of the engineering cements have the further property of being able to set under water and of being unaffected by the action of water (after setting in air). Cements having this property are said to be "hydraulic."

As a general rule cements are formed by mixing together earths containing an excess of lime with others which consist largely of silica and alumina. These are intimately mixed, wet or dry, heated to a high temperature, and ground to a fine powder. It is found that a cement made in this manner possesses the above properties of setting and hardening in or out of water.

Natural Cements—There are some earths which contain the necessary ingredients in themselves in such proportions as will yield a passable cement. When such a material is burnt and ground the result is spoken of as a "natural cement." The constituent earths are limey or marley clays or clayey limestones. The natural cements of this kind which occur in Great Britain are of relatively small importance when compared with the true Portland cements.

Roman Cement—This cement, whose name has no local significance, has been used in Britain for some hundreds of years, and is prepared from natural materials containing the requisite elements. The particular materials used in the manufacture of this cement are found in nodules dredged up on the coasts of Essex and Kent: these contain about 60 per cent. of lime as well as some clay. They are simply burnt at a bright red heat. The following is a typical approximate analysis of Roman cement, which may be classed as a limey clay.

	Per cent.
Silica	19·5
Alumina	10·0
Lime	44·5

	Per Cent.
Ferric oxide	7·5
Manganese dioxide	1·6
Magnesia	2·9
Insoluble siliceous matter	5·9

Also sulphuric acid, carbonic acid, water, and alkalies.

This cement sets very quickly owing to the high percentage of alumina. As compared with Portland cement the manganese is unusually high. The iron gives it its characteristic reddish colour. The hydraulic properties for both salt and fresh water are excellent. Roman cement may almost be described as a natural cement in which the clay preponderates.

HYDRAULIC LIME—This is a kind of natural cement (clayey lime), in which the percentage of lime is high. Among cements of this class are the Blue Lias lime of Britain and the French hydraulic limes. These are very different in composition. That of Blue Lias is somewhat as follows:

	Per cent.
Silica	14·2
Alumina	6·8
Lime	63·4
Ferric oxide	2·3
Magnesia	1·5
Insoluble siliceous matter	2·4

Also sulphuric acid carbonic acid, water, and alkalies.

The French hydraulic limes contain nearly 22 per cent. silica, rather less than 2 per cent. alumina, and 74 per cent. lime.

The excess lime in hydraulic limes must be slaked before using, by sprinkling or other means. In British practice the preliminary slaking is done by the user. In France it is carried out by the vendor, systematically and on a large scale. After this slaking of the superfluous lime, the material remains as a true cement containing an excess of slaked lime. It sets very slowly, but in the end becomes as hard as good Portland cement. This slow-setting property is useful in some classes of work, such as foundations.

The above include the chief "natural" cements familiar to British engineers, but these are all of far less importance than the "artificial" cements. These embrace the well-known and universally employed "Portland cement" and the puzzuolanic or slag cements. These bear a general similarity to the natural cements, but their ingredients are carefully selected and weighed, so as to ensure certainty in composition and uniformity.

PORTLAND CEMENT—A brief definition of the composition and manufacture of Portland cement is given at the commencement of the British Standard Specification for this material. Being clear and concise, it is quoted here:

"The cement shall be manufactured by intimately mixing together *calcareous* and *argillaceous* materials, burning them at a clinkering temperature and grinding the resulting clinker, so as to produce a cement capable of complying with this specification."

The above tells briefly what happens in the manufacture of the cement, but makes no attempt to give detail either of the process or the resulting material.

In Great Britain the substances employed in the manufacture of Portland cement are chalk (calcareous) and clay (argillaceous), or these in other forms, such as limestone and shale or marl. Local conditions vary and the particular natural substances which happen to be available must be used, provided they contain the proper ingredients.

The essential materials needed to produce Portland cement are carbonate of lime and clay containing alumina and silica. The clay is found in the mud of several river estuaries, notably those of the Thames, Humber, and Medway, where the greater part of the cement is manufactured, and to this clay is added chalk. The composition of a couple of the samples of the earths employed are given below :*

Clay	Per cent.
Insoluble siliceous matter	26·67
Silica	31·24
Alumina	16·60
Ferric oxide	8·66
Lime	0·25
Magnesia	1·91
Soda	1·00
Potash	0·45
Sodium chloride	1·86
Water, organic matter, and loss	11·36
	100·00

Chalk	Per cent.
Silica	0·92
Alumina+Ferric oxide	0·24
Lime	55·00
Magnesia	0·36
Carbonic anhydride	43·40
	100·00

In manufacturing Portland cement from the chalk and clay these are mixed in the proportion of about 3 to 1. This yields a percentage of calcium carbonate, in the dried mixture, of about 75. This is got by multiplying the (lime+carbonic acid) by 3 and dividing by 4. The essential process of making cement from these materials, after being prepared in the above quantities, consists in first mixing, wet or dry, burning the mixture so formed, and grinding the resulting clinker.

Mixing—At one time the mixing of the constituent earths was in all cases effected by a wet process, which consisted in

* Mr. Blount in "The Encyclopædia Britannica."

placing the materials in a wash mill and supplying to them a large quantity of water. In this mill a number of harrows are drawn by horizontal arms attached to a central vertical spindle round the chamber which forms the body of the mill. These harrows or rakes take the material which has been supplied in the right proportions, mix up the various constituents among themselves, and at the same time thoroughly assimilate them with the water. The resulting fluid or slurry, referred to as the "slip," is allowed to flow into large settling tanks. The earthy matter held in suspension settles on the tank bottom as a sediment: and the water is then drawn off and the thin paste remaining collected and spread out on floors to dry by means of artificial heat. Thence it is removed to the furnace to be burnt. In the more modern plan, which has in the majority of cases replaced the above, the constituent materials are first mixed in a wash mill so that the fluid resulting contains from 40 to 50 per cent. of water: afterwards the slurry is introduced to the grinding of millstones, whose effect is not only still further to reduce the size of the grains but more completely to bring together the different component materials. This conduces to a more uniform cement and a greater freedom from isolated portions of lime. It is of the utmost importance in making good cement that the materials be intimately and uniformly mixed. Drying is effected by waste heat from the kilns.

Kilns and Grinding—There are a number of types of kiln used for the calcining of the slurry before it is ground into the required cement. These include the ordinary "chamber" kiln; the old-fashioned "bottle" kiln, now practically out of date; the "stage" kilns of the Dietsch type; and the Hoffman or ring kiln.

Of these, the chamber kiln may be taken as typical. It is still largely used. The kiln itself is somewhat similar in shape to an ordinary old-style lime-kiln, with curved conical sides and a fire-grate at the bottom. On this grate are placed materials for starting the fire, consisting of wood and coke; then a layer of dried slurry, another layer of coke, and so on, until the kiln is fully charged. A kiln of this type may take as many as four or five days to complete the burning, about two of these being taken up in starting the combustion, or rather getting it properly going. An average charge will use about 12 tons of coke to 50 tons of slurry, and from this charge should result some 30 tons of clinker.

After removal from the kiln the clinker is reduced to the fine powder of the required cement. The grinding is effected by millstones, or, in recent years, by such improved appliances as ball mills, edge runners and tube mills. The main point to be

considered in connexion with the grinding of Portland cement is that the resulting grains shall be extremely small, so that the final product may be said to have a "floury" consistency. This is extremely important, and it will presently be seen that the Standard Specification insists on all cement being very finely ground and on the cement being subjected to very rigid tests in this direction.

In the chamber kiln economy of heat is effected by carrying the hot gases from the combustion chamber, or kiln proper, to the chimney by way of a long horizontal passage, on the floor of which is spread out the moist slurry. As it dries it cracks and splits up into slabs which are suitable for the kiln.

The Dry Process—A kiln which works on a principle very different from that described is the "rotary" kiln, at one time the chief type used in the United States and which is now being gradually introduced into use in Great Britain and the Continent.

The rotary kiln consists of a long cylinder of boiler plate, about 6 ft. diameter and 60 or more ft. long, lined with refractory material. This kiln is supported on rollers on which it may be rotated slowly, and is placed in a slightly inclined position so that, as it rotates, the materials inside, besides being stirred up by the turning of the kiln, gradually pass along towards the lower end. The main heat is applied at the lower end, either by petroleum flame of high temperature or by the flame of powdered coal. The products of combustion pass up the kiln and heat and dry the fresh slurry as it comes down from the upper end. The raw material is fed in at the upper end either in the form of slurry or as a dry powder ground moderately fine, and passes slowly down the length of the kiln, gradually absorbing heat as it approaches the lower end. When it arrives there it is partly burnt but not clinkered: this final clinkering is completed by the high temperature of the flame arising from the petroleum jet or the blast of powdered coal.

When the hot clinker reaches the bottom end of the kiln it is shot into the upper end of a cooling cylinder, which has the same general form as the main kiln, but is smaller. In the cooling cylinder cold air enters at the lower end, and gradually becomes heated as it passes towards the upper end, where it is utilized in the combustion of the hot flame of the kiln. Here, after passing a water spray, the clinker enters grinding rolls and thence to a second cooling cylinder. A rotary kiln of this design should turn out about 200 tons per week as against 30 tons for a chamber kiln.

The vitrified clinker, by whatever kiln produced, is extremely hard and of a greenish black colour. When it comes from a rotary kiln, both colour and composition appear uniform, but

that from a stationary kiln is often far otherwise in appearance, and there may be a good deal of clinker which is only partially burnt. This unburnt clinker may contain free lime or unsaturated lime compounds. The existence of such is fatal to good cement, and all partially burnt clinker ought to be picked out. Sometimes 1 or 2 per cent. of gypsum or plaster of Paris is permitted to be added to the finished cement to modify the setting time. Nothing else may be added.

The following* is given as a typical analysis of Thames or Medway cement :

Silica	22·0
Insoluble residue	1·0
Alumina	7·5
Ferric oxide	3·5
Lime	62·0
Magnesia	1·0
Sulphuric anhydride	1·5
Carbonic anhydride	0·5
Water	0·5
Alkalies	0·5

Other Artificial Cements—Puzzuolanic cement is made from a mixture of a material containing a high percentage of silica and alumina with lime, such as blast-furnace slag with a certain proportion of slaked lime. The slag is reduced to a granulated condition before mixing by being run in its molten condition into water. The lime is burnt separately in an ordinary lime-kiln. The broken slag, with or without further grinding, is mixed with the lime and the whole ground to a fine powder in ball mills and tube mills or other grinding appliances. The slag should have ingredients as follows : Silica with alumina, not more than 49 per cent. ; alumina, 13 to 16 per cent. ; 42 per cent. of lime ; and less than 4 per cent. of magnesia. The proportions used in the mixture are 3 parts by weight of granulated slag to 1 part of lime.

A slag cement usually contains 19 per cent. to 29 per cent. silica, 11·5 per cent. to 17·5 per cent. alumina, and 50 to 54 per cent. lime, with some other smaller quantities of the other constituents.

The cement made in this way is supposed to be little inferior to the standard Portland cement.

The puzzuolanic or slag cements are slow-setting, but are found to develop great strength as time elapses.

The hydraulic cements, both Portland and slag, have their chief uses as binding materials for concrete and mortar which

* Mr. Bertram Blount.

may or may not after setting be exposed to the action of water. Lime mortar, made with sand and slaked lime, has not the property of resisting the effect of the presence of water, and it is only when a cement containing silicates of alumina as well as lime is used that the desired water-resisting property is attained.

* Standard Specification for Portland Cement (Summary), 1920

Each sample for testing is to consist of approximately equal portions selected from twelve positions within a heap or from twelve heaps, or from twelve bags or barrels. When the number is less than twelve then from each bag, barrel or other package. When more than 250 tons of cement is to be sampled at one time separate samples shall be taken from each 250 tons or part thereof. Not more than 250 tons shall be stored in such a manner that it cannot be separately identified and sampled and separated in bulk from the remainder, e.g. in the event of storage of more than 250 tons in a silo, provision shall be made by which each 250 tons, or any part of 250 tons in excess thereof, shall be isolated from the remainder, and sampled at different points.

The vendor shall afford every facility, and provide all labour and materials, for taking and packing the samples for testing and for subsequently identifying the cement sampled.

The tests and chemical analyses hereinafter mentioned shall (unless otherwise provided for in the contract between the vendor and purchaser) be made at the expense of the purchaser, but no charge shall be made by the vendor for the cement used for samples or for carriage thereon.

TESTS—(*a*) Fineness, (*b*) chemical composition, (*c*) tensile strength (neat cement), (*d*) tensile strength (cement and sand), (*e*) setting time, (*f*) soundness.

(*a*) *Fineness*—Sample of 100 grammes (approx. $\frac{1}{4}$ lb.) to be sifted for 15 minutes on each of the following sieves, in the order given :

(1) Residue, on a sieve $180 \times 180 = 32{,}400$ meshes per sq. in., shall not exceed 14 per cent.

(2) The residue, on a sieve $76 \times 76 = 5{,}776$ meshes per sq. in., shall not exceed 1 per cent.

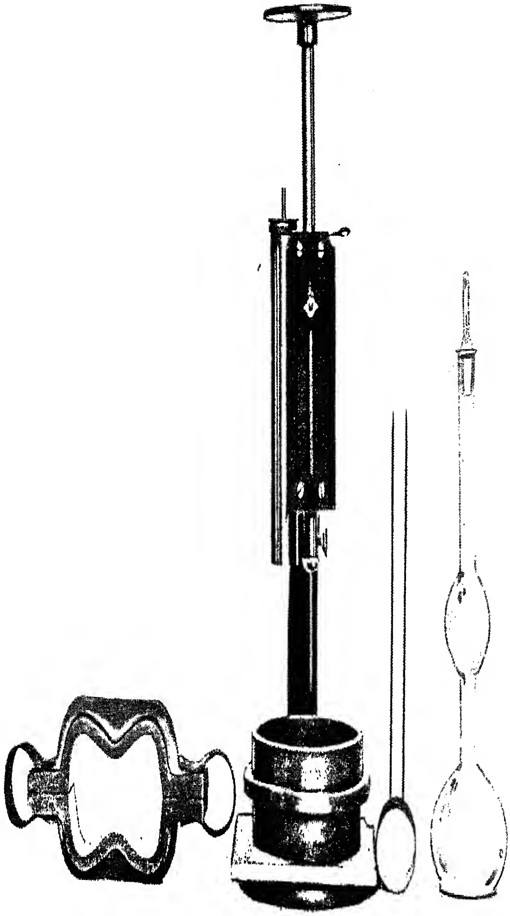
Sieves to be of cloth from wire 0·0018 in. diam. for fine, and 0·0044 in. for coarse. The wire cloth to be woven (not twilled).

In the 1915 specification the specific gravity test was included, and it was specified that the specific gravity should not be less

* Abstracted by permission of the British Engineering Standards Association from B.S. Specification for Portland Cement, 1920. Price 1/2 post free.

PLATE XX

CEMENT TEST APPLIANCES



(a) BRIQUETTE MOULD

(b) VICAT NEEDLE

(c) FOR LE CHATELIER SOUNDNESS TEST

(d) BOTTLE FOR SPECIFIC GRAVITY DETERMINATIONS

than 3·10. The bottle which was to be used is shown on Plate XXVI (*d*).

(*b*) *Chemical Composition*—The proportion of lime to silica and alumina calculated (in chemical equivalents) by $\frac{CaO}{SiO_2+Al_2O_3}$ shall not be greater than 2·85 nor less than 2·0. The percentage of insoluble residue shall not exceed 1·5 per cent.; that of magnesia shall not exceed 3 per cent.; and the total sulphur (as SO_3) shall not exceed 2·75 per cent.; the total loss on ignition shall not exceed 3 per cent.

(*c*) *Tensile Strength* (*Neat Cement*)—Mixture to be plastic when filled into the moulds (Plate XXVI (*a*)). These must rest on non-porous plates. In filling, the operator's hands and the ordinary gauging trowel are alone to be used. No ramming or hammering in any form will be permitted nor any other instrument but the trowel used. Clean apparatus to be used, and the temperature of the test room to be from 58° F. to 64° F.

Briquettes to be kept in damp atmosphere for 24 hours after gauging, then to be removed from the moulds and placed in water until the time for breaking arrives. Water is to be changed every 7 days.

Briquettes are to be tested at 7 and 28 days after gauging, six for each period. The breaking strength shall be the average strength of six. In testing the load to commence from zero and to increase at the rate of 100 lb. in 12 seconds.

The breaking strength at 7 days shall be not less than 450 lb. per sq. in. The strength at 28 days shall be not less than

$$\text{Strength at 7 days} + \frac{40{,}000 \text{ lb.}}{\text{Strength at 7 days.}}$$

(*d*) *Tensile Strength* (*Cement and Sand*)—The mixture to consist of 1 part by weight of cement to 3 parts by weight of sand. To be gauged with sufficient water to wet the whole mass without any excess. After filling a mould a small heap of the mixture shall be placed in the mould and pressed down with the "standard spatula" until level with the top of the mould. This is to be repeated a second time and the mixture patted down until water appears on the surface. To be kept in air for 24 hours, then in clean water until taken out for testing. Testing as for neat briquettes.

Breaking strength at 7 days to be not less than 200 lb. per sq. in. After 28 days strength shall be

$$\text{Strength at 7 days} + \frac{10{,}000 \text{ lb.}}{\text{Strength at 7 days.}}$$

The standard sand shall come from Leighton Buzzard and pass through a 20 to the inch mesh sieve but be retained by one of

30 meshes. For these the wire to be respectively 0·0164 in. and 0·0108 in. The sand should be well washed.

(e) *Tests for Setting Time*—The two gradations are "Quick" and "Slow."

Unless a specially quick-setting cement is specified or required, it shall have an initial setting time of not less than 20 minutes and a final setting time of not more than 10 hours.

If a specially quick-setting cement is specified or required, it shall have an initial setting time of not less than 2 minutes, and a final setting time of not more than 30 minutes.

The initial and final setting times shall be found in the following way:

The Vicat mould (Plate XXVI (*b*)) to be filled with cement gauged as before and smoothed off on the top. The needle is then to be lowered gently until just touching the surface of the block and suddenly released. This is repeated until the needle on being released does not penetrate the block completely. The time when this occurs from the time of filling the mould is the initial setting time referred to.

For the final set an attachment is made to the needle whereby a circular sharp edge is added to the end. The final set shall be taken to have been attained when the end of the needle makes an impression but the attachment does not. When scum forms on the block the under side may be used for the final set.

(*f*) *Tests for Soundness*—Le Chatelier test, with the apparatus shown (Plate XXVI (*c*)): In conducting the test the mould of the apparatus is to be placed on a small piece of glass and filled with cement gauged as before, care being taken to keep the edges of the mould together during filling. The mould shall be covered with another glass plate, a weight applied to keep all together, and the whole submerged in water at 58° F. to 64° F. and left there for 24 hours. The distance between the indicator points is now measured, the mould again submerged, the water brought to boiling-point in 25 to 30 minutes and kept boiling for 6 hours.

The mould is now removed from the water, allowed to cool, and the distance between the points again measured: the difference between the two measurements represents the expansion of the cement. When the cement has been aerated for 24 hours the expansion shall not exceed 10 mm. In the event of the cement failing to comply with the above, a further test shall be made with another portion of the same cement, which portion has been aerated for 7 days, that is, spread out 3 in. deep at temperature 58–64°.

NOTES ON THE ABOVE SPECIFICATION—The tests mentioned in the above specification are generally made by persons who

devote the whole or greater part of their time to the carrying out of this or similar work. In this way they attain a familiarity with the habits of the cement as well as the necessary manual skill desirable. It is important that the instructions in the Specification be carried out rigidly. The following notes may be useful.

(*a*) *Fineness*—The importance of fineness in good cement was thoroughly investigated some years ago by Mr. Butler (p. 450). His experiments went to show that the fineness has considerable effect on the time of setting, and on the "adhesive" strength, that is, the strength when mixed with sand. This—the really important strength—is found to be higher with finer cement, though the "cohesive" strength of the neat cement may be slightly lower.

The size of wire in the sieves should be measured by the user. This is best done with the gauze in its frame by means of a micrometer microscope. Makers should not be allowed to supply sieves with twilled instead of plain weaving. This point should be carefully observed when new sieves are supplied. When all the very fine cement has been shaken through a sieve it will be seen that the residue is very much darker in tint, approximating to that of the original clinker.

(*b*) *Chemical Composition*—These must, of course, be made by trained chemists.

(*c*) and (*d*) *Tensile Strength of Cement*—This is the most universally carried out of all the tests specified. Many engineers and most contractors possess the apparatus necessary and carry out tests on all cement used. The three important points to be watched in tensile tests are *gauging*, *filling the moulds*, and the *holding grips used in the machine*.

The gauging should be effected on a slate or iron slab, the water being added gradually but quickly. The mixing is made more complete by rubbing the paste hard down on to the slab with the trowel, and turning it over. When the paste has been thoroughly and uniformly mixed its consistency should be such that when made into a large ball and dropped from a height of 1½ ft. on to the slab it shows no sign of cracking. If it cracks it is too dry. The ball may be cut up into suitable portions with the trowel and these severally packed into the moulds by hand, the top surfaces being smoothed off with a trowel. At one time small wooden rammers were used for packing the moulds and even metal pieces of the same shape as a loosely-fitting briquette were sometimes used with a hammer to pack the cement the more tightly. But these expedients, many of them unfair, are now forbidden, and there is far less margin left for the faking of tests. Probably the most satisfactory grips are those made by Messrs. Fairbanks, U.S.A.

The following are important as showing the typical kind of results to be expected from average cements:

* BUTLER'S RESULTS

Cement	How Treated.	Fineness. Residue per cent. on Sieves of Meshes per lineal in.			Setting Properties.		Percentage of Water used for gauging Briquettes.		Tensile Strength in lb. per sq. in.									
									Neat Cement.					Three parts Sand to one part Cement.				
		180	76	50	Initial Set. Min.	Set Hard. Min.	Neat.	Sand.	7 days.	28 days.	3 months.	6 months.	12 months.	7 days.	28 days.	3 months.	6 months.	12 months.
F	As received from manufacturer	33·0	16·0	4·0	15	90	21·66	7·81	483	572	623	662	653	183	276	383	440	482
F	Reground extremely fine	2·5	ni	nil	4	60	25·00	9·38	498	541	538	531	506	347	452	564	599	637
G	As received from manufacturer	35·0	20·0	8·0	10	90	20·00	7·81	495	618	62	694	759	187	245	334	377	392
G	Reground extremely fine .	1·0	nil	nil	1	4	25·00	8·13	540	474	560	466	477	282	363	494	595	617
H	As received from manufacturer	28·0	11·0	4·0	8	60	19·16	7·81	445	493	584	663	706	167	230	312	373	399
H	Reground extremely fine .	0·8	nil	nil	2	5	26·66	8·13	433	501	514	482	535	287	364	508	585	599
I	As received from manufacturer	39·0	15·0	2·5	20	120	20·0	8·59	592	639	736	791	751	240	297	389	425	410
I	Reground extremely fine .	0·8	nil	nil	2	10	30·17	11·26	417	394	459	476	498	387	465	560	585	618

* Butler, "The Finer Graining of Portland Cement," Min. Proc. I.C.E., vol. cxxxii.

The following is a sample set of results :

Date..................

MATERIALS TESTING LABORATORY

Report on Mechanical Tests of a sample of Portland Cement

submitted by........................ *Test No*..............

Description.	Fineness. Residue per cent. on Sieves of Meshes per lineal in.		Tensile Strength in lb. per sq. in.				Setting Time by Vicat Needle.		Le Chatelier Soundness Test.	Remarks.
			Neat Cement.		3 Standard Sand to 1 Cement.		Minutes.		Millimetres Increase.	
	180	76	7 Days.	28 Days.	7 Days.	28 Days.	Initial.	Final.		
	13	0·8	460	580	280	302	17	42	9	
			465	590	255	305				
			450	576	250	304				
			455	572	252	307				
			468	583	256	310				
			462	588	280	306				
		6)	2,760	3,489	1,573	1,834				
			460	$581\frac{1}{2}$	$262\frac{1}{6}$	$305\frac{2}{3}$				
British Standard Requirements	14	1	450	539	200	250			10	

Test carried out by...

197. Concrete—Portland cement concrete is made up of broken stone, stone-dust, or sand, and cement. The proportions in which the constituents are mixed are generally expressed in volumes. The stone and sand taken together are referred to as the "aggregate." For some kinds of work broken bricks are used in place of stone.

For reinforced-concrete work the best material for the solid part of the aggregate is undoubtedly pieces of broken hard stone such as granite, trap, or millstone grit: the rough surfaces of this last combined with its strength make it specially valuable.

For reinforced concrete the size of the pieces must be limited to that which will pass through a specified hole—$\frac{3}{4}$ in., $\frac{7}{8}$ in., or 1 in. For other work the sizes may be greater, especially when in relatively large masses.

A strong concrete is obtained when the proportioning is 4 : 2 : 1 for stone : sand : cement, and these figures may vary up to such weak concretes as 5 : $2\frac{1}{2}$: 1 and 5 : 3 : 1.

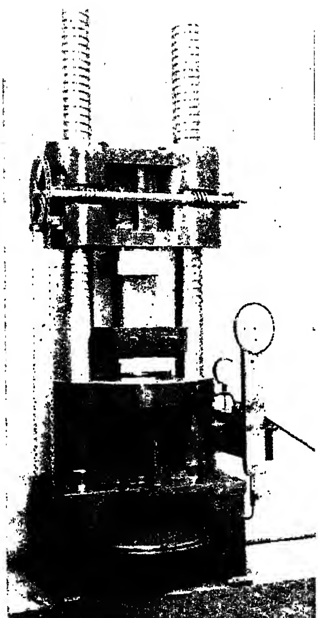
The strength of good concrete depends largely on the quality of the cement used, and this strength ought to increase steadily with the lapse of time.

Voids, or the total space between the stones, are measured by using a tank specially arranged for the purpose. Such a tank is 1 cu. ft. in volume with the top open and a drain tap at the bottom. In using it the tank is first filled up to the top with the pieces of broken stone, the tap is closed and the vessel filled up with water until the water surface comes up to the top of the tank, which has been placed in a level position. The tap is now opened and the water drained off. The volume of this water is the volume of the voids. When the stone is hard and the water drained off to the last drop, the measurement will be sufficiently accurate. When soft and porous the stone should be given a preliminary soaking before the measurement is made, so that no part of the voids volume may be lost by absorption.

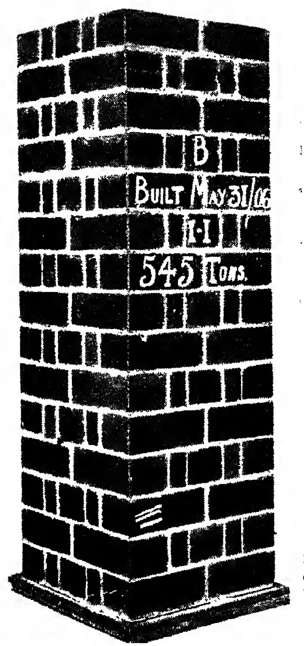
The volume of the voids so obtained is also the volume (dry) of the sand and cement, and is just sufficient to fill up all the space between the pieces of stone. When the volumes of the constituents have all been measured the latter are placed together in their correct volumes and mixed either by hand or in a mixing machine. Water is then added, its amount being fixed by experience and the kind of work for which the concrete is being prepared.

198. Testing the Strength of Concrete—This most frequently takes the form of crushing tests of cubes formed in moulds, the usual sizes being 4 in., 6 in., 9 in., and 12 in. cubes. The moulds are filled and the material is allowed to set hard before the specimens are removed from the moulds. They should then

PLATE XXVII.



(*a*) 500-TON AMSLER COMPRESSION MACHINE



(*b*) BRICK PIER LOADED AT THE COLLEGE OF TECHNOLOGY, MANCHESTER

be kept in a moist atmosphere until the time for testing arrives. This space of time may vary from a week to a year or more. The longer the concrete has been kept, the stronger it should be. A suitable machine for crushing tests is Amsler's 500-ton press, working with oil (Plate XXVII (*a*)). The manner of placing the cubes of concrete (or stone or brick) between the platens of the testing machine, used by the authors, is the following: The top of the cube is smeared with a thick layer of plaster of Paris paste, and a square of rough millboard, rather larger than the face of the cube, is placed upon it. The cube is now turned upside down and the process repeated on the face opposite to the first. It is then placed between the platens (previously greased), and about half a ton of load applied and kept there for ten minutes. This should all be done quickly, and will be found to give a hard uniform bed. Thick cardboard has been used, and also thin timber, but the above method is probably the best, as well as the most convenient.

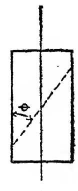


FIG. 234.

In Germany the loaded faces are ground so as to be quite plane and in this state are placed against the platens of the machine without any attempt at an intervening bed to equalize the stress. Though specimens for crushing most often have the form of cubes, the correct form for these should be a prism with a height equal to about $2\frac{1}{2}$ times the transverse dimension, so as to allow of a free natural-shear fracture across the specimen, as in Fig. 234. Where the cube form is used, the planes of shear are slightly steeper than in the prismatical and the form of the broken specimen is as shown in Fig. 235. The shear planes, which make an angle of about 30 deg. with the axis, intersect about half-way up, the intersection being a line instead of a point. These lines of intersection form a square (in the case of a square prism), and this becomes the ultimate dividing surface between the upper and lower pyramids remaining after completion of the test. Plate XXVIII (*a*) is a photograph of an actual crushed cube, where the fracture surface develops as in the last figure. With the taller specimens the pyramids are more nearly complete and terminate in points.

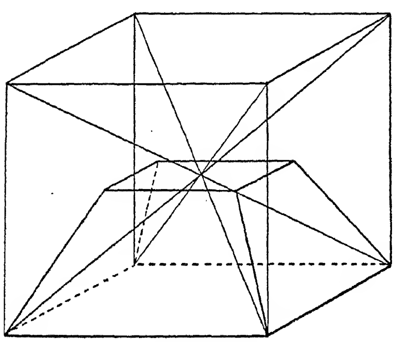


FIG. 235.

In some cases the fracture takes the form of a wedge terminating in a chisel edge. This is shown in the case of a round sandstone pillar on Plate XXVIII (*b*). For a given material the cubical specimen gives a higher crushing stress than the prism.

The modulus of elasticity of concrete in compression, which must be known when designing beams and struts, is taken as lying between 1,000,000 and 4,000,000 lb. per sq. in.

For reinforced work it is generally the ratio

$$\left(\frac{\text{modulus for the steel}}{\text{modulus for concrete}}\right)$$

which is required.

199. Reinforced-Concrete Work—The following data relative to reinforced-concrete beams and columns should be useful:

The beams referred to are rectangular with reinforcing bars near the bottom.

y is the distance of the neutral axis from the top edge of the beam, d the distance from the top to the centre of the reinforcement, b the breadth of the beam, and m the ratio $\frac{E_s}{E_c}$.

C is the maximum stress in the concrete.

S is the maximum stress in the steel.

a is the area of steel cross-section.

M is the bending moment on the section.

E_s is Young's modulus for the steel and E_c that for the concrete.

$$y = -\frac{am}{b} + \sqrt{\frac{am}{b}\left(2d + \frac{am}{b}\right)}$$

$$S = \frac{3M}{a(3d - y)}$$

$$C = \frac{6M}{by(3d - y)}$$

In the case of the columns W is the total load carried.

A is the part of the cross-section which consists of concrete.

a is the cross-section of the reinforcement.

m is the modulus ratio $= \frac{E_s}{E_c}$

$$S = \frac{Wm}{am + A}$$

$$C = \frac{W}{am + A}$$

The values adopted by several countries are, for E_c,

France	3,750,000 to 2,000,000
Italy	3,000,000
United States	2,500,000

	Modulus Ratio m.	Corresponding E_c.
France	8 to 15, according to manner and weight of reinforcement.	$3{\cdot}75$ to $2{\cdot}0 \times 10^6$
Italy	10	$3{\cdot}0 \times 10^6$
United States . . .	12	$2{\cdot}5 \times 10^6$
Germany.	15	$2{\cdot}0 \times 10^6$
Austria	15	$2{\cdot}0 \times 10^6$

The working stresses allowed on reinforced concrete under compression when being used in beams is often 500 lb. per sq. in., and in pillars 300 lb. per sq. in.

Many other experiments have been carried out and their results published, but those in the next table will serve to show the kind of values to be expected.

In the Institution of Civil Engineers' Report by the Committee on Reinforced Concrete, it is stated that the average values of the modulus of elasticity of concrete mixed in the usual proportions are approximately, for stresses up to 500 lb. per sq. in.: Limestone concrete, $4{\cdot}0 \times 10^6$ lb. per sq. in.; gravel, $2{\cdot}5 \times 10^6$ lb. per sq. in.; cinder, $1{\cdot}0 \times 10^6$ lb. per sq. in.

In the discussion on the writer's paper * some interesting values were given by Mr. C. F. Marsh as calculated from one of the diagrams in the paper. This gave the average longitudinal strains of four of the columns tested.†

* Popplewell, Min. "Proceedings," Inst. C.E., vol. clxxxviii, 1911–12, Pt. II.

† Discussion on above paper.

EXPERIMENTAL VALUES FOR MODULUS OF ELASTICITY OF CONCRETE

Material.	Age. Weeks.	Crushing Strength. Tons per sq. ft.	Elastic Modulus. Millions of lb. per sq. in.
* Dock gravel and finely ground cement (5 to 1)	13	211	1·63
Dock gravel and ordinary ground cement (5 to 1)	34	225	2·55
Dock gravel and ordinary ground cement (5 to 1)	15	70	1·75
Dock gravel and ordinary ground cement (5 to 1)	27	101	1·83
Broken sandstone and ordinary cement	53	141	{1·54 1·86
† Penmaenmawr granite, Leighton Buzzard sand, stone-dust, and ordinary cement (5 to 1) . . .	5	—	2·28
Ditto, ditto.	5	—	2·35
Ditto, ditto.	27	143	3·00
Ditto, ditto.	29	(average)	3·04
Ditto, ditto.	31	—	3·08
‡ Concrete like the last but no stone-dust	8	150	1·22
Ditto, ditto.	22	(average)	1·53
§ 4 parts trap rock, 2 coarse sand, and 1 part Alpha cement	13	—	2·32
Ditto, ditto.	13	—	1·85
Ditto, ditto.	13	—	1·54
Ditto, ditto.	13	—	1·84
Ditto, ditto.	13	—	1·72
Ditto, ditto.	13	—	0·73
Ditto, ditto.	13	—	1·65
The following are also of interest:			
‖ 4 stone, 2 sand, and 1 part cement	13	—	3·46
Ditto, ditto	26	—	4·50
6 parts stone, 3 sand, 1 cement .	13	—	3·75
Ditto, ditto.	26	—	2·81

* Min. "Proceedings," Inst. C.E., vol. clxi.
† Min. "Proceedings," Inst. C.E., vol. clxxvii.
‡ Min. "Proceedings," Inst. C.E., vol. clxxxviii.
§ Tested by Professor Burr, New York.
‖ From paper by Professor Hatt in "Engineering Record," May 18, 1902.

200. Other Test Results—The figures in the following tables represent some results of experiments carried out in recent years. Many of them were on concrete intended for use in reinforced construction.

The quantities in Table I refer to material tested for Mr. L. G. Mouchel in connexion with Hennebique reinforced work.

It will be noticed that the cubes give higher result than the pillars, whose fracture is less restricted.

* TABLE I

Material.	Age. Weeks	Approximate Limit of Proportionality.	Cracking Stress.	Crushing Stress.	Elastic Modulus.	
		Tons per sq. ft.	Tons per sq. ft.	Tons per sq. ft.	Tons per sq. ft.	Lb. per sq. in.
Concrete cube, dock gravel and finely ground cement (5 to 1)	13	65	183	211	105,000	1,633,000
Concrete cube, dock gravel and ordinary cement (5 to 1)	34	95	196	225	164,000	2,550,000
Concrete column, dock gravel and ordinary cement (5 to 1)	15	40	70	70	113,000	1,750,000
Ditto	27	40	101	101	118,000	1,830,000
Concrete column, broken sandstone and Portland cement (5 to 1)	53	75	141	141	{ 99,000 120,000	1,540,000 1,860,000

The figures in the following table were derived from tests of concrete having the following composition:

Broken Penmaenmawr granite, screened to pass a ¾ in. mesh and to be retained by a ½ in. mesh	27 cu. ft.
Leighton Buzzard sand, unscreened	13½ ,, ,,
Portland cement	6½ cwt.

This gives a concrete containing about 5 parts of stone and sand.

* Popplewell, Min. "Proceedings," Inst. C.E., vol. clxi.

* TABLE II

Cubes from Column.	Age.	Compressive Strength.			
		First Crack.		Crushing.	
	Months.	Lb. per sq. in.	Tons per sq. ft.	Lb. per sq. in.	Tons per sq. ft.
B	3	2,180	140	2,780	179
C	3	1,570	101	1,910	123
	3	1,520	98	1,870	120
D	6	2,560	165	2,640	170
	6	2,980	192	2,980	192
E	6	2,180	140	2,260	145
	6	2,140	138	2,360	152
F	6	1,520	98	1,880	121
	6	2,115	136	2,180	140

In the case of column A the cubes were omitted.

The figures in Table III are given by Mr. C. S. Meik for concrete used chiefly in reinforced work.

† TABLE III.—STRENGTH OF CONCRETE (MR. MEIK'S RESULTS)

Material and Proportions.	Size of Sieve.	Months.	Size of Block.	Crushing Stress.	
				Lb. per sq. in.	Tons per sq. ft.
Gravel and sand, $1 : 1\frac{1}{3} : 2\frac{2}{3}$ . .	$\frac{3}{4}$ in.	3	6 in.	1,182	76·32
Lock ballast, 1 : 4	$\frac{3}{4}$,,	3	6 ,,	2,257	162·72
Pit ballast, 1 : 4	$\frac{3}{4}$,,	3	6 ,,	3,046	195·84
Guernsey granite and sand, $1 : 1\frac{1}{3} : 2\frac{2}{3}$	$\frac{3}{4}$,,	3	6 ,,	3,685	236·1
Gravel and sand, $1 : 1\frac{1}{3} : 2\frac{2}{3}$ . .	$\frac{3}{4}$,,	6	6 ,,	1,440	92·2
Lock ballast, 1 : 4	$\frac{3}{4}$,,	6	6 ,,	2,453	157
Pit ballast, 1 : 4	$\frac{3}{4}$,,	6	6 ,,	3,640	233·3
Guernsey granite and sand, $1 : 1\frac{1}{3} : 2\frac{2}{3}$	$\frac{3}{4}$,,	6	6 ,,	3,998	256·3
Gravel and sand, $1 : 1\frac{1}{3} : 2\frac{2}{3}$ . .	$\frac{3}{4}$,,	9 and over	6 ,,	1,541	99·4
Lock ballast, 1 : 4	$\frac{3}{4}$,,	12	6 ,,	2,456	157
Pit ballast, $1 : 1\frac{1}{2} : 3\frac{1}{3}$	$\frac{7}{8}$,,	20	12 ,,	4,130	265
Guernsey granite and sand, $1 : 1\frac{1}{3} : 2\frac{2}{3}$	$\frac{3}{4}$,,	9	6 ,,	4,467	286·6

* Popplewell, "Experimental Columns of Reinforced Concrete," Min. "Proceedings," Inst. C.E., vol. clxxxviii.

† "Preliminary Report of I.C.E. Committee on Reinforced Concrete."

* TABLE IV.—MR. DEACON'S TESTS OF 9 IN. CUBES

Age.	Crushed between	Lb. per sq. in.	Tons per sq. ft.
36 months	Millboards	2,883	185·4
35 "	"	2,917·2	(187·6)
34 "	"	2,888	(185·7)
32 "	"	2,803·66	180·3
29½ "	"	2,516	161·8
28½ "	"	2,215·9	142·5
28½ "	"	2,819·21	(181·3)
25½ "	"	1,737	111·7
23 "	"	2,719·7	174·9
20 "	"	2,544	163·6
17 "	"	2,911	187·2
5 "	"	1,903·32	122·4
2 "	"	1,408·8	90·6
77 days	"	1,743·15	112·1
70 "	"	1,309·78	84·23
56 "	"	1,475·7	94·9
43 "	"	1,774·25	114·1
35 "	"	1,665·4	107·1
28 "	"	2,208	141·99

The second and third and last eleven had Parian cement on the top face.

† The following figures were obtained by Mr. J. H. Ellis, when testing Portland cement concrete used in warehouse wharves at Shanghai, where the proportions were 4 beach gravel, 2 sand, to 2 cement. The proportions for the foundations were 5, 2½, 1.

TABLE V

Series.	Number of Samples.	Average Crushing Strength in lb. per sq. in.				
		28 Days.	56 Days.	84 Days.	168 Days.	1 Year.
1	45	2,165 (26 tests)	2,581	2,811	3,070 (19 tests)	—
2	56	1,923	—	2,540	3,333	—
3	32	2,251	—	—	—	4,137

201. Tests on Grip in Reinforced Work—The figures in the tables below indicate the magnitude of the forces (often called "adhesion") with which set concrete is able to grip round bars of steel. Age increases the force of the grip: so does the small-

* Figs. in brackets refer to specimens which cracked but did not crush. From Unwin's "Testing of Materials."

† Ellis, Min. "Proceedings," Inst. C.E., vol. clxxxviii.

ness of the bar. In the results below one of these influences is seen acting against the other almost to the extent of complete compensation. These bars, used by Mr. Popplewell, were plain round metal bars as received from the maker.

* RESULTS OF GRIPPING TESTS, IN WHICH SPECIMENS OF ROUND STEEL WERE PUSHED THROUGH PRISMS OF CONCRETE 6 IN. SQUARE IN SECTION AND OF VARIOUS LENGTHS

Size of Bar.	Length of Prism.	Age of Concrete.	Frictional Resistance in lb. per sq. in. of Surface.	Average.
In.	In.	Months.		
$\frac{3}{16}$	6	4	422	587
,,	7	,,	471	
,,	8	,,	675	
,,	9	,,	481	
,,	10	,,	590	
,,	11	,,	790	
,,	12	,,	660	
$\frac{3}{4}$	6	9	660	635
,,	7	,,	642	
,,	8	,,	680	
,,	9	,,	520	
,,	10	,,	505	
,,	11	,,	695	
,,	12	,,	743	

In the second table below are the results of withdrawing tests of small bars (about $\frac{3}{16}$ in.) of circular section. They were withdrawn from old sleepers which had been in use for several years on the Western Railway of France. The bars in this case were not perfectly straight.

† No.	Resistance to Slipping.		Observations.
	Kg. per sq. cm.	Lb. per sq. in.	
1	41·6	592	Steel broke outside the concrete.
2	50·0 1st trial	711	No slip under the load applied.
	81·2 2nd trial	1,155	Steel slipped.
3	74·0	1,053	Steel broke outside the concrete.
4	87·1	1,239	Steel slipped.
5	92·0	1,308	Steel broke outside the concrete.
6	81·3	1,156	Steel slipped.
7	57·6	819	Steel slipped.

*† Popplewell on "Reinforced Columns." Discussion, Mr. Popplewell and Mr. de Vesian, Min. "Proceedings," Inst. C.E., vol. clxxxviii.

202. *Machine for Testing Reinforced-Concrete Beams (60 ton, 14 ft. span)—In this the load is applied upwards by oil pressure on a well-fitting ram without packing. The pressure is shown on a gauge, placed near the cylinder to avoid lag in the pressure. The cylinder as well as the end supports is attached to a couple of heavy steel girders. The readings on the deflection meter are taken on a scale divided into twentieths of an in., and these can be divided according to the leverage of the pointer. This machine by Messrs. Amsler-Laffon has been used by the authors for some years, and has been found both useful and convenient.

For reinforced-concrete beams it is used as shown in Fig. 236 with the load applied at two points, between which the bending moment is a uniform one.

203. Artificial Paving Flags—In recent years a valuable product has come into general use in the shape of the artificial flag used to replace the natural flag used for footpaths and pavements generally. The great majority of these are made at stone quarries from the small pieces of the stone obtained after the breaking of the stone and during the process of screening. Such pieces have a size varying from one that would pass a $\frac{1}{2}$ in. mesh screen to one that will pass through a $\frac{3}{4}$ in. screen. These pieces are embedded in a matrix of Portland cement mixed with fine dust from the breaking of the stone, or with sand, or both. The material is really a Portland cement concrete, with a low percentage of broken stone and having a very uniform composition.

This mixture is placed in moulds, into which it is in most cases held under pressure. When set and removed from the moulds these flags possess a smooth or slightly ribbed surface at the top which is to be exposed to wear, and a rougher surface below. The shape is rectangular, generally 2 ft. wide, and 2 ft., $2\frac{1}{2}$ ft., or 3 ft. long; of course, special sizes may be used when occasion arises, but those mentioned above represent the more usual sizes turned out at the quarries. The thickness aimed at is $2\frac{1}{2}$ in., but it is found to be difficult to get them exactly to the intended size, and there is always a small discrepancy between the intended and the actual thickness. The precise method of manufacture varies according to the quarry, but in most cases the result is a hard, strong material with a fracture showing uniformity.

To ensure satisfactory delivery of artificial flags of any particular make, a certain number of the flags are taken from each batch and tested. For this purpose two main tests are employed, and sometimes a third is added. The two first are for strength to resist cross-breaking, by which many of these flags may be expected to fail; and the power to absorb water, which should

* From Min. "Proceedings," Inst. C.E., vol. clxxvii.

give a definite idea as to the compactness or hardness of the material. As a rule the strongest flags and those most likely to resist wear have a low absorption. The third test is for wear or abrasion.

TEST FOR TRANSVERSE STRENGTH—This may be carried out conveniently in some such machine as the one shown on Fig. 236. This consists of a cast-iron table resting on the top of a ram which can be forcibly raised by means of oil pressure beneath. The flag rests on a pair of rollers placed the standard distance apart, generally $1\frac{1}{2}$ ft. or 2 ft. for the larger flags. As the table rises and with it the flag, its middle line is pressed upon by a third roller fixed to a rigid frame which passes across the machine above the table.

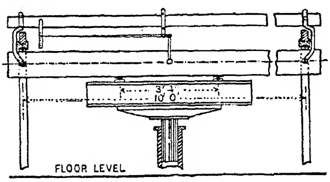


FIG. 236.

As friction is eliminated in the ram the load on the middle of the flag is proportional to the pressure of the oil in the cylinder. The authors have always used strips of thick felt to distribute the pressure evenly between rollers and flag. When this has been comfortably adjusted until there is no "rock" in the flag, the load is gradually increased until the flag fails by breaking across, careful note being made of this load as it is reached. The whole scheme is shown on Fig. 237. The two halves are removed from the machine, and the thickness measured, to the nearest 0·01 in., at five points along the fracture, and an average taken. The standard span which has been adopted by the Corporation of Manchester is 2 ft., and if the actual span is greater or less than this, allowance is made in the Report, and the breaking load which would have been needed to break the flag if it had been of the standard span is calculated. The same applies to the thickness.

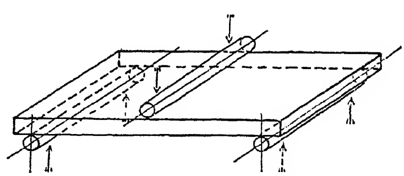


FIG. 237. Scheme for Test of Artificial Flag.

An example of the usual calculation is given below:

This flag was 24 in. wide, 24 in. long, and tested on a span of 18 in.

The central breaking load was observed to be 5,965 lb.

It is desired to find the load which would have been required if the testing span had been 2 ft. or 24 in., and the thickness

$2\frac{1}{2}$ in. instead of the actual thickness of 2·54 in. This is called the "equivalent breaking load," and is equal to—

$$(\text{The actual breaking load}) \times \frac{1\cdot5}{2\cdot0} \times \frac{(2\cdot50)^2}{(2\cdot54)^2}, \text{ or}$$

$$= 5{,}965 \times \frac{1\cdot5}{2\cdot0} \times \frac{(2\cdot5)^2}{(2\cdot54)^2} = 4{,}335 \text{ lb.}$$

In carrying out this test care is necessary in setting the flag so that the pressure through the rollers shall be quite even; also that the load be slowly and steadily increased until fracture occurs quite suddenly, and it will be needful to watch carefully the pressure gauge—if such is employed—so as not to miss the maximum point; or to have a maximum indicator to show this point.

If a lever machine is used for the test there need be little trouble in fixing the maximum load point.

Absorption Test—This should be carried out previously to the transverse test. It consists in first weighing the flag, then immersing it in water for a specified time, most generally 24 hours, and again weighing it after the soaking. Before the second weighing all superfluous water is to be removed. Absorption $= \frac{\text{Final weight} - \text{Initial weight}}{\text{Initial weight}} \times 100$. This absorption varies enormously in flags of different qualities, having values between 0 and 20 per cent., but being rarely more than 0·9 per cent. in flags of the best qualities.

The following are a few typical results:

Equivalent Breaking Load on a Flag, 24 in. wide, 18 in. span, $2\frac{1}{2}$ in. thick.	Make.	Absorption after 24 hours' Immersion, per cent.
3,664	Grit aggregate.	—
5,130	Welsh granite.	—
2,440	(Porous.)	1·78
3,120	,,	1·0
2,310	Clinker body—Cement face.	—
3,230	,, ,, 1 in. ,, $1\frac{1}{2}$ in.	—
4,450	,, ,, ,, ,, $1\frac{1}{2}$ in.	—
3,290	Shap, Pink	0·00
3,500	,, Blue, hand made	0·75
4,640	Hard flag.	0·90
4,330	Pressed flag.	0·25

Tests for Abrasion—In order to compare the power of flags to resist wear when in daily use the following experiment has

been used :—A block of cast iron 12 in. × 9 in., weighing 200 lb., was placed on the flag and reciprocated continuously for 8 hours, by a crank and connecting rod, at an average speed of 42 ft. per minute. It was found that for one flag the amount of dust rubbed off in this time was 1/200lb. Before the test the rubbing surface of the iron was rough-planed in a direction at right-angles to that of movement. As the flag weighed 45 lb. the above = 0·01 per cent. For comparison, other flags were tried, with different results. Bauschinger made abrasion tests on artificial paving bricks ; his plan was to hold the bricks, with a certain pressure, against the face of a rotating table, and provide an unlimited supply of emery powder between the brick and the table. Abrasion was measured as the weight lost by the brick after a certain number of revolutions.

This is a more definite and scientific way of proceeding than the one described above as used by the authors, but the former was found to work satisfactorily for simple comparative determinations, and Bauschinger's method precluded the use of whole or half flags, and necessitated the cutting of small pieces 4 in. square. Such a specimen was weighted with about 4 lb. per sq. in. The emery was fed in at the rate of 20 gm. for every 10 revolutions. The speed was 20 revolutions per minute, and 200 revolutions completed an experiment. The following are a few typical results :

* Material Tested.	Average compressive Strength, lb. per sq. in.		Loss of Volume in cu. in. caused by Abrasion.
Granite	22,400	dry	0·24
		wet	0·46
Basalt	34,200	dry	0·19
		wet	0·47
Clay-slate	26,000	dry	0·16
		wet	0·35
Limestone	20,500	dry	1·10
		wet	1·41
Sandstone	17,600	dry	0·81
		wet	0·64
Brick and tile	—	dry	0·38
		wet	0·75
Artificial flag made with Portland cement	—	dry	0·51
		wet	1·82
Asphalt paving	—	dry	0·61
		wet	1·62

* Johnson, "Strength of Materials."

204. Brickwork—The question of the strength of brick masonry, as apart from the strength of the bricks composing it, is important to engineers. The engineer has to deal with brickwork in the lump, and the strength of this is dependent on influences other than the strengths of the individual bricks.

THE BRICKS—Originally devised, in all probability, as a cheap substitute for blocks of stone, which required to be quarried and cut to shape before they could be used, bricks made from soft natural earth, moulded to shape and baked hard, have been used for building and engineering work from very early times.

The advantages which artificial bricks possess over natural stone blocks are facility in production, cheapness, simplicity and quickness in use, and uniformity. Against these advantages it may be said that the average brick is not so strong for load-carrying purposes as is the average building stone; but this objection is easily overborne by the above favourable qualities.

There are three main classes of brick which may be employed by engineers, namely, bricks of the "blue Staffordshire" type, the red engineering brick of the "Accrington" class, and the common stock brick made in all districts where clay is obtainable. In addition to these are fire-bricks, also used by engineers, but for purposes other than weight-carrying. The qualities desired in a brick are, in greater or less degree, strength to carry the loads imposed upon it; durability to resist the effect of the atmosphere in which it is placed; power to withstand frost; resistance to abrasive action; bed surfaces to be of such a kind that with the help of the cementing material slipping is made impossible; as well as some minor properties which are applicable only in special cases. The possession of any or all of these depends on:

(1) The natural materials used in making the brick.
(2) The method of manufacture.
(3) The extent of the burning.

MATERIALS USED IN BRICKMAKING—The natural earths used in brickmaking are of three principal kinds (in Great Britain), namely:

Pure clays. These are made up almost wholly of silica and alumina, to which may be added smaller quantities of iron oxide, lime, magnesia, and other salts.

Sandy clays, called "loams."

Limey clays or "marls."

Pure clay is useless as a material to form the paste which is afterwards burnt to make bricks, and, to prevent cracking when drying and being burnt, it is necessary to add some portion of a lighter earth or sand. The looseness of loam is often remedied by the addition of lime, which on burning forms a flux. A good

brick earth should contain its ingredients in approximately the following proportions:

One-fifth, alumina; three-fifths, silica; one-fifth made up of soda, potash, lime, iron, magnesia, and manganese.

It is these last minor ingredients which affect the colour of the resulting brick, when taken in conjunction with the temperature of burning. A brick made from clay containing iron but no lime or other similar substances, and burnt at a bright red heat, comes out red, the precise tint depending on the quantity of iron. This red colour is reduced to a dark blue when burnt at a higher and vitrifying temperature; this is what happens in the case of the "Blue Staffordshire" bricks, which are made from a clay containing a high proportion of iron oxide and are burnt at an extremely high temperature which renders some of the material fusible. These blue bricks may be vitrified right through or on the surface only. The extent to which vitrification is allowed to proceed may have very important effects on the value of bricks which are expected to resist the influences of long-continued damp and great variations of temperature, or of long-continued vibration. Other colours, such as yellows, buffs, and pale reds, result from the use of various marls and malms, but the use of such is of little importance to the engineer, who is chiefly concerned with questions of strength and durability.

On the metallurgical and steam-raising side of engineering, where bricks are needed to resist extremely high furnace temperatures without damage, the clay used is one almost free from lime, magnesia, and similar salts, but high in silica. The fireclay from which these bricks are made is found immediately beneath the coal in the coal measures. Its composition varies with locality, the best fire-bricks consisting almost wholly of silica.

BRICKMAKING—The methods used in the manufacture of bricks differ in details according to locality, but the general schemes are alike in all districts. In the old method the material, whether pure clay, malm or marl, is first dug up and spread out for exposure to the weather, to be afterwards mixed with water and ground in a mortar mill to the condition of a thick paste or mud. This method, though still used, has been largely replaced by other and quicker processes. In this old "soft paste" plan the clay, intimately mixed with water, is allowed to settle by being spread out flat and so gets rid of much of its water; after this it is pressed into open rectangular moulds by hand; the clay bricks are then removed from the moulds and stacked up to dry.

For "wire-cut" bricks made by machine, the paste is allowed to flow from the grinding mill, and is forced through a rectangular opening on to a table, in the form of a slab of rectangular section

and indefinite length. As this moves along the table it passes across a number of tightly stretched vertical wires, so placed as to cut the main slab into several narrower slabs, each having the width of a brick. At regular intervals the flow is stopped and a cross-movement of the wires cuts off one-brick lengths. The result is a number of bricks of stiff clay paste; these are only approximately true in form, their shape depending on the setting of the wires and on the movement of the slab. These are stacked up until sufficiently dry and ready for burning.

Nowadays all the better bricks, especially those used for facing buildings and for engineering work, are made from a much drier paste or from clay powder only slightly moistened. They are formed by being pressed into moulds in some form of brick-making machine. A ground shale, moulded under great pressure, is used for some of the best engineering bricks.

The two kinds just mentioned are called respectively "pressed bricks" and "shale bricks."

The "burning" of the bricks, whether made by hand or machine, is sometimes carried out in "clamps," built of dried bricks into walls and spaces, until they attain a roughly rectangular form and finally are completely surrounded by a layer of old bricks. Fuel is placed within this and combustion started and allowed to proceed for a week or more, until all the bricks are burnt through. A later plan, by which all the best bricks are now burnt, is to perform this operation in a permanent kiln kept for the purpose in the brickyard. One of the best known of these is of a hollow arched section, with its plan curved round to form a hollow circle. There are several openings leading to the inside. The burning is carried out in sections, the kiln being divided into compartments, one to each opening. Thus when a compartment is ready for burning, the fuel, in the form of small coal, is placed on the top of the bricks and the gases of combustion proceed downwards, to be finally led away to the chimney. While this is going on the next compartment, already charged, may be absorbing waste heat and getting thoroughly dry and ready for burning. The compartment beyond this is meanwhile being charged. Of the two compartments on the other side of the burning compartment, the first may be cooling down and the second being emptied. Thus in a kiln of this type work goes on continuously and much economy is thereby effected. The resulting bricks are uniformly burnt. The product resulting from the above processes of brickmaking, in order to be a "good" brick, should have sharp edges and angles as well as plane and parallel faces; it should be hard and yield a clear ringing sound when struck; it should absorb less than 10 per cent. of its weight of water; when fractured its material should appear as of uniform,

compact, and fine texture; its specific gravity should be not less than 2; when subjected to a crushing load in a testing machine it should fail suddenly by shearing and not by a process of gradual crumbling. The qualities specified above all appear in a more marked degree in machine-made or pressed than in hand-made or wire-cut bricks.

205. Tests for Brickwork—When used for purely architectural purposes the strength of brickwork to resist load is of small importance, when compared with its appearance and its suitability for withstanding the disintegrating influences of the atmosphere; but when employed in a strictly engineering sense, the safe stress which a given sample of brickwork may be expected to withstand is the essential quality to be considered. The requisite information as to the actual strength of brick masonry has been very meagre in past years, but in several directions the whole question is now in a more healthy condition and promises to improve further. The total number of strength tests carried out and published is certainly small, but the carrying out of such experiments has been costly and dependent on private effort. At the same time a small number of tests have been made, with results which ought to serve as general guides to practising engineers. A second point in favour of the designer is the increasing uniformity of material and the general improvement in quality, especially of cementing mortars.

Brickwork or brick masonry is a composite material whose strength depends upon many things, including the strengths of the individual bricks, the strength and quality of the mortar used, the manner in which the building has been carried out, and its

The Bricks—The following data refer to—(1) The actual crushing strength of a number of bricks of well-known makes; (2) tests of the strengths of several typical mortars; (3) crushing strength of brick masonry composed of the bricks and mortar so tested; (4) experiments as to the elastic properties of the above.

In the United States transverse tests are applied to bricks and brickwork as well as crushing tests, but in Great Britain this is not usual, beyond breaking sample bricks across for the purpose of examining the fracture and the material.

In the crushing tests of individual bricks by the writers, given on p. 469, each specimen was placed in the testing machine with a layer of plaster of Paris between it and the platens of the machine. This manner of distributing the load uniformly is replaced in a few cases of the results quoted from other authorities by layers of cardboard, thin wood, or lead; the writers have always used plaster of Paris or Portland cement.

Where bricks are provided with "frogs" or panels, it is the practice to fill these up level with the main surface of the brick with plaster or cement, if possible several days before testing.

In the tests below, each brick was tested whole, and for this purpose was set on its normal bed.

TABLE I.—CRUSHING STRENGTH OF BRICKS

Description. Bricks.	Dimensions, inches.	First Crack. Tons per sq. ft.	First Crack. Lb. per sq. in.	Crushing. Tons per sq. ft.	Crushing. Lb. per sq. in.	Authority.	Remarks.	
Yellow London stock . . .	9 × 4·2	—	—	115	1,790	Unwin		
Aylesford common (pink) .	8·9 × 4·4	79	1,230	205	3,180	,,	Aver. of 2	
Aylesford pressed (red) . . .	9·1 × 4·3	71	1,110	141	2,180	,,		
Rugby common (red) . . .	9·5 × 4·2	158	2,450	190	2,960	,,	Between boards	
Leicester wire-cut (pale red) . .	9·1 × 4·2	115	1,790	229	3,560	,,		
Gault wire-cut (white) . .	8·7 × 4·1	111	1,720	173	2,680	,,		
Manchester wire-cut (light red).	8·8 × 4·3	87	1,355	264	4,100	Popplewell		B
"Engineering" (red) . . .	9·0 × 4·3	110	1,710	290	4,500	,,		
Manchester common (red) . .	9·0 × 4·3	74	1,150	120	1,970	,,	Panel	
Ditto	8·8 × 4·2	84	1,310	212	3,300	,,	Plain	
Common red engineering .	9·0 × 4·5	160	2,490	280	4,250	,,	Aver. of 6	
Shawforth plain	— —	227	3,530	353	5,500	,,	,, ,,	
,, panel	— —	152	2,360	327	5,085	,,	,, ,,	
Blue Staffordshire	— —	240	3,730	750	11,660	,,	,, ,,	
Colliery shale bricks (red) .	9·0 × 4·1	67	1,040	220	3,420	,,	Mean of 6	
Shale bricks, Enfield red . .	8·8 × 4·2	205	3,190	496	7,710	,,	,, ,,	
Accrington (Huncoat Plastic) (red) . . .	$8\frac{3}{4} \times 4\frac{3}{8}$	118	1,840	250	3,890	,,	Mean of 3	C
Ruabon pressed	8·8 × 4·3	361	5,610	(361)	(5,610)	Unwin*	Not crushed	
Digby Colliery, Notts . . .	9·3 × 4·1	248	3,860	(353)	(5,500)	,,	Not crushed	
Candy pressed .	8·8 × 4·3	80	1,245	381	5,930	,,		
Common Blue Stafford . .	8·9 × 4·3	240	3,740	(353)	(5,500)	,,	Not crushed	
Pressed Blue Stafford . .	9·0 × 4·3	—	—	275	4,280	,,		
Blue Stafford .	9·0 × 4·5	82	1,280	356	5,550	Popplewell		D
Blue Brindle, Staffords . .	8·9 × 4·3	204	3,180	485	7,520	,,	Mean of 9	E

STRENGTH OF MORTAR—As the compressive strength of the mortar has an important bearing on the strength of the brickwork of which it forms a part, it will be well here to examine a few results of tests upon mortar.

In several of the results of brickwork tests given in Table III,

* Unwin's "Testing of Materials."

the strength of the mortar used was determined separately with the following results :

TABLE II.—MORTARS

Description.	Dimensions of Specimens.	First Crack. Tons per sq. ft.	First Crack. Lb. per sq. in.	Crushing. Tons per sq. ft.	Crushing. Lb. per sq. in.	
Black mortar (lime, sand, and ground destructor clinker) (28 weeks old) . .	4 in. × 4 in. × 4 in.	31·8	495	31·8	495	E
Portland cement mortar (3 sand–1 cement) (24 weeks old)	4 in. × 4 in. × 4 in.	10	155	10	155	F
Portland cement mortar (5 samples).		Crushed. (7 weeks old.)		Crushed. (20 weeks old.)		
1–1	4 in. × 4 in. × 4 in.	79	1,229	98	1,524	A
2–1	,, ,, ,,	32	498	51	771	
3–1	,, ,, ,,	27	420	28	435	
4–1	,, ,, ,,	20	311	24	373	
5–1	,, ,, ,,	16	249	18	280	

As a general rule cement mortar increases in strength as the ratio of cement increases, and also increases with age, if the cement

TABLE III.—BRICKWORK

Kind of Bricks.	Kind of Mortar.	Age, weeks.	First Crack. Tons per sq. ft.	First Crack. Lb. per sq. in.	Crushed. Tons per sq. ft.	Crushed. Lb. per sq. in.	
Accrington. (388 tons per sq. ft. crush)	sand/cement 1–1	6	97	1,509	125	1,944	A
	2–1	6	91	1,415	159	2,473	
	3–1	6	79	1,229	125	1,944	
	4–1	6	90	1,400	116	1,804	
	5–1	6	69	1,073	125	1,944	
These figures refer to the 5 cement mortars already given.							
Common wire-cut .	Lime and sand	4	35	544	53	824	B
,, ,, ,, .	Port. cement (3–1)	39	64	995	84	1,306	
Accrington . . .	Black clinker and lime	4	78	1,213	96	1,493	C
Blue Staffords . .	Port. cement (3–1)	39	69	1,073	98	1,524	D
Brindled Staffs. . .	Black clinker and lime	12	120	1,866	165	2,567	E
,, ,, . .	,, ,,	25½	110	1,711	185	2,855	
,, ,, . .	Port. cement (3–1)	12	132	2,053	169	2,629	F
,, ,, . .	,, ,,	26	90	1,400	161	2,504	
,, ,, . .	Lias lime	12	38	591	119	1,740	
,, ,, . .	,, ,,	26	41	638	102	1,588	

is good and the sand satisfactory. In the figures just given in Table II the 3–1 mortar in the set of five is more than twice as strong as the sample in the previous case, although only 7 weeks old. This is probably caused by the differences in the sand used in the two cases, because the cement was good (475 lb. per sq. in. at 7 days).

Table III gives results of crushing tests made on columns of brickwork.

The above-mentioned brickwork piers were all 1½ ft. square in section (two bricks) and 4 ft. high; for testing they were set between the platens of a 900-ton compressive testing machine in neat Portland cement. Another result obtained by the writers was:

Common wire-cut (2 bricks square: 16 courses high) .	Port. cement (1–1)	3 days	176	2,738	189	2,940	
Best common brick (3 sq. ft. area of section: 2⅝ in. high)	(2 clinker 1 sand, 1 lias lime)	3 weeks	24	373	80	1,244	

This last is a sample of ordinary everyday brickwork such as is put into wall and similar work.

Baker, in his book on "Masonry Construction," quotes results of tests made on brick masonry at the Watertown Arsenal Laboratory. The piers were 12 in. square and varied in height from 1 ft. 4 in. to 10 ft. Various mortars were used, with the following results:

Mortar used.	Experiments	Crushing Strength.	
		Lbs. per sq. in.	Tons per sq. ft.
1 lime, 3 sand	15	1,508	97
1 lime, 3 sand, 1 Rosendale cement	1	1,646	106
1 lime, 3 sand, 1 Port. cement .	1	1,411	91
1 Rosendale cement, 2 sand . .	1	1,972	127
1 Portland cement, 2 sand . . .	8	2,544	163
,, Portland cement	1	2,375	153

The bricks used had an average strength of nearly 15,000 lb. per sq. in., or 964 tons per sq. ft.

The above piers were from 1½ to 2 years old when tested. In those piers which were tested at different heights it was found, of course, that the shorter piers were stronger than the high ones. The experiments showed in a general way that the strength of

a pier laid in either lime or cement mortar and 10 ft. high is two-thirds the strength of a similar pier 1 ft. high.

An important point to be noted in all the above experiments is the ratio which the strength of the brickwork bears to the strength of the bricks. It will be seen that the bricks used are denoted by certain letters of the alphabet and the same are applied to the corresponding mortars and piers; this will help identification.

RATIOS OF STRENGTH OF BRICKWORK TO STRENGTH OF BRICKS AND OF MORTARS USED

Reference Letter.	$\frac{\text{Strength of Brickwork}}{\text{Strength of Brick.}}$	$\frac{\text{Strength of Brickwork.}}{\text{Strength of Mortar.}}$
(A)	0·322 0·410 0·322 0·300 0·322	1·27 3·12 4·46 4·00 6·95
(B) Lime mortar. . . . P.C. mortar	0·200 0·317	— —
(C)	0·332	3·00
(D)	0·276	9·8
(E) Black mortar . . .	0·380	5·800
(F) (3–1) P.C. mortar . .	0·332	16·1

Diagrams were plotted showing curves of compressive stress and amount of shortening suffered by the several columns tested. These shortenings were measured by a Martens' mirror extensometer (already shown applied to the brickwork) reading to $\frac{1}{50,000}$th in.

An inspection of these curves makes it clear that up to a certain stress (P.L) the diagram takes the form of a straight line, thus pointing to an elastic condition of the material. It was found that within this limit removal of the load meant a very

small amount of permanent set; after three or four repetitions and removals of stress the set disappeared almost entirely.

Thus brickwork, with any pretension to being called "good," when sustaining working stresses which are far below the (P.L.) stresses of the diagrams, may be regarded as an elastic material. From the measurements taken during the experiments the following values have been calculated for the modulus of elasticity of brickwork, bricks, and mortar, as well as the observed values of the elastic limit stresses.

Reference Letter.	Approx. Elastic Limit (P.L.). Tons per sq. ft.	Modulus of Elasticity.	
		Tons per sq. ft.	Lb. per sq. in.
(B) 3–1 P.C. mortar . .	25	90,000	1,400,000
(C)	24	(51,400)	(800,000)
(D)	55	121,000	1,880,000
(E) Black mortar . . .	40	149,800	2,330,000
(F) (3–1) P.C. mortar . .	30	122,000	1,895,000

The modulus of elasticity of the bricks was: Common wire-cut, 1,760,000; Accrington (Huncoat Plastic), 5,920,000; Blue Staff., 5,280,000; Blue Brindle, 2,830,000, all lb. per sq. in. Also the modulus for the black lime mortar was 352,000; and for P.C. mortar (3–1), 131,500.

From the above data several general laws appear to be evident. These are:

(1) The strength of brick masonry is largely dependent on the quality of the mortar used in its construction. A first-class brick with a poor mortar will not give so good a brickwork as a medium-quality brick with a good mortar.

(2) Where Portland cement is used for the mortar, its strength, and therefore the strength of the brickwork, increases as the ratio of cement to sand increases.

(3) It is noticed that as the ultimate crushing strength increases so also does the elastic strength and the modulus of elasticity; thus showing that these qualities have some bearing on the power of the brickwork to sustain loads. It is important that where a mass of brick masonry is to be built for the purpose of carrying a considerable amount of load, the kind of brick used should be selected, not because it possesses excessive strength, but for the reason that, while distinctly of good quality, it has the further important advantages of surface durability and a very low absorption figure for moisture, and therefore greater immunity from the effects of frost.

This quality of durability is often of more importance than initial strength, and care is required in making a proper selection. Some bricks with a hard clinker surface are often porous beneath the outer crust and fail to withstand exposure in damp positions. At the same time a brick which is primarily good will retain much of its strength even after many years of exposure in such positions as culverts or drainage works generally.

Absorption—The test for this is carried out in the same way as that described for artificial flags, namely, by immersing the brick in water for a specified length of time, say, 24 hours, and weighing it before and after immersion. The gain in weight divided by the initial weight gives the absorption fraction, generally expressed as a percentage. In the following three examples of absorption results the above method was used:

Brick.	First Crack.		Crushing.		Absorption, per cent.
	Tons per sq. ft.	Lb. per sq. in.	Tons per sq. ft.	Lb. per sq. in.	
Blue Staffs. (panel) .	85	1,324	270	4,200	0·75
Brindled Staffs . .	175	2,720	250	3,880	3·70
Wire-cut Staffs. . .	240	3,730	430	6,690	0·40

Old Bricks. Taken out after many years' service in the brickwork of a sewage culvert. Unfortunately there was no record of the original quality of these, so that no definite comparison can be made.

	105	1,630	140	2,180	—

206. Safe Compressive Stress on Brickwork—It is possible that the values generally selected for this purpose err on the conservative side and that stresses higher than those generally adopted might very well be used, especially for the better grades used in engineering work.

From the preceding Tables the ultimate strengths of three grades of brickwork are estimated as follows:

Brickwork.	Crushing Strength.	
	Tons per sq. ft.	Lb. per sq. in.
Brindled Staffordshire, in best black clinker or P.C. mortar (3–1) (6 months old)	160	2,488
Best Accrington Plastic, in P.C. mortar (6 weeks old)	125	1,944
Common wire-cut, in clinker and lime mortar (3 weeks old) . . .	80	1,244

Unwin recommends a factor of safety of 20 for brickwork subjected to steady loads ; this gives for the above samples safe stresses of 8, $6\frac{1}{4}$, and 4 tons on the sq. ft.

Using a somewhat lower factor, 16, the corresponding working stresses are 10, 7·8, and 5. These last practically coincide with the leading United States' practice of 10, 8, and 5 tons for, respectively, best brickwork in (2–1) P.C. mortar, good brickwork in (2–1) P.C. mortar, and for ordinary brickwork in lime mortar.

Baker thinks these too low and suggests 30 and 20 tons as possible for respectively "good bricks laid in good P.C. mortar" and "reasonably good bricks in lime mortar."

The majority of engineers will agree that these last figures appear high and mean a very low factor of safety.

The factor of safety 8 gives safe stresses of 20, 15·6, and 10, and these would appear to be well below one-half the elastic limit stresses quoted above. The values employed and suggested are summarized thus :

Safe Stresses on Brickwork

Quality.	I.	II.	III.
Crushing of typical samples	160	125	80
Approximate elastic limits	55	40	25
Unwin (factor of safety, 20)	8	$6\frac{1}{4}$	4
Factor of safety = 16	10	7·8	5
Good American practice	10	8	5
Suggested increase to (Factor of safety, 8)	20	15·6	10
Baker's proposed values	30	—	20

The above values, suggested by the writers, of 20, 15·6, and 10 tons per sq. ft. may appear at first sight extreme, but a little

consideration will show that they are not so unreasonable as may at first appear to be the case.

The brickwork to which they refer is all supposed to be good of its kind, of the sort usually put into engineering construction, built under careful inspection, of sound and well-selected materials.

The three grades specified above may be said to refer generally to:

I. The best possible bricks, laid in P.C. mortar (2–1 or 1–1).

II. Good sound engineering bricks laid in P.C. mortar (3–1 or 2–1).

III. Common bricks in P.C. (3–1) lime mortar.

In support of these proposed values it may be pointed out that the pressure on the base of a tall chimney in Glasgow is normally 9 tons per sq. ft., but gets as high as 15 tons per sq. ft. on the leeward side in gales.

A few years ago Mr. Popplewell caused to be built an experimental brickwork pier whose cross-section was a square of two bricks or $1\frac{1}{2}$ ft. side. The bricks were laid in English bond, and the pier was 15 courses high (Plate XXVII (*b*)). The bricks were the best Accrington Plastic with shallow frogs; on breaking with a trowel fracture appeared to show a bright-red, uniform, sound material. These bricks were set in a (1–1) mortar of good Portland cement. Ample time was allowed for hardening of the mortar before testing was attempted, and both top and bottom surfaces were laid on the machine platens in neat cement.

In the test the compressive load was steadily increased until a maximum of 545 tons was reached. It was not convenient at the time to increase the load beyond this point on account of pipe-joint trouble. Up to this load there was no sign of fatigue, either by creaking or by visible crack. Mr. Popplewell at this point released the load and removed the pier from the machine.

Before being taken out of the machine, this pier was carrying, without sign of distress, a load of 545 tons, equivalent to $545/2\frac{1}{4} = 242{\cdot}2$ tons on the sq. ft. It is probable that a little more would have been sufficient to destroy the pier. The bricks were good, but not exceptionally so; the mortar was what one would expect in first-class engineering work, and the pier was built by a thoroughly competent bricksetter. If this result is compared with the higher of the two safe stresses suggested above, it is seen that this load actually carried with safety bears to the suggested stress a ratio of $242 : 20 = 12{\cdot}1 : 1$. The load of 545 tons was not the crushing load, but the load actually carried with safety; if it had been allowed to remain on the pier for any length of time it is possible that gradual collapse might have supervened. The stress attained in the above case may appear somewhat

remarkable, but the result serves to show what may be done with brickwork carefully built with materials in no way specially good, but of a sound average quality. It further tends to strengthen the writers' contention that a working stress of from 15 to 20 tons per sq. ft. ought to be possible on good engineering brickwork.

207. Strength of Stone—Various kinds of stone are used by the engineer for building purposes and for the construction of road surfaces. In addition to these uses stones of various kinds are employed in the manufacture of cement, and in the formation of artificial paving flags.

The usefulness of a given kind of stone for purposes connected with building, whether for enclosing walls, for the various parts of stone bridges, abutments and foundations for metal bridges, the walls of quays, or structures in the nature of pillars for carrying loads, depends on its strength, durability, and power of resisting the absorption of water. Its appearance is a secondary consideration, although many of the complete engineering structures of stone in all parts of the world are undoubtedly pleasant to look upon, both by reason of the quality of the stone used and of the beauty of the design.

Constituents of Stone—The principal chemical elements which enter into the formation of the stone used for engineering purposes are the following:

Aluminium in the form of its oxide, alumina, in clay, felspar, and mica.

Silicon, also as an oxide, silica, which occurs pure, in quartz and in flint, and also mixed with most earthy bases.

Calcium, as the oxide lime (slaked and unslaked) or as a carbonate as limestone.

Iron occurs as an oxide in many of the rock constituents.

Magnesium, as in the oxide in magnesia stones, like hornblende and soapstone, and as a carbonate, as magnesian limestone.

Classification—The different kinds of stone used in engineering work may be classified as follows:

Granite—This is at the same time the strongest and the most durable of stones, and also the hardest and most capable of taking a polish.

This stone is of volcanic origin, is crystalline in structure, and consists of varying proportions of quartz, felspar, and mica. The second of these—grey, red, or yellow—imparts its prevailing tint to the granite. Sometimes hornblende and talc replace the mica.

The disadvantages of granite as a stone for building purposes are its high cost, the greater difficulty in working it when com-

pared with the other stones, and the fact that it resists fire badly. The poorer kinds often fail through the disintegration caused by frost or by decay of the felspar.

Basalt, Trap, and Greenstone—These are not unlike granite in composition and qualities. They consist chiefly of hornblende, felspar, and augite; they are very strong and capable of resisting wear, but are not easy to cut and dress. Well-known examples are the stones generally spoken of as "Welsh granite," with which many of our streets are paved.

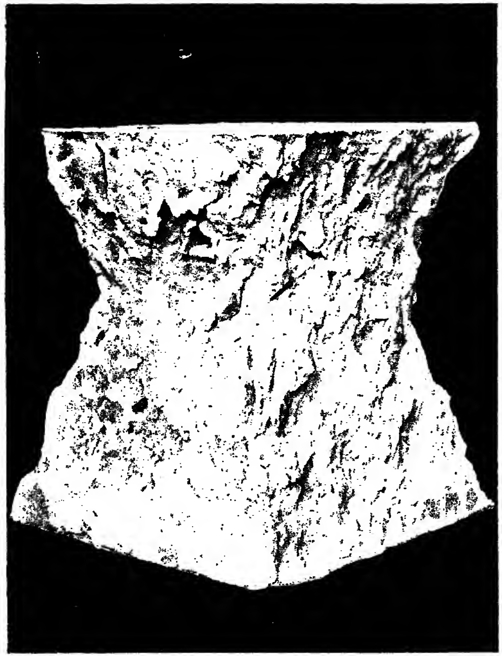
Clayslate—Mainly composed of quartz and mica, this forms a fine-grained, compact building stone, and when it can be split along planes of cleavage normal to the natural bed, yields flags and slates.

Sandstones—Of stratified formation, with grains of silica held together with a cement material, of a siliceous, argillaceous, or limey nature. Where the cement is nearly pure silica, the stone is hard and durable.

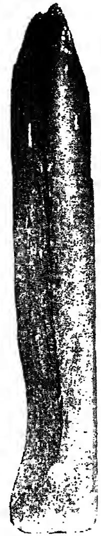
Limestones—The best of these is marble, which is a nearly pure carbonate of calcium. But the majority of limestones in use for building purposes are formed of grains of carbonate of lime held together in a siliceous or calcareous matrix. They go to form very useful building stones, being fairly strong and easy to work; their relative softness, however, militates against their durability, and with many varieties the surface is found to crumble away gradually, especially in towns. On Plate XXVIII (*c*) is shown the result of a test on a cube of Portland stone.

The above represent briefly the kinds of stone in general use by engineers. Their origin is in most cases known and the time elapsed from their formation varies widely in different cases. Some rocks, as granite, are of plutonic origin, having been first rendered fluid and afterwards allowed to solidify. These volcanic rocks are markedly crystalline in the granites, but less so in basalt and trap. The sandstones are of stratified origin and have been formed by the settling under water of particles removed by denudation from either volcanic rocks or old stratified rock, which has itself been formed in the same manner at a previous age. The grains of the stratified rocks are held together in a matrix of cementing material; the grains are often pure silica, especially in the strongest and most durable stones. Sometimes the stratification is very apparent. In other cases the stone has the appearance of a perfectly isotropic material and the planes of stratification are not clear. In the case of some rocks there is a second direction of cleavage at right-angles to the direction of stratification, such secondary cleavage having been induced by the pressure of the overlying rocks. This occurs in slate rock. The limestones are formed by precipitation from solution in

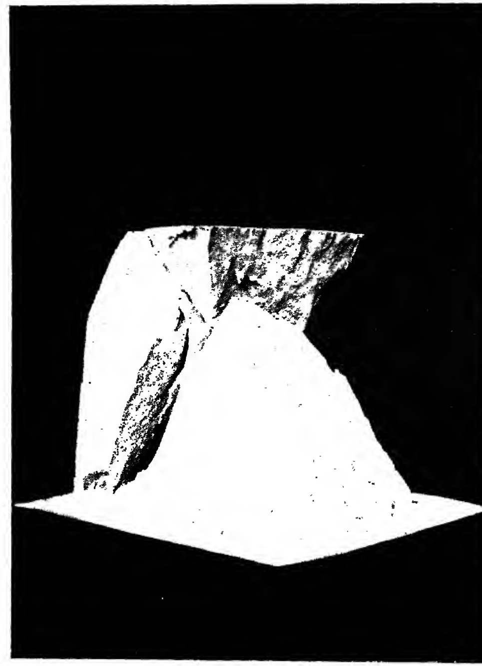
PLATE XXVIII.



(*a*) FRACTURE OF CONCRETE CUBE



b) RESULTS OF COMPRESSION OF SANDSTONE PILLAR



(*c*) RESULT OF COMPRESSION TEST ON PORTLAND STONE

water or by stratification from minute skeletons or shells, held together by a limey or siliceous matrix.

Qualities of Good Stone—Stone used for engineering purposes must possess sufficient strength to withstand the forces to which it may be subjected. It must also be capable of withstanding the deleterious effects of minute quantities of certain gases in the atmosphere; it must not be capable of absorbing more than a very small portion of water, which may be expected to cause disintegration after freezing; and it should be easy and cheap to work.

Strength of Stone—In engineering work it may be said that all the stone used, with a few exceptions, is subjected to pressure. In the best kind of masonry care is taken that this pressure is distributed uniformly on all the beds carrying load. In order to find the amount of load necessary to crush a given sample of stone, a specimen is tested in a compressive testing machine, care being taken that the specimen is bedded approximately as it would be in practice. The manner of carrying out such a test and the necessary precautions have already been discussed (p. 453). It may, however, be repeated here that of the different ways of bedding the specimen in the machine, the best is to have the bed surfaces plane and parallel and to insert a thin layer of plaster of Paris or a sheet of thin millboard. A layer of wood is too soft and generally useless. Lead, which has been used for this purpose, spreads under the load and helps to tear the material asunder.

Unwin made some experiments to verify this last point, as shown in the following results:

*Experiments on the Bedding of Stone Compression Specimens (4 in. Cubes)

Description.	Crushing. Tons per sq. ft.	Remarks.
Portland .	516·4	Between two millboards on each face.
,, .	469·8	One plate of lead on each face.
,, .	408·8	One plate of lead on each face, ⅛ in. smaller than face all round.
,, .	300·0	Three plates of lead on each face.
Yorkshire Grit	712·1	Between two millboards on each face.
,, ,,	716·9	Cemented between two strong iron plates with plaster of Paris.
,, ,,	504·4	One lead plate on each face.
,, ,,	322·3	Three plates of lead on each face.

* Unwin's "Testing of Materials."

These experiments appear to show conclusively how the use of lead plates for bedding reduces the crushing stress. The same remark applies in a lesser degree to soft materials like wood and leather. Where millboard is used, it should be thin. Probably the best bedding material is a very thin layer of neat Portland cement, but this requires several days to harden. The German plan is to grind the faces plane and use no setting material.

The shape of the specimen has a material effect on the crushing stress. Stone has been most generally tested in the form of cubes, rectangular prisms of various ratios of sectional area to height, and solid cylinders. In actual work stone is most often used in the form of rectangular solids, and it would seem obvious to make test pieces of similar shape. There are two main rules which control this point:

(1) Prisms of geometrically similar form have the same strength per unit area, prisms being regarded in the general sense and including cylinders.

(2) In prismatic specimens having different heights, greater height means less strength.

For square prisms less in height (h) than cubes Vicat gives the equation—

$$f = \frac{a}{h} - b$$

where f is the strength per unit area, and a and b are constants for the material. Where the height is greater than the length of side of the square, s, Bauschinger gave the following :

$$f = 1 - n\frac{s}{h}$$

This equation is supposed to be valid up to $h = 4s$ or $5s$.

The table on following page gives a number of crushing results collected from various sources.

Porosity or Absorption—The determination of porosity is made by first weighing the sample, then allowing it to soak in water for a specified length of time, and, lastly, reweighing after removal of superfluous moisture. There will be a gain in weight shown by the second weighing ; call this w. The absorption will be expressed as $\frac{W'-W}{W}.100 = \frac{w}{W}.100$, where W is the original weight of the sample, and W′ its final weight.

It is generally found that 24 hours are sufficient for saturation, though some experimenters allow the stone to soak for several days. On page 482 are examples of the results of absorption tests. The specimens were 3 in. cubes and the time allowed for soaking 24 hours.

CRUSHING STRENGTHS OF VARIOUS KINDS OF STONE

Description.	Crushing. Tons per sq. ft.	Crushing. Lb. per sq. in.	Authority.
Granite.			
Aberdeen red (2 in. cube) . . .	1,614	25,100	Unwin.
Aberdeen red ($2\frac{1}{8}$ in. cylinder, $3\frac{1}{4}$ in. high)	1,357	21,110	,,
Aberdeen grey (2 in. cube) . .	1,412	21,960	,,
Aberdeen grey ($2\frac{1}{8}$ in. cylinder, $3\frac{1}{4}$ in. high)	1,162	18,075	,,
Black and white speckled, paving (7 in. × $3\frac{1}{4}$ in. × 6 in. high) . .	233	3,624	Popplewell.
Paving sets (8), mostly about 4 in. cubes, approximately . .	359	5,584	,,
(The low result (average) for the above granite paving blocks was no doubt due to the fact that their surfaces were quite rough, and the plaster setting was not hard enough to counteract the roughness. The best were over 600 lb. per sq. ft.)			
Quartz	1,270	19,744	Mallet.
Mount Sorrel	832	12,940	Fairbairn.
Basalts.			
Penmaenmawr (2 in. cube) . .	1,086	16,893	Fairbairn.
Hornblendic Greenstone . . .	1,580	24,577	Wilkinson.
Felspathic Greenstone . . .	1,106	17,204	,,
Slate.			
Welsh Clayslate (3 in. cubes) .	733–1,052	11,400–16,364	Unwin.
Slate	1,205	18,744	Mallet.
Irish Slate, Valencia (3 in. cubes)	720	11,200	Wilkinson.
Irish Slate, Killaloe (3 in. cubes)	1,974	30,706	,,
Sandstones.			
Cooper Sandstone, U.S. . . .	975	15,163	Watertown Arsenal Reports.
Kibble Sandstone, U.S. . . .	666	10,363	,, ,,
Worcester Sandstone, U.S. . .	627	9,762	,, ,,
Craigleith, Edinbro' (2 in. cubes)	504	7,840	Royal Commission Report.
Darley Dale (2 in. cubes) . .	455	7,077	,, ,,
Morley Moor (2 in. cubes) . .	318	4,946	,, ,,
York Grit (3 in. cubes) . . .	712	11,075	Unwin.
Red Mansfield (3 in. cubes) . .	609	9,473	,,
Darley Dale (9 in. × 6 in. × 5 in. high)	435	6,766	Popplewell.
Red Alton (3 in. cubes) . . .	309	4,806	Unwin.
Shawforth stone, Rochdale (10 in. × 10 in. × 5 in. high) . . .	Withstood 440 without crack	6,844	Popplewell.
Limestones.			
Ancaster (2 in. cubes) . . .	150	2,333	Royal Commission Report.
Bath, Box ground (2 in. cubes) .	95	1,477	,, ,,
Bolsover (2 in. cubes)	484	7,529	,, ,,
Portland	239–392	3,873–6,097	Rennie.
Portland (2 in. cubes) . . .	430	—	Popplewell.
Portland (4 in. cubes) . . .	516	8,026	Unwin.
Portland (2 in. cubes) . . .	250	3,888	Royal Commission Report.
Purbeck (1 in. cubes)	587	9,131	Rennie.
Mt. Vernon Limestone, U.S. . .	492	7,647	Watertown Arsenal Report.
North River Bluestone, N.Y. .	1,475	22,947	,, ,,
Creole marble (Georgia) . . .	865	13,466	,, ,,
Etowah marble (Georgia) . .	903	14,052	,, ,,
Marble Hill marble (Georgia) .	740	11,505	,, ,,
White Italian marble	1,400	21,777	Rennie.
White statuary marble . . .	206–389	3,204–6,051	,,

ABSORPTION TESTS OF SAMPLES OF STONE

Stone.	Per Cent. of Water absorbed.	Authority.
Granite	0·2–0·3	Wray.
Roofing slate	0·5	,,
Coal-measure sandstone	1·0–7·0	,,
Old red sandstone	1·6– 3·8	,,
New red sandstone	6·0–10·0	,,
Mansfield	2·6– 4·8	,,
Lias sandstone	3·0– 8·0	,,
Craigleith	8·0	Royal Commission
Kenton	9·0	,, ,,
Mansfield	10·4	,, ,,
Portland	13·5	,, ,,
Ancaster	16·0	,, ,,
Bath	7·0	,, ,,
Oolitic limestone.	4·0–12·0	Wray.
Purbeck limestone	4·0– 2·0	,,
Magnesian limestone	3·5	,,

CHAPTER XVIII

MISCELLANEOUS MATERIALS

These include leather belting, fibrous belting, composite belting, fibrous ropes, both of cotton and the coarser fibres, woven cotton belting, indiarubber, paper, belt fastenings, chains, sheet metal, commercial lead, and material of the nature of porcelain. The essential qualities of all or any of these are revealed by experiments, most of which are possible in a general testing laboratory, though not all.

208. Leather Belting—Apart from questions of softness or hardness, flexibility or non-flexibility, the principal test applied to a sample of leather belting is for tensile strength and ductility. Specific gravity may also be determined. The strip tested should be not less than 20 in. long and may with advantage be longer. It is found that the ordinary wedges used for steel bite into the leather and often cut through along their front edges. To obviate this it has been found best to make the teeth in the form of simple V indentations cut across the piece at right-angles to its length, at the same time gradually reducing the depth of the teeth from their full height at the back down to zero at the front. This tapering of the wedge teeth is best effected by first forming the teeth to their full height and then grinding away towards the outer end. The result of this arrangement is that the penetration is deeper towards the inner end and consequently the tightness of hold tapers off from a hard grip at the inner end to nothing at the front (Fig. 238). With a little care wedges with this tapering grip will be found quite satisfactory and will not cut the leather.

The gripping is sometimes helped by the insertion through the belt of two pins, which are pulled against the backs of the wedges.

The elongation is measured between lines drawn on the leather with either ink or lead pencil, across the specimen. It may be measured during the loading up to a load close to fracture, or it may be measured after fracture to get the permanent stretch. The difference between these yields information as to the elasticity. Below are data obtained from a large number of tests of belts of varying makes.

LEATHER

Tensile strength, lb. per sq. in.	4,000 to 8,000.
Elongation before fracture, on 10 in.	6 per cent. to 20 per cent.
Elongation after fracture, on 10 in.	1·5 per cent. to 5 per cent.

COMPOSITE BELTING

Tensile strength, lb. per sq. in.	5,600 to 7,600.
Elongation after fracture, on 10 in.	1·5 per cent. to 10 per cent.
Elongation up to fracture, on 10 in.	18 per cent. to 20 per cent.

The strength of leather is greater when the hair has been left on than when removed.

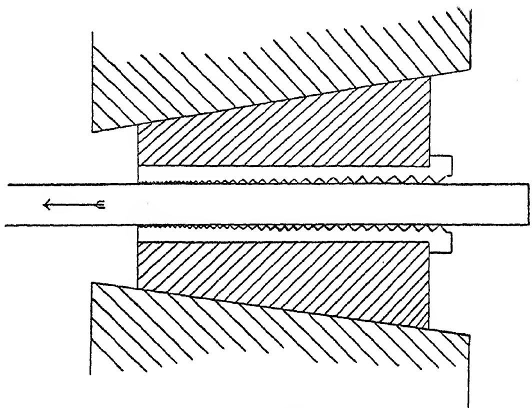


FIG. 238.

209. Fibrous Ropes—There are difficulties in the testing of fibrous ropes caused by the holding of the ends. The ideal plan for testing a rope would be to coil it several times round a pair of large pulleys, and either hold the pulleys in a testing machine or rotate them forcibly in bearings, until fracture of the rope took place. The general plan is to use a pair of conical holders; such are used for wire ropes and provide enlarged ends to rest in the cones. Knots are of little use, as fracture generally takes place close to the knot.

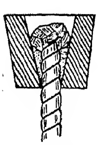


FIG. 239.

A "Turk's head" (Fig. 239) formed at each end of the length to be tested will give good results if these are properly made. It will sit better in its cone if there is a tight wrapping of yarn close against the heads. This may be still further helped by saturating the ends with thin Portland cement, or cement and thin glue.

When the load comes on the rope failure often begins by the breaking of a few individual yarns forming a strand or possibly the breaking of a complete strand. There is much stretch, but it

PLATE XXIX.

TURK'S HEAD AT END OF COTTON ROPE

is difficult to measure after fracture. The following are some average results :

MANILLA ROPE

6,000 to 10,000 lb. per sq. in.

A photograph of an actual specimen is given on Plate XXIX.

210. Cotton Driving Ropes—Very great care is needed when preparing the ends of cotton ropes to prevent pulling out of the cotton yarns which, when bundled together, go to form the rope yarns. The most common form of failure is that which takes place when one strand is broken. The breaking stress taken on the original cross-section has an average value of a little more than $2\frac{1}{2}$ tons per sq. in. or from 5,500 to 6,500 lb. per sq. in. Some authorities give 8,000 lb. per sq. in. as the breaking strength of cotton ropes.

211. Woven Cotton Belting—In testing the strength of wide cotton belts a good plan is to have a pair of T-shaped pieces of

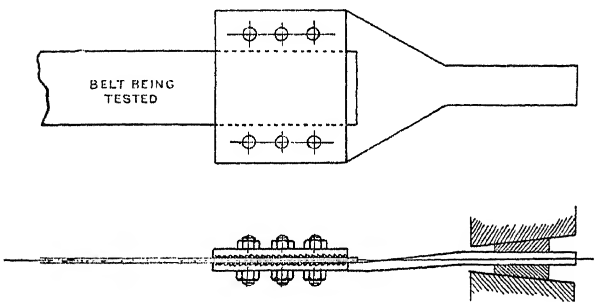


FIG. 240.

steel plate, with the long straight pieces held in the ordinary steel grips, with the T-ends projecting in front of the shackle. Holes are drilled through the T-piece, a plate of precisely the same size is bolted to it, with the end of the belt gripped between the T and the loose plate. Greater security is obtained if the belt is wound a couple of times round the plate before the bolts are put in place. It is found that, as the pull increases, thinning of the belt follows and continuous tightening of the bolts is needed to maintain the grip. This is shown in Fig. 240.

212. Paper—Rings of paper are sometimes used in a state of compression to form calendering rolls, and it is necessary to perform preliminary experiments with given samples of paper before using them in the rolls. For this purpose a compression testing machine must be used.

Also, the quality of paper is decided by the tensile test of a

strip, carried out in a small tension machine provided with fine-cut wedges.

213. Sheet Metal—Tests of sheet metal are made in tensile testing machines. The specimens ought to be parallel, narrow, and fairly long, preferably with enlarged ends. The width and thickness as well as the load measurements are easily made, and from these the results are calculated in the manner usual in larger pieces.

214. Commercial Lead—It is often necessary to make comparative tests of two or more samples of lead for the purpose of ascertaining which is the more suitable for use in the processes of "squirting," rolling, or wire-drawing. The difference between lead which is quite sufficiently pure and that which is not sufficiently pure to withstand the mechanical treatment is not revealed by mere chemical analysis but requires some delicate mechanical test, generally in the nature of some kind of indentation.

215. Porcelain—The rapid growth in electrical work, especially in "wireless," has had the effect of making porcelain a very important engineering material. The tests used are both tensile and compressive. For the former of these the specimens are of cylindrical form at the breaking section, with a very gradual increase in diameter towards the holding ends. For compression, cylindrical pieces are used with height about $2\frac{1}{2}$ times the diameter. When testing an equalizing material must be placed between the ends and the platens.

Cross-breaking tests can also be applied.

The following remarks are important:

MECHANICAL PROPERTIES—*Compressive Strength*—Boyd* carried out mechanical tests on porcelain and found that the compressive strength of a column 16 in. long and 1 in. in diameter was about 20,000 lb. per sq. in.

The results obtained by Clarkson† were as follows:

The test pieces were in the shape of a cube.

His results show that in compression unglazed porcelain seems stronger than glazed porcelain, and the strength of porcelain in compression is from ten to twelve times its strength in tension.

Cross-section in sq. in.	Lb. Compression.	
	Glazed.	Unglazed.
0·6	4,600	8,400
0·8	6,900	11,200
1·0	9,000	14,000

* Boyd, J. E., "Am. Soc. of Mech. Eng.," 38, pp. 222–226.

† Clarkson, R. P., "Electrical World," pp. 25–27, July 7, 1910.

Clarkson * carried out, later, further tests and found that the compressive strength varied from 5,780 to 17,580 lb. per sq. in. for glazed porcelain and from 7,200 to 19,450 for unglazed.

Kempton † gives some figures of the compressive strength of porcelain, according to which it varies from 10,000 lb. per sq. in. for a cylinder 2 in. long and 2 in. in diameter to about 50,000 lb. per sq. in. for a cylinder 2 in. long and $\frac{7}{16}$ in. in diameter. The reason for lower strength in large pieces is ascribed to lack of heat penetration in firing, although the influence of fibrous structure is also noted.

Bleininger ‡ finds that the compression strength varies from 50,000 to 65,000 lb. per sq. in.

Barringer § gives a much lower value of 20,000 lb. per sq. in.

Langton ‖ gets about 33,000 to 45,000 lb. per sq. in. as the compressive strength of porcelain. Some pieces tested at the College of Technology, Manchester, yield a high value, as much as 67,000 lb. per sq. in. Some unpublished data give about 20,000 lb. per sq. in.

Tensile Strength—Boyd ¶ gives the tensile strength of porcelain as about 3,000 lb. per sq. in. The method of test employed by Boyd failed to develop the full tensile strength of the material, and, judging from bending tests and the form of diagram, he concludes that the real tensile strength is about twice as great as the figure given above. In the discussion that followed on Boyd's paper, Barringer remarked that the mechanical strength of porcelain is greatly affected by temperature changes.

Clarkson ** used, for the purpose of mechanical tests, standard briquettes recommended by the American Society of Civil Engineers for tensile tests of cement. He gives the following results for different cross-sections of porcelain :

Cross-section in sq. in.	Lb. Tension.	
	Glazed.	Unglazed.
0·6	750	Less than 400
0·8	1,045	650
1·0	1,340	1,000

* Clarkson, R. P., "Electrical World," p. 1222,, May 18, 1911.
† Kempton, W. H., "A.I.E.E. Journal," pp. 967–977, May, 1910.
‡ Bleininger, A. V., "Met. and Chem. Engineering," pp. 589–594, May 15, 1917. § Barringer, L. E., April 15, 1917.
‖ Langton, J., "I.E.E. Journal," p. 214, 1912.
¶ Boyd, J. E., "Am. Soc. Mech. Eng.," 38, pp. 222–226.
** Clarkson, R. P., "Electrical World," pp. 25–27, July 7, 1910.

From the above results, it would appear that in tension glazed porcelain is about 25 per cent. stronger than unglazed. Glaze, the authors are of opinion, has the same effect as scale in the case of cast iron.

Clarkson * carried out further tests, at a later date, on test pieces of disc shape and $\frac{1}{4}$ in. thick at the centre and tapering slightly to the edges. The tensile strength of porcelain tested was found to vary from 870 lb. to 2,142 lb. per sq. in. for glazed porcelains and 587 lb. to 1,923 lb. per sq. in. for unglazed specimens. Kempton † gives much lower values for the tensile strength of porcelain. According to his results, it varies from 539 lb. per sq. in. to 897 lb. per sq. in. Some tests that were carried out in Manchester College of Technology, and referred to by Marsden at the discussion on Kempton's paper, give higher values for tensile strength. It ran from 3,316 lb. per sq. in. for a rectangular piece with a section of 0·2 sq. in. down to about 2,000 lb. per sq. in. for a rectangular piece of 0·6 sq. in. section.

Bleininger ‡ gives 1,300 lb. per sq. in. for the tensile strength of porcelain, while, according to Barringer,§ the value runs from 200 to 1,800 lb. per sq. in.

HARDNESS BY SHORE SCLEROSCOPE

Test Number.	Number of Experiments.	Mean Value.	Maximum Value.	Minimum Value.
1	24	95·8	102	89
2	20	83·9	87	80 ‖
3	30	92·9	95	90
4	44	98·9	102	94
5	24	99·1	102	97
6	20	83·9	88	80 ‖

216. Indiarubber Buffers—The usual dimensions of these, as used in railway-carriage suspension, are 5 in. diameter, 2 in. thick, with a hole through the centre 2 in. in diameter. To test this perforated disc it is placed between the platens of a compressive testing machine, after the dimensions have been carefully measured. A certain specified load is imposed, the thickness of

* Clarkson, R. P., "Electrical World," p. 1222, May 18, 1911.

† Kempton, W. H., "A.I.E.E. Journal," pp. 967–977, May, 1910.

‡ Bleininger, A. V., "Met. and Chem. Engineering," pp. 589–594, May 15, 1917.

§ Barringer, L. E., April 15, 1917.

‖ Specimens 0·40 in. long, 2·5 in. diameter, and hardness taken on flat surfaces only. All other specimens 2·5 in. long and 2·5 in. diameter and hardness taken on both flat and curved surfaces.

the buffer under this load is noted (possibly $\frac{1}{4}$ in.), after which the load is removed and the buffer remeasured. This remeasurement will show that a considerable amount of permanent set has taken place. It is the extent to which this permanent set is to be limited which forms an important section of the specification. The purer the rubber the more complete its recovery after loading.

Another test consists in subjecting the buffer to a large number of rapidly applied blows, and noting the effect on the material. This may be carried out with a smithy steam hammer by placing the buffer within a short metal tube, so as to limit the amount of compression; but a better plan would be to carry out the test in a specially designed machine.

If there are no signs of deterioration visible after this treatment the test is considered satisfactory. In the specification

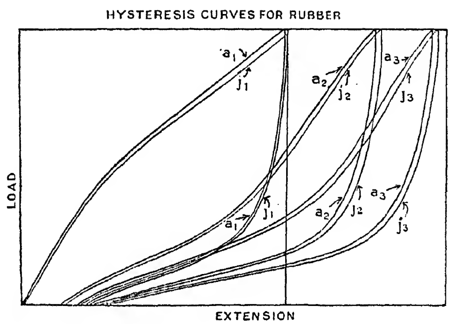


FIG. 241.—Test Pieces cut from 2,500 megohm Cable covering and put through three cycles of Extension and Retraction.

Curves a_1, a_2, a_3 are made up of four lines which practically coincide, each due to a separate test ece.

Curves J_1, J_2, J_3 are due to a single test piece cut from the same cable covering as the others, but containing a longitudinal joint.

laid down, the number of repetitions is given as well as the final length.

Mr. Alfred Barnes (formerly Professor of Physics at the Manchester College of Technology) constructed a machine to draw automatic diagrams of load and strain for strips of indiarubber under repeated tensile stress. The area of the hysteresis loop measures the quality for each sample (Fig. 241).

217. Chains—In connexion with chains of all kinds, both those made with links and those built up with plate links connected by pins, testing is necessary. Not only is this for the purpose of directly finding the total breaking load, but, especially in case of the second class, with the object of obtaining detailed information relating to the strength properties of the material forming the

different parts. The best class of plate link and pin chains have their parts machined with extreme accuracy from material whose strength properties are well understood. Consequently the total strengths of such chains can be foretold with a high degree of certainty.

The ends of test lengths of chain of the ordinary kind are held in the rounded portion of a pair of U-pieces of metal slightly stronger than that in the chains themselves, the straight portions lying in the ordinary wedge grips.

In testing plate chains, two end portions are specially prepared, one held in the wedge grips at each end. Each of these holders is machined to resemble one of the sets of links. It only requires a pin to connect the chain which is to be tested. The above list of miscellaneous tests is always growing and can never be regarded as complete so long as there is progress in engineering.

CHAPTER XIX

REPEATED AND REVERSED STRESSES

WHEN engineering members are subjected to steady loads in one direction only, the experimental information provided by the ordinary testing machine is sufficient, but when the same kind of load is repeated many times, or where the load is continually reversed, special experiments are needed. Wöhler* was the first to discriminate between the effect of (1) a steadily applied load, (2) a repeated load, and (3) a repeated reversal. He found that (2) would break with a smaller load than (1), and that, under the conditions of (3), the breaking load would be smaller still. Wöhler's simple repetition tests were made in specially designed machines having the loads applied through levers and controlled by springs. The speed gave about 70 repetitions per minute.

For reversals of stress he applied known loads to the outer ends of circular cantilevers which were meantime being rotated at about seventy times per minute. There were also repeated bendings and repeated twistings. The results of all these were published about the year 1870. Many of them are still utilized in the design of bridges.

Many of Wöhler's experiments have been repeated, and some new methods have been introduced. Of these, the most important are—First, the dynamic plan suggested by Professor Osborne Reynolds and used in a modified form by Dr. T. E. Stanton; second, the rotating cantilever or beam, used by Sir B. Baker; third, the rotating-beam experiments made at University College and by Mr. Popplewell; fourth, the repeated reversals of torsion by Mr. W. Mason; and fifth, the electric resonance tests, carried out by Professor Hopkinson.

In this last the specimen is so placed as to support a heavy block of soft iron, which acts as armature to an electro-magnet which is served by a current of high frequency. When the speed of the alternations approaches the natural speed of longitudinal vibration of the specimen, it is possible to keep it in continuous vibration through a certain range of stress (see Fig. 242). The strains are measured by a mirror apparatus.

An important series of experiments was carried out in the

* "Zeitschrift für Bauwesen," 1860, 1863, 1866, 1870.

laboratories of University College, the specimens being arranged as rotating beams.

The authors of these pages carried out a series of experiments under very slow speed of reversal in order to foretell, if possible, the limiting range of stress, and their results were afterwards compared with their own tests with rotating beam and with direct stress.

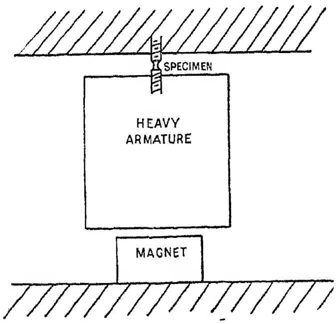


FIG. 242.

In all the above experiments the bar was stressed from a minimum load to a maximum. The algebraic difference between minimum and maximum stresses is called "the range of stress."

In each experiment limiting stresses were fixed and the machine allowed to run on until fracture occurred.

The two results found were therefore—*the limiting stresses employed*, and *the number of repetitions required to break the bar.*

For each material it was possible, from the above results,

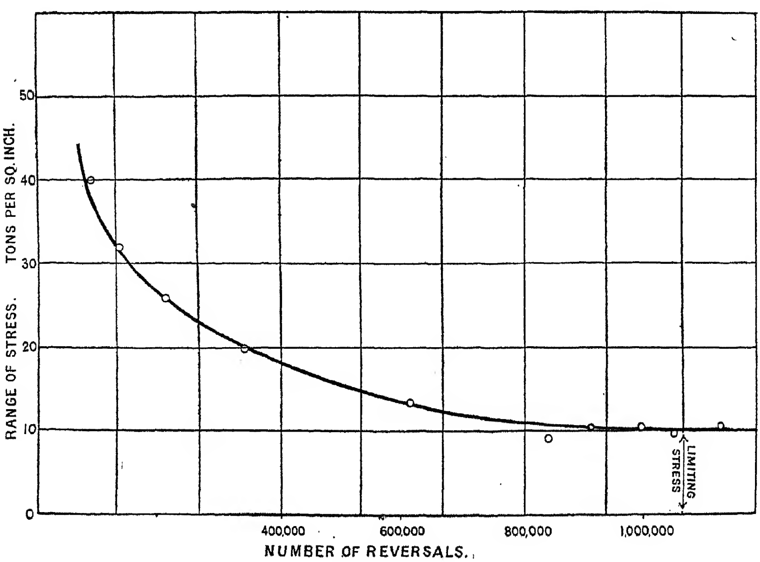


FIG. 243.

TABLE OF EXPERIMENTS ON ENDURANCE OF STEEL AND IRON

Experimenter.	Nature of Experiment.	Materials.	Number of Cycles per Minute.
Wöhler *	Revolving cantilever	Phœnix iron	60–80
		Homogeneous iron	60–80
		Vickers' axle steel	60–80
		Firth's tool steel	60–80
Baker	,,	Soft steel	50–60
		Fine drift steel	50–60
Rogers	,,	0·32 per cent. C steel	400
		Same (annealed)	400
Stanton and Bairstow †	Direct tension and compression (Reciprocating weight)	Wrought iron	800
		Piston rod steel	800
Reynolds and Smith ‡	,,	Mild steel (annealed)	1,337–1,917 (20·9) (12·4)
		Cast steel (annealed)	1,320–1,990 (20·1) (13·1)
University College, (Eden, Rose, Cunningham) §	Rotating beam	Drawn W.I.	250
		,,	620
		,,	1,300
		Drawn mild steel	250
		,,	620
		,,	1,300
		Mild steel	Up to 7,000
Hopkinson ‖ . .	Electric resonance	(Mild steel and carbon tool steel)	Up to 350
Popplewell ¶ . .	Slow rotating beam. Direct-stress testing machine		One cycle in 24 hours
Bairstow ** . .	Direct-stress testing machine.	Mild steel and wrought iron	Up to 1,200

* "Testing of Materials." Unwin. (Wöhler's figures.)

† "Resistance of Iron and Steel to Reversals of Direct Stress," Stanton and Bairstow. Proc. I.C.E., 1906, vol. clxvi.

‡ "On a Throw Testing Machine for Reversals of Mean Stress." Professor Osborne Reynolds and J. H. Smith. Phil. Trans. R.S., 1902.

§ "Endurance of Metals." E. M. Eden, W. N. Rose, and F. L. Cunningham. Min. Proc. Inst. M.E., 1911.

‖ Royal Society, Proceedings, vol. lxxxvi, Professor Bertram Hopkinson.

¶ "Slow Reversals of Stress and the Ultimate Endurance of Steel." W. C. Popplewell. Min. Proc. I.C.E., 1914.

** Royal Society Phil. Trans., vol. ccx. Mr. L. Bairstow.

to plot a curve of (range of stress) against (number of repetitions). In these, each bar of the material provides one point of the curve. One such curve is shown in Fig. 243. Many of the experiments so far carried out are summarized in table on page 493.

The following table gives the detailed results of two of Wöhler's experiments under repeated tensile loads :

	Experiment.	Stress Applied. Max.	Stress Applied. Min.	Range of Stress.	Number of Repetitions to Fracture.
		Tons per sq. in.			
Iron Axle made by the Phœnix Co.	No. 1	22·92	0	22·92	800
	,,	21·01	0	21·01	106,910
	,,	19·10	0	19·10	340,853
	,,	17·19	0	17·19	409,481
	,,	17·19	0	17·19	480,852
	,,	15·28	0	15·28	10,141,645
	,,	21·01	9·55	11·46	2,373,424
	,,	21·01	11·46	9·55	(4,000,000)
	,,	17·10	0	17·10	37,828
Krupp's Axle Steel.	No. 2	38·20	0	38·20	18,741
	,,	33·40	0	33·40	46,286
	,,	28·65	0	28·65	170,170
	,,	26·14	0	26·14	123,770
	,,	23·87	0	23·87	473,766
	,,	22·92	0	22·92	(13,600,000)
	,,	21·95	0	21·95	(13,200,000)
	,,	38·20	23·87	14·33	(1,801,000)
	,,	38·20	19·10	19·10	(12,100,000)
	,,	38·20	16·70	21·50	(12,000,000)

The bracketed figures refer to unbroken bars.

When an experiment on a set of bars of the same material has been completed and the curve of repetitions and applied stresses plotted, it becomes possible to deduce the essential conclusion. It is seen that as the repetitions increase and the limiting stresses diminish, the curve becomes flatter, suggesting that the number of repetitions is approaching infinity. The actual maximum number is always very great, but there is sometimes a question whether the curve would not have become flatter if the total number of repetitions had been greater. But when the number is much greater than 1,000,000, the curve is usually flat enough to justify a reading of the "limiting stress," that is

to say, the range of stress which this particular material may be expected to withstand without injury during an unlimited number of repetitions under the conditions fixed by the experiment. With the usual allowances for exigencies, a "safe stress" so found should be quite valid.

218. Rotary Beams compared with Direct Stress—It is always found that limiting stresses ascertained by the former of these methods are higher than those for the same material by the direct-stress method. The conditions as to strain are quite different. With the rotating beam or cantilever, there must be a thin zone of highly strained material near the surface, while that nearer the centre is wholly or in part elastic and therefore able to give some support to the outer layers. In the direct-stress conditions the stress and strain are uniform over the entire section.

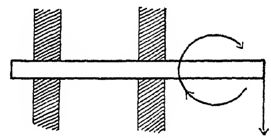


FIG. 244.

The arrangement of the rotating cantilever shown on Fig. 244 is quite simple and not difficult to provide for. Here the stress intensity varies from the support to the load, which may be either a dead weight or a spring.

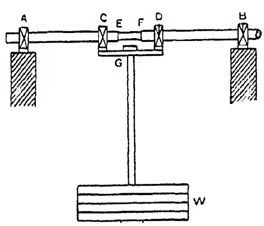


FIG. 245.—Popplewell's Rotating-beam Device.

A and B are the main ball bearings.
a is distance between end bearing and middle bearing.
C and D are the ball bearings through which the load is applied.
E and F are the limits of the reduced portion.
G is the frame carrying load.
W is the load.

A uniform stress at all sections of the portion experimented upon is obtained by adopting the plan used by Mr. Popplewell, as well as in the University College experiments, of having a beam in place of a cantilever and applying the load at a pair of points (see Fig. 245). In these experiments the approach of failure was indicated by a tendency to wobbling caused by the increased deflection. The fracture was roughly square across the shaft.

When increasing the speed to 350 revolutions a minute a critical region is passed through and great care has to be observed.

Mr. Popplewell's REVERSALS OF DIRECT STRESS were got by using a small crank and connecting-rod arrangement. The sliding block held one end of the specimen, its other end being in a socket attached to the middle of a flat plate spring. Upward movement of the slide caused upward deflection of the spring and a consequent pull on the specimen. In a similar way downward movement caused a push. The amount of deflection of the spring, and consequently the load, was kept the same through-

out, and stress variations were obtained by changing the diameters of the specimens and so their areas.

In the DYNAMIC plan used by Osborne Reynolds and Dr. Stanton, the specimen (s) was held between the crosshead of a crank and connecting rod (see Fig. 246).

219. Elastic Limits and Ultimate Endurance—After considering the following elastic phenomena and noting their bearing on endurance results, it seems possible to predict, with some degree of certainty, the limiting stress of any given sample of steel, by simply carrying out certain experiments in an ordinary testing machine and so avoiding much of the tediousness of actual endurance tests to destruction.

When a bar is loaded for the first time in either tension or compression, the "elastic" limit found under these conditions is spoken of as the "primitive limit."

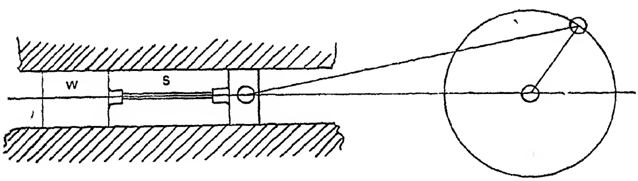


FIG. 246.

If the bar is loaded beyond this point, the load removed, and a second limit found, this second limit will be at a load higher than the first. Loading in the same direction beyond the limit has the effect of *raising* the limit. If, on the other hand, the bar had been loaded beyond the primitive limit, but with the opposite kind of stress, it would have been found that the limit had been *lowered*.

That is to say, stress repeated in the same direction raises the limit. Stress repeated beyond the limit in the opposite direction lowers it.

When a bar is subjected to a series of alternating stresses of varying magnitude, it is possible to arrive at a pair of limits the same in tension as in compression. These have been called "natural" limits, and Dr. Unwin has suggested that they may be not very far removed from the actual "limiting stresses" for the material.

Further lapse of time at the normal temperature appears to have the effect of restoring the material to its original condition, or possibly to a stress higher than the primitive limit.

A set of such experiments made by Mr. Popplewell on a mild-steel bar is given on page 498. The complete figures are

plotted on Fig. 247. These figures appear to suggest that uniformity of applied stresses conduces to uniformity of "artificial limits." Further experiments revealed the information that the higher the applied stresses the lower the artificial limits, and vice versa. So that by making the applied stresses lower

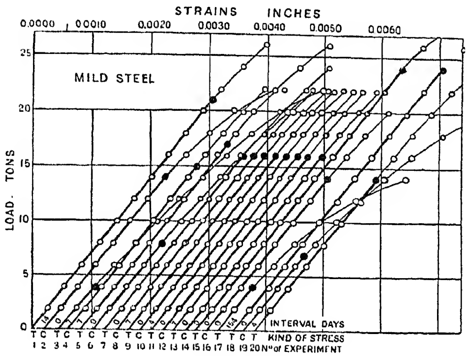


FIG. 247.

and lower, the artificial limits will become gradually higher, until the two values coincide. Thus, a bar was given 10 tons per sq. in. for ten times, alternately in tension and in compression. This was found to give an artificial limit. Next, the same process was followed at 11 tons per sq. in. tension followed by compression, and a new artificial limit determined which was found to be lower than the last one. This process was continued until half a dozen points were obtained, sufficient for plotting. This is seen in Fig. 248. Here the centres of the small circles represent artificial limit points. A curved line is drawn through these points and continued to cut the diagonal line 0C in D. At every point in 0C, the applied stresses and artificial limits have equal values, so that in the curve, as the applied stress points get lower, the artificial limits become higher, until a point is reached when they are equal. This is at D. The value at D may be called the "ultimate limit." This is the stress below which there is no induced artificial limit, and it is thought that it may be the same as the ultimate stress when found from endurance experiments.

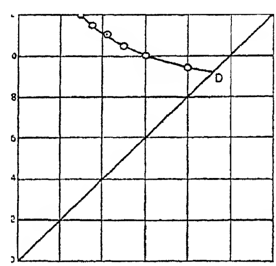


FIG. 248.

MILD-STEEL BAR

Number of Experiment.	Interval between Experiments.	Proportional Limit.		Maximum Stress.	
		Tension.	Compression.	Tension.	Compression.
	Days.	Tons per sq. in.	Tons per sp. in.	Tons per sq. in.	Tons per sp. in.
1	Original condition	10·20	—	12·62	—
2	14	—	6·80	—	10·69
3	At once	1·94	—	10·69	—
4	7	—	7·28	—	10·69
5	7	8·25	—	12·62	—
6	At once	—	3·88	—	10·69
7	8	(7·76)	—	10·69	—
8	At once	—	(7·76)	—	10·69
9	7	(7·76)	—	10·69	—
10	At once	—	(7·76)	—	10·69
11	4	(7·76)	—	10·69	—
12	At once	—	(7·76)	—	10·69
13	7	(7·76)	—	10·69	—
14	At once	—	(7·76)	—	10·69
15	229	11·65	—	13·60	—
16	At once	—	6·80	—	9·71
17	,,	1·94	—	9·71	—
18	154	11·65	—	14·56	—
19	At once	—	3·40	—	11·65
20	4	6·80	—	11·65	—

It seems that such experiments under slow reversal of stress may suggest the ultimate values likely to be obtained from a safe range of stress and to enable endurance results to be foretold. The reasonableness of this suggestion has been demonstrated by subjecting steel bars to the slow reversals experiments and at the same time subjecting other bars of the same material to actual endurance experimental tests. The results yielded in the two cases were very near to one another.

220. Effect of Speed on Endurance—From a careful examination of the details of the load-strain conditions as shown on the diagrams, as well as from the results of published endurance tests, the opinion is held by many engineers that high speed of stress reversal is in favour of endurance, and that where the reversals are rapid the specimen will last longer than when the speed of reversal is slow.

The whole question appears to depend on the speed with which early semiplastic strains and their release can take place.

The diagram Fig. 249 (*a*) represents the course of a very slow cycle of reversed stress. From A to PL the line is straight

and the conditions are elastic. From PL to C the line is curved, the slowness permitting the full strain cC to take place.

After passing C the load is removed, and the line CD, representing this removal, is a straight elastic line. The second half of the cycle DFA is a repetition of the first.

The second figure (*b*) represents a quicker cycle between the same limits of stress. The difference between this and (*a*) is shown in the position of the PL which, in this second case, appears higher than in (*a*). This is owing to the fact that semiplastic

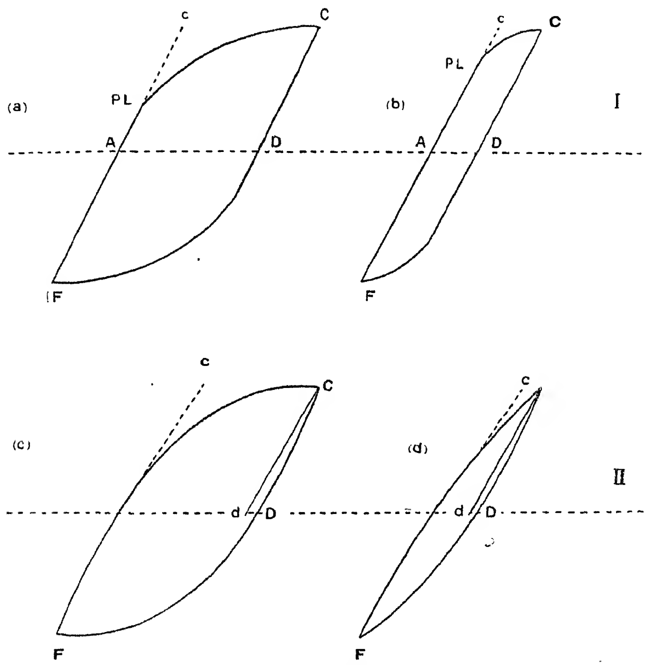


FIG. 249.

strain or yield, beginning at the P limit, requires time for its development, and cannot be expected to have made itself clearly evident quite so soon when the applications and removals of stress are rapid. These effects of speed on strain can be observed with an extensometer when stress is imposed or removed.

A pair of automatic diagrams under the above conditions would be more as shown in (*c*) and (*d*), with PL points less apparent and the release lines not quite parallel to the original application lines.

This means that the PL settles down to the artificial limit at a higher stress where the speed is high than where it is low. Therefore fracture will be later. The question may be regarded from another point of view with the same result.

On Fig. 249 the plus disturbance to the material is C*c* and the minus disturbance is D*d* due to recovery of strain. The minus disturbance is very small as compared with the plus. At the higher speed C*c* is less than before, as also is D*d*, but the total area of the loop is reduced, indicating that the total disturbance is less and that the material has consequently a better chance of life. This evidence is all of a theoretical nature, but there is also the evidence gained from actual endurance tests carried to destruction. In the table already given, it is seen that the speed was varied in the tests of Reynolds and Smith, the University College experiments, and those of Professor Hopkinson.

The balance of the evidence obtained from these endurance tests appears to support the theoretical view just given. Hopkinson, Stanton, and University College give results which at least do not show deterioration following higher speeds. Against this Reynolds and Smith found the greater strength at the lower speeds. This discrepancy has not been accounted for, and we must accept the above balance in favour of greater strength with high speeds, or perhaps, more precisely, that within the speeds so far attained (1–7,000 per minute) the speed has no very marked effect on endurance.

CHAPTER XX

CHOICE OF MATERIALS AND FACTORS OF SAFETY

THE use of certain materials for certain known branches of engineering work has, in course of many years and through long experience, become recognized as standard practice. They have proved their true value by the way they have fulfilled expectations. When a designer is called upon to select material for the work he has in hand he is at once guided by known custom and his choice is generally limited to two or three possibilities, and often he has to consider the relative merits of a very few materials. Many qualities have to be inquired into, such as strength, stiffness, heaviness, ease in machining, hardness, durability, and many others, of which the most important is cost. For example, the choice may have to be made between steel and brickwork on the one hand and reinforced concrete on the other. The one with the lower price will most likely secure the contract, though speed of construction will also count. So long as other things are reasonably satisfactory, the cheaper material must ultimately get the job, though an engineer who is worth his salt will aim at turning out the most satisfactory work possible, even at some slight financial sacrifice. His aim should always be to find the best possible material for the work in hand. Consequently he ought always to be on the look-out for new materials and new ways of doing things. Since engineering began there has been steady progress, and this must continue.

221. Choice of Materials—The choice of the most suitable material for any given purpose, as well as all the properties it should possess, is nowadays in a large number of cases settled for the engineer by "standard specifications." Much trouble is thus saved.

At the same time there are many general qualities of promising materials which may have an important bearing on their choice for new and untried work, and some of these are worth discussing. For example, it may be necessary to decide between the relative merits of cast iron and a steel casting for certain work. Cast iron is easily shaped to the desired form; it is cheap; it is more certainly free from blow-holes than steel castings: but it is less strong. The question to be answered by the engineer before he

can decide which metal to use becomes one of weighing advantages against disadvantages and striking a balance. Use the cheapest material, but let it possess all the qualities essential to the work.

It should be borne in mind that the gentlemen who frame specifications are largely of past experience and naturally take conservative views. This interferes with progress. If such views had always persisted, we might now be living in tents, and have to swim across rivers, but engineers are fortunate in the fact that standard specifications are frequently being revised and stiffened. This is especially true of Portland Cement, and has resulted in a marked improvement in quality. All encouragement should be given to the development of new materials, and their qualities should be carefully examined and tested.

222. Working Stresses—When a designer is called upon to fix the correct dimension of some mechanical part or structural member which has to resist straining action, it is an essential condition of success that he shall have reliable information about the strength of the material he is about to use. It is common practice for the designer, when he has calculated, or otherwise found by means of some graphical method, the magnitude and nature of the load which is expected to come on the part in question, to provide an area of section of such a size that when the load is divided by it the value of the "safe working stress" is the result; or the required sectional area is equal to the total load divided by the working stress. The working stress is almost universally made to depend on the ultimate strength of the material, and is obtained by dividing the ultimate strength by a number which varies according to the conditions surrounding the case in question. The figure thus made use of is referred to as a "factor of safety," and is so framed as to allow for all conditions which are known, and for a good many which are only partly known or can merely be guessed at. A proper choice of this factor, when all arguments *pro* and *con* have been carefully weighed, is the essential step in fixing upon the stress to be applied to a unit area of the given material when under the specified conditions.

This plan of deriving the working stress from the tenacity by dividing it by a factor of safety is not right in principle unless the factor is made to depend in part on the results of actual endurance tests. It will be shown below that the factor of safety depends in part on a number of miscellaneous circumstances whose existence and probable effects are matters of practical experience in the use of a material, and whose allowances are purely empirical; and partly on the conditions of loading, which are known. Where the stresses are repetitive or fluctuating, the results of past experiments are utilized to provide those portions of the qualifying factors which provide for the conditions of

loading. The applied load on a material under working conditions is either "dead," that is, steadily applied, or is "live," or "repetitive," or "fluctuating."

Most of the public authorities, such as the Admiralty, War Office, Board of Trade, Lloyd's, as well as local authorities and the railway companies, have set their seal on the current practice by insisting on specifications on ultimate strength as a measure of suitability. The railway authorities have recognized the importance of properly allowing for the effects of live and fluctuating loads, and have insisted on the use of factors of safety which take these into account by the use of specified formulæ.

Attempts have been made to utilize the elastic limit and the yield-point stresses for determining safe stresses, but experimental results have shown pretty clearly that these stresses are very unreliable, and that their magnitudes depend greatly on previous treatment of the material, so that the above plan has never been adopted.

In any given case the two main questions are: (1) What load is likely to be applied to the part whose dimension is required? and (2) What is the greatest stress which the material can withstand under the conditions in question?

When these two questions have been answered satisfactorily, the problem is solved.

In most cases the answer to the first is not difficult to find. It may not be quite complete nor rigidly correct, but it is generally possible to obtain a very fair approximation.

Assuming, then, that the first question has been answered satisfactorily, how is the one relating to safe stress going to be dealt with?

There are two plans possible. In order to arrive at a satisfactory working stress, the more usual plan, as already mentioned, is to find the ultimate or breaking strength of the material and to divide this by a constant called a factor of safety. The other is to make the safe working stress dependent, wholly or in part, on endurance tests.

As a matter of fact, neither plan is ever made use of quite apart from the other, but it will be convenient here to consider them separately in the first instance.

Assuming that the safe working stress for a given material is obtained by dividing the ultimate strength by a factor of safety, the circumstances which affect the choice of this factor for steady loads, and which must be taken into account, are the following:

(*a*) Incomplete or Inaccurate Estimation of Loads, or neglect of some part of the straining action.

(*b*) Possible Errors in Estimation—It is hardly fair to saddle the factor of safety with errors in estimating the straining

loads or calculating the dimensions, but both kinds of error do creep in and often it may be wise to allow for such possibilities.

(*c*) LESSENING OF ORIGINAL DIMENSIONS caused by legitimate wear or corrosion.

(*d*) DIFFERENCES BETWEEN THE PROPERTIES OF THE MATERIALS THEMSELVES, as well as variations in the quality of different samples of one material.

(*e*) ERRORS IN WORKMANSHIP, by which essential dimensions are left too small and sectional areas thereby reduced.

The above reasons for making allowances in the factor of safety refer to contingencies which are pretty certain to exist but about whose magnitudes there is necessarily a good deal of ignorance on the part of the designer.

These are influences which are apart altogether from differences in the manner of loading and are intended to be applied in the first instance to dead loads.

There are other conditions of quite a different kind, called the loading conditions, which may or may not affect the factor of safety. These are dealt with in the following way.

(*f*) DEAD AND LIVE LOADS—The simplest kind of loading is that in which the load causing stress, whether in tension, compression, or shear, is slowly and steadily applied; and is either removed and reapplied in the same way after long intervals of time, or is allowed to remain on the material after its first application. This is a *Dead Load.*

When, however, the load is allowed to come on the material suddenly, but without an actual blow, it is referred to as a *Live Load.*

What happens in the application of a simple live load is shown by the following instance.

A beam is supported at its two ends and a weight gently lowered by a rope until it just touches the upper surface of the middle of the beam without deflecting it. The rope is now cut, with the result that the beam suddenly feels the effect of the downward pressure without any actual blow being struck.

The effect of such sudden application without blow is to cause elastic strain—in the above example, deflection—which is double what it would be if the load were allowed to come very slowly to its position of repose.

As the strain is in this way doubled by the load being a live one, so the stress must also be doubled.

Therefore, to find the dead load which would produce the same effect as a given live load, it is necessary to multiply the latter by 2. This is the "equivalent dead load."

(*g*) LOADS REPEATED THROUGH KNOWN RANGES—The case of a load, many times repeated, either through a range extending

from 0 to a maximum value, or from a given minimum to the maximum, often occurs in practice and is extremely important, especially in relation to railway bridges. The greater portion of what is actually known about this branch of the subject has been derived from the published results of experiments carried out by Wöhler and his contemporaries. Although these were made some forty years ago and upon materials not quite the same as those in use at the present time, the general laws deduced are very much to the point, and continue to form the bases on which are constructed many of the formulæ employed by engineers in this and other countries.

The main facts revealed are:

There is some stress, called a "limiting stress," which a given kind of steel or wrought iron is capable of resisting during an unlimited number of repetitions. This application may be from a lower to a higher stress, in the same direction; or it may be from a given tensile stress to a compressive stress of equal magnitude. When a stress is repeatedly applied through a range extending from zero to a maximum, the limiting value will be lower than where the range is from a higher minimum. Thus the safe stress depends not only on the range, but also on the actual values of the limiting stresses, that is, upon their ratio.

This is extremely important, and shows the necessity of taking into account not only the range through which the material is stressed, but the limiting stresses also.

(*h*) Reversed Loads, through ranges extending from a certain tension to an equal compression, or from one load to another of unequal value and of the opposite kind.

This case has been investigated both by the earlier experimenters of forty years ago and by several in quite recent years. The result is that there is a great deal more information available on the subject than on simple repeated stress. Not only so, but the later experiments have been made with materials such as are in constant use at the present day.

(*i*) Blows whose Magnitudes are Known, many times repeated. At the National Physical Laboratory, Dr. Stanton has carried out a series of experiments to find the limiting stress of material subjected to repeated blows.

In choosing suitable values for factors of safety, practical experience in the use of the material in question has always played an important part, and rightly so. An inspection of the following figures, very generally employed, will make it clear how the factor varies even for similar conditions of loading. Many of these variations are remarkable, and some difficult to account for.

TABLE I.—FACTORS OF SAFETY

Material.	Factors of Safety for—			
	Dead Load.	Live or varying Load.		Structure liable to Shock.
		Stress of one kind only.	Reversed Stresses.	
Cast iron	4	6	10	15
Wrought iron and steel	3	5	8	12
Timber	7	10	15	20
Brickwork and masonry	21	30	—	—

The values given in the above table may be taken as representing fairly common practice.

One important material not included is reinforced concrete.

223. Reinforced Concrete—At the present time many of the Governments of European countries and the United States of America have carefully considered the question of design in reinforced concrete and have laid down very definite rules relating to questions of design, and framed regulations for carrying out work.

The working stresses are such that the average factor of safety taken from a large number of cases is about 5, when the concrete is young, but probably not much more than half as much when the concrete has become fully matured.

The factor of safety for the steel is in most cases 4 or 5. The reasonableness of these coefficients when compared with the 20 for brickwork and masonry, in Table I, is very striking, and will make some users of these latter materials wonder whether it is not possible for them to employ higher working stresses. The reason why the factors of safety for reinforced concrete are comparatively low is that the whole question as it affects this material has been very carefully considered, not once but often, and by many capable authorities.

Specifications for cement have been steadily made stiffer, the effect on strength of variations in aggregates and proportions has been investigated experimentally, and there has been wholesale collation of details regarding the behaviour of completed structures and the effect of lapse of time. These have all assisted designers in making their estimates and calculations, and at the same time have helped to give them the desired confidence.

It is quite possible that some of the factors of safety quoted are higher than they ought to be, or at any rate higher than they need be under good conditions, but when they are examined in detail it will be seen that, taking everything into account,

there is little to find fault with in those belonging to the most important materials. For example, in the case of steel, the factor of safety given in the table (p. 506) for dead loads is 3. If the average strength of structural mild steel were 27 tons per sq. in., the safe load with the above factor would be 9 tons per sq. in. for steady loads. As a matter of fact, the safe stress used in actual framed structures of mild steel is less than the above figure, but not very greatly so. It may be 6 or 7 or 8 tons per sq. in. Suppose it to be 7: the difference between this figure and the value of the primitive elastic limit of mild steel is not great. So that the margin allowed to cover such things as variation in quality, defective workmanship, errors in calculation, and so forth, is small.

224. Effects of Repeated and Reversed Stresses—In the tables of factors of safety in common use, such as the one quoted above, there are two distinct sets of figures included, namely, those in the first column applicable to cases where the loads are steady; and those in the three remaining columns, which are the values in the first column modified to allow for the manner of loading. The figures in the first column may be said to have grown into existence partly as the result of experience and partly as a matter of considered judgment based on intimate knowledge of the materials employed.

By dividing the ultimate or breaking stress by the proper factors in the first column a figure is obtained which represents the stress which may be safely applied to the material when the load is a steady one gradually imposed.

Assuming these allowances to be fair for gradually applied loads, modified values are given in the remaining columns, which modifications are intended to take into account other ways of applying the loads. These include "Loads Repeated through known Ranges," "Loads completely Reversed," and "Loads accompanied by Shock." So that, having assumed that the factors in the first column do give reasonable allowances for the contingencies described, it is then only necessary to multiply each factor in the first column by a figure which will account for differences in loading. The constants employed for this purpose are obtained from the results of experimental research, and are quite apart from those in the first column.

In Table II are given a number of results obtained by Bauschinger * and Wöhler. In this table the figures in the second column (maximum stress) are stresses which have been applied several million times previous to or without fracture, and are here considered as limiting stresses. The third column contains the minimum stress in each case, and it will be seen that this

* 13th Heft of Mittheilungen.

minimum varies from zero to a complete reversal of the maximum. In Wöhler's results several of the minima are of the same kind as the maxima, but of course less in magnitude.

The fourth column contains the "ranges" or differences between the quantities in columns (2) and (3).

In the fifth column are figures representing the results of dividing the numbers in column (2) by the ultimate strengths of the material in question.

TABLE II.—RESULTS OF EXPERIMENTS ON REPETITION AND REVERSAL

BAUSCHINGER'S FIGURES (ALL STRESSES IN TONS PER SQ. IN.)

Material and Ultimate Tenacity.	Maximum Stress (limiting).	Minimum Stress.	Range of Stress.	Ratio, $\frac{\text{max.}}{\text{ult.}}$
Wrought-iron plate (25·2) . .	9·85	0	9·85	0·39
	13·1	0	13·1	0·52
	16·4	0	16·4	0·65
Steel plate (28·5)	16·0	0	16·0	0·56
Bar iron (26·6)	13·2	0	13·2	0·50
	13·8	0	13·8	0·52
	16·4	0	16·4	0·62
	17·2	0	17·2	0·65
Mild-steel boiler plate (26·6) .	18·4	0	18·4	0·72
	16·4	0	16·4	0·62
	16·4	0	16·4	0·62
	18·7	0	18·7	0·70
Thomas steel axle (40·1) . .	16·3	0	16·3	0·41
	19·7	0	19·7	0·50
Thomas steel rail (39·0) . .	16·4	0	16·4	0·48
	19·7	0	19·7	0·50
Wrought iron (21·1) . . .	+7·65	−7·65	15·3	0·36
	15·8	0	15·8	0·75
	21·0	11·5	9·5	0·99
Cast-steel axles (49) . . .	+13·38	−13·38	26·76	0·27
	23·0	0	23·0	0·47
	38·2	16·7	21·5	0·70
Untempered cast-steel springs (55)	23·9	0	23·9	0·44
	33·5	11·5	22·0	0·61
	38·3	19·1	19·2	0·70
	43·0	28·7	14·3	0·80

These represent several typical sets of experiments. Inspection of the figures shows a number of extremely important facts which may be taken as representing general laws These are :

(*a*) When the stress is simply repeated many times between a maximum and zero, the limiting stress to withstand an unlimited number of repetitions varies from 0·40 to 0·75 the ultimate strength, with an average value of 0·57. This is represented in the empirical formulæ following as the round fraction $\frac{2}{3}$.

(*b*) Where the stress fluctuates between a certain maximum value and a minimum of the same numerical value but of opposite kind, that is, undergoes complete reversal, the ratio is very near $\frac{1}{3}$, and this figure is used in the formulæ.

(*c*) Where the range is between a given maximum and a minimum of the same kind, the ratio increases with the minimum.

(*d*) Further, it is seen that if the above ratios represent the conditions for the structural materials, the reversal constant for the harder steels is somewhat lower, say $\frac{1}{4}$.

So that when the intended condition of loading are sufficiently well known it becomes possible to say what the constants of the formulæ are to be, taking no account of the contingencies represented by the factors of safety given in the second column of Table I. When the factors for dead loads have been decided upon, the Wöhler-Bauschinger ratios are introduced for the purpose of converting the factors for dead loads into corresponding factors for repeated and reversed loads.

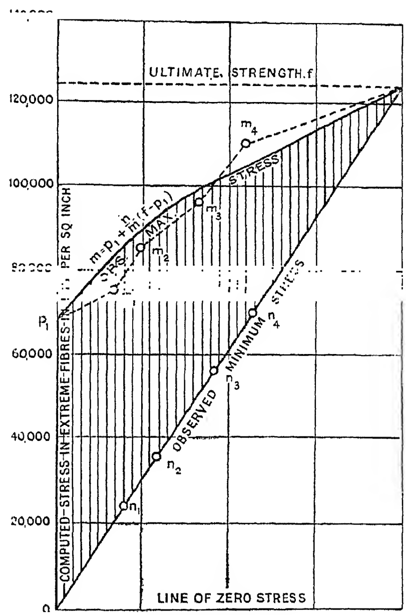


FIG. 250.

225. The Launhardt-Weyrauch Formula and its Use—From the plotted results which represent Wöhler's and Bauschinger's experiments, several formulæ have been constructed for the use of engineers, and especially intended for railway bridges. One of these is the formula of Launhardt, dealing

with stresses of one kind only when acting between fixed limits.

Another, by Weyrauch, was designed to deal with reversed stresses.

The manner of obtaining working stresses in the two cases above will be appreciated by examining Figs. 250 and 251.*

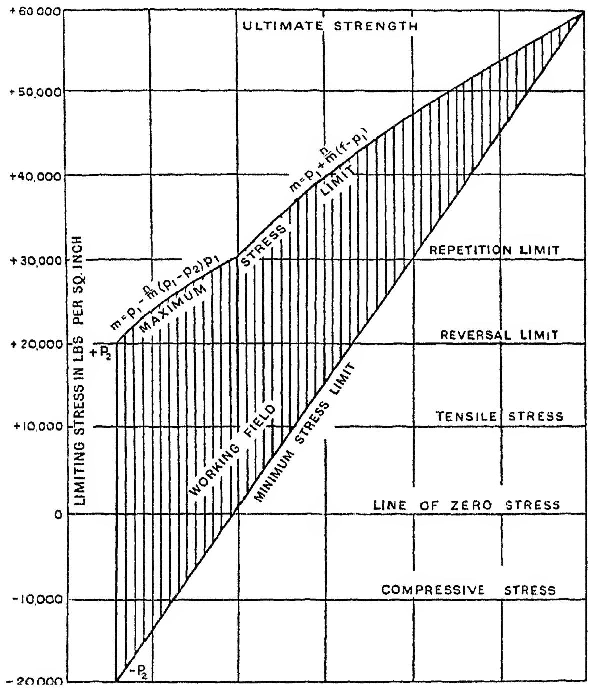


Fig. 251.

These two have been combined in the Launhardt-Weyrauch formula, which is intended to deal with both the above cases, and has thus a more general application than either of the individual formulæ mentioned above.

The combined formula is as follows, where f_{max} is the required limiting stress under which the material can withstand an

* In these diagrams m = maximum stress.
p = repetition limit when n = o.
n = minimum stress.
f = ultimate static strength.
(See Johnson's *Materials of Construction*, pp. 543, 544.)

unlimited number of repetitions or reversals of stress, f_1 is the value of f_{max}, or the repetition limit, when the lower stress f_{min} is zero, and f is the ultimate strength in tension under a steadily applied load. When f_{min} is compressive its sign changes from plus to minus.

$$f_{max}=f_1+(f-f_1)\frac{f_{min}}{f_{max}} \quad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

In the above formula it is assumed that

$$f_1=\tfrac{1}{2}(f+f_2) \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

where f_2 is the limiting stress for complete reversal.

In the case of the structural materials, mild steel and wrought iron, it has been found from the experimental results that $f_1=\frac{2}{3}f$ approximately, and inserting this value in (ii) the result is obtained that $f_2=\frac{1}{3}f$.

When these values are inserted in the combined formula, it becomes

$$f_{max}=\frac{2}{3}f\left(1+\frac{1}{2}\frac{f_{min}}{f_{max}}\right. \qquad \text{(iii)}$$

In Table I (page 506), containing factors of safety usually employed, it may be remembered that the factor chosen for structural steel and wrought iron subjected to steady load is 3.

Applying this to (iii) it is found that the working stress for repetition and reversal becomes

$$f_w=\frac{1}{3}f_{max} \text{ or } =\frac{2}{9}f\left(1+\frac{1}{2}\frac{f_{min}}{f_{max}}\right), \text{ or}$$

$$=\frac{2}{9}f\left(1+\frac{R}{2}\right),$$

where R is the ratio of minimum to maximum stress.

For example, take the case of mild steel under steady load :

here the working unit stress is $=\frac{2}{9}\times 27\left(1+\frac{R}{2}\right)$

$$=6\left(1+\frac{R}{2}\right) \qquad \text{(iv)}$$

when for simplicity f is taken as 27 tons per sq. in.

When the load is a steady one, the ratio of min. to max. is unity and the working stress attains its greatest possible value for this material, of 9 tons per sq. in. In many cases where the combined formula is employed a value 5 is used in place of the 6 above. Here the working unit stress

$$=5\left(1+\frac{R}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(v)}$$

This particular form of the combined equation is widely used both in Europe and in America, and yields results as follows:

When $R = 1$, working unit stress for steady load $= 7{\cdot}5$ tons per sq. in.

When $R = 0$, working unit stress for repeated load $= 5{\cdot}0$ tons per sq. in.

When $R = -1$, working unit stress for reversed load $= 2{\cdot}5$ tons per sq. in.

The repeated load mentioned above refers to repetition between zero and a maximum, and the reversed load applies to a complete reversal from a certain minus load to a plus load of the same numerical value.

In Cain's formula, widely used for repeated stresses without reversal,

$$\text{working unit stress} = \text{w. u. stress for dead loads} \times (1 + R).$$

From what has been said, and without further complication, it is clear that the process of fixing a suitable working stress for use with a certain material, which is being used under given conditions, is by no means a simple one, and it is not surprising that there are often wide differences between the values arrived at by different engineers.

In the first place about half a dozen unknown contingencies must be estimated and allowed for. Then comes the question of dead and live load. Further, the load may be repeated through a known range: or there may be a repetition of complete reversal of stress: or the material may be subject to blows, repeated either slowly or very quickly. Most of all of these points require consideration and the exercise of sound judgment in coming to decisions as to reasonable allowances to be made.

Existing knowledge as to the effect of repetition and reversal is in a not very satisfactory state, and yet engineers are content to go on making the best of a bad job, and for a great deal of their data rely on incomplete experimental results of more than forty years ago! At the same time it is a great testimonial to the extreme care with which these particular experiments were made that they are trusted so implicitly at the present time. Possibly it was not without bitter experience of calculating working stresses that Sir Maurice Fitzmaurice, in his Presidential Address to the Institution of Civil Engineers, referred to Engineering as an inexact science. It is certainly very largely made up of compromises.

The factors of safety given in Table I are seen to differ according to the manner of loading, as shown by the difference between the figures in the three vertical columns, in the manner

discussed. But there are also differences between the figures in the five horizontal lines of figures. These last are made necessary by the peculiarities of the four differing types of material. In the first column, containing the factors to be used for dead loads, the smallest factor is that of 3 for wrought iron and steel. This is easy to understand in view of the fact that a good deal more is known about the mechanical properties of wrought iron and steel than of any of the materials mentioned, and, in addition, these are materials whose guaranteed qualities may fairly be relied on. The factor for cast iron is a little higher at 4, and this increase can easily be understood when it is remembered that cast iron is not so uniform as wrought iron and steel, added to which there is always the possibility of unsound castings. In timber, whose qualities are notoriously lacking in uniformity, the factor is higher still at 7, a very reasonable decision. But why the factor for brickwork and masonry should be placed as high as 20 is not so clear, especially at the present time. These are not working stresses, but factors with which to reduce the ultimate strengths; and at the present time sufficient ought to be known about the ultimate strengths of these materials to allow a somewhat smaller factor to be used.

The combined formula given above, in which the safe working stress $= 6.\left(1+\frac{R}{2}\right)$, derived from $\frac{1}{3}.\frac{2}{3}.27.\left(1+\frac{R}{2}\right)$, may be considered as made up of three factors, and written as

$$= \left(\frac{1}{3}\right)\times(27)\times\frac{2}{3}\times\left(1+\frac{R}{2}\right), \text{ or}$$

$$= E \times Q \times L, \text{ where}$$

E is the exigency factor, which here $= \frac{1}{3}$,
Q is the quality factor, which here $= 27$, and
L is the loading factor, which here $= \frac{2}{3}\left(1+\frac{R}{2}\right)$.

The distinction will be appreciated on reference to Table I.

Here the quality factor is the ultimate tensile strength of the material, the exigency factor is the $\frac{1}{3}$ already discussed as being intended to allow for a number of unknown or partly known conditions already specified, and the factor concerned with the manner of loading is $\frac{2}{3}.\left(1+\frac{R}{2}\right)$.

If the exigency factor, E, is allowed the same value for all kinds of material, which seems fairly reasonable, in view of the existence of the quality factor, it will have the uniform value of $\frac{1}{3}$, and the remaining factors depend only on the ultimate strength and the effect of the manner of loading.

The remaining factors which enter into the composition of the working unit stress are the constant $\frac{1}{3}$, the ultimate strength and the $\frac{1}{3}\left(1+\frac{R}{2}\right)$, called here the "loading" factor. This last depends for its value on the manner of loading which is used, whether dead, repeated, or reversed. The product of the three factors just referred to and the ultimate strength gives the required safe working stress.

The employment of a factor of safety with which to provide a suitable safe working stress to resist blows has been mentioned; but as this is a matter for special consideration and not capable of being settled by the use of a single coefficient, there is no need for further reference to it.

The formulæ described and referred to above, or similar formulæ, have been elaborated and largely used in calculating sectional areas of the parts of steel railway bridges, with the aid of an "impact coefficient." This very important subject provides material for a much longer special discussion than is possible here. Although the experiments on repetition of stress mentioned above have not been repeated within recent years, a large number of endurance experiments with complete reversal of stress have been carried out by Dr. Stanton and Mr. Bairstow; Messrs. Eden, Rose, and Cunningham; by Professor Hopkinson; and by the writers of this book. These later results, though not providing direct information on stress repeated over fixed ranges, do at the same time give some indirect help.

Stresses in Railway Girders—In calculating the dimensions of a tension bar of a steel bridge or of one of its main booms, several methods are employed, the rules being for the most part different in the various countries. The first matter of real importance for the consideration of the designing engineer is that of the nature and magnitude of the loads imposed on each part of the bridge. These loads in the case of railway bridges are due to the pressures of the locomotive wheels on the rails which are laid across the bridge, to the weight of the structure itself, as well as to straining actions caused by unsymmetrical loading and unforeseen twists. These extra loads are in some cases due to the attachment of the cross-girders when such are employed, instead of carrying the rails directly along the tops of the main girders.

When the actual maximum loads likely to come on any and every part of the bridge have been ascertained, the forces which they cause in the individual members of the bridge, the boom loads from the bending moments, and the loads in the diagonal bracing bars from the shearing forces, can be calculated.

If the loads so calculated were all due to steadily applied wheel loads the parts could be dimensioned quite simply by dividing all loads by a suitable factor of safety and thus obtaining the area of cross-section. But all that part of the external load on a railway bridge which is not the weight of the bridge itself is a moving load. The mere fact of its being a moving load and not a dead load has considerable influence on the calculation of the dimensions, and, in addition, a good deal depends on the character of the moving load, as to whether it moves slowly or quickly, smoothly or bumpily, or whether there is likely to be anything of direct impact. In defining these various points a great deal of knowledge of the actual conditions is necessary and a fund of sound judgment.

There are two principal ways of dealing with the question of impact in steel bridges. Both are only approximate and full of compromise.

At one time it was common practice in Great Britain to use a low safe stress covered by a high factor of safety, say 6·5 tons per sq. in., and to ignore all loads to which the word impact could possibly be applied. This plan had the advantage of delightful simplicity, but on the whole it probably erred in the direction of making stresses too low, thus causing excessive consumption of material with the consequent monetary waste.

As an improvement on this simple plan, the two types of method mentioned above have come into use.

One of these, which has been called the "range" plan, is to calculate the working stress by means of one of the formulæ based on the repetition and reversal experiments.

For bridge work this range of stress formula is less frequently used than one of the impact formulæ. The choice of method depends on the custom followed in individual offices, most of which are the offices of railway companies, though there may be a number of private firms as well.

The range-of-stress type of formula is framed on the assumption that every part of a steel railway bridge is subjected to repeated stresses which take place between limits that can be ascertained with a fair amount of certainty in each given case, and also on the assumption that such parts would behave in a similar manner to the bars tested under repetition of stress by Bauschinger and Wöhler and confirmed by Baker.

Launhardt's and Wöhler's combined formula may also be given as safe working stress $= 4{\cdot}66\left(1 + \frac{R}{2}\right)$ where R is (i) the ratio of the minimum stress to the maximum stress on the part in question. This is the formula employed by one of the British

Railways and is of the typical shape of many others used for similar purposes.

Another Railway uses the following formula, which is found to give results which are not widely different from those of (i).

$$\text{Safe working stress} = \frac{4}{\left(1 - \frac{R}{2}\right)} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Another British Railway recognizes no impact and no stress variation in so definite a manner as to be allowed for in a formula, but overrides all such considerations by using constant stresses for specified purposes, as the following figures show:

Safe working stress for	main girders	= 6·5 tons per sq. in.
,, ,, ,, ,,	cross-girders	= 5·0 ,, ,, ,,
,, ,, ,, ,,	rail bearers	= 4·0 ,, ,, ,,

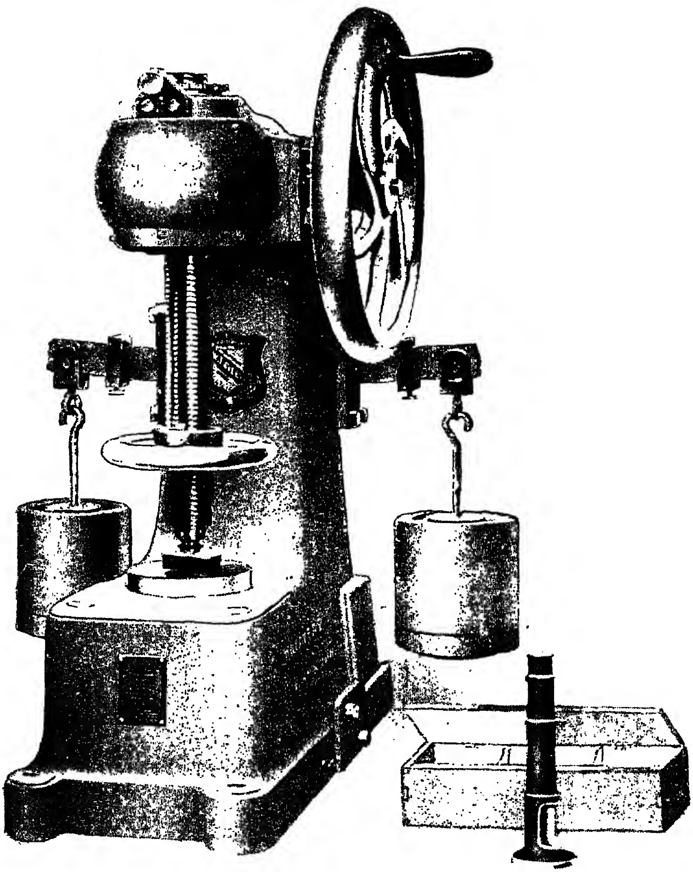
In bridges designed after the three plans above the moving load is reduced to its equivalent dead load and added to the dead load arising from the weight of the structure. This will give the maximum load per ft. of span, and with it may be calculated the maximum forces endured by the different parts of the bridge, some of which will be tensile and some compressive. Next, the forces in all the same parts caused by the minimum load, which is the dead load due to the weight of the structure, is calculated. The minimum and maximum forces in all parts will now be known, and the ratio $R = \frac{\text{Min.}}{\text{Max.}}$ can be found in each case, and the safe stresses calculated from (i) or (ii). In this connexion it is to be noted that some of the forces may be minus.

Where the third of the above plans is adopted it becomes simple to take the safe stresses already fixed, and, making use of them, calculate the sizes of the parts in the usual way.

In all cases so far mentioned the term "impact" has been ignored, but in many railways in Britain, India, and America it is made the chief factor in bridge design.*

* Anderson on Railway Girders: contribution to the Discussion by Mr. C. Gribble. M.I.C.E., vol. cc.

PLATE XXX.



MACHINE FOR BRINELL TEST (AVERY)

CHAPTER XXI

HARDNESS

Tests for determining the relative hardness of a number of engineering materials are based on the amount of force needed to displace a portion of the material from the surfaces of the specimens, either by ploughing a scratch or by making an indentation.

226. The Scratch Test—This is generally considered to provide the most satisfactory gauge for hardness, partly because it may be applied to materials of all kinds, from the hardest to the softest.

In Professor Turner's Sclerometer * the scratch is made by drawing the sample across a scratching diamond, which is fixed near one end of a balanced horizontal lever. The sample is drawn across the diamond, a small weight being added to give the necessary pressure. A predetermined standard scratch having been decided upon, the width of the given scratch is measured with a micrometer microscope and the weights varied until the standard width is arrived at, so that some multiple of this weight provides the hardness number.

There are other and cruder forms of the same instrument. In some of these the weight is a constant one and the hardness number is a multiple of the width of the scratch. So that the hardness number in the scratch test depends on the *width* of a groove.

In the *Brinell Test*, now very generally used, the circular indentation caused by a hardened steel ball 10 mm. diameter being forced into the surface by a load of 3,000 kilogrammes, is measured. From the *diameter* thus obtained and the diameter of the ball, it is possible to calculate the segmental spherical area of the depression. Call this A and the load P. Then the Brinell hardness number is $B=\frac{P}{A}$ where A is in sq. mm. For softer materials, as copper and brass, a lighter load of 500 kilogrammes is employed.

The load may be applied in any convenient machine adapted to tests in compression, though there are a few machines on the market specially designed for this purpose, such as that made by Messrs. Avery (Plate XXX).

* Johnson's "Materials," p. 384.

The diameter of the impression must be measured with a micrometer microscope.

The area A of the curved surface of the depression is as follows where r is the radius of the ball and D the diameter of the depression :

$$A = 2\pi r\left(r - \sqrt{r^2 - \frac{D}{4}}\right)$$

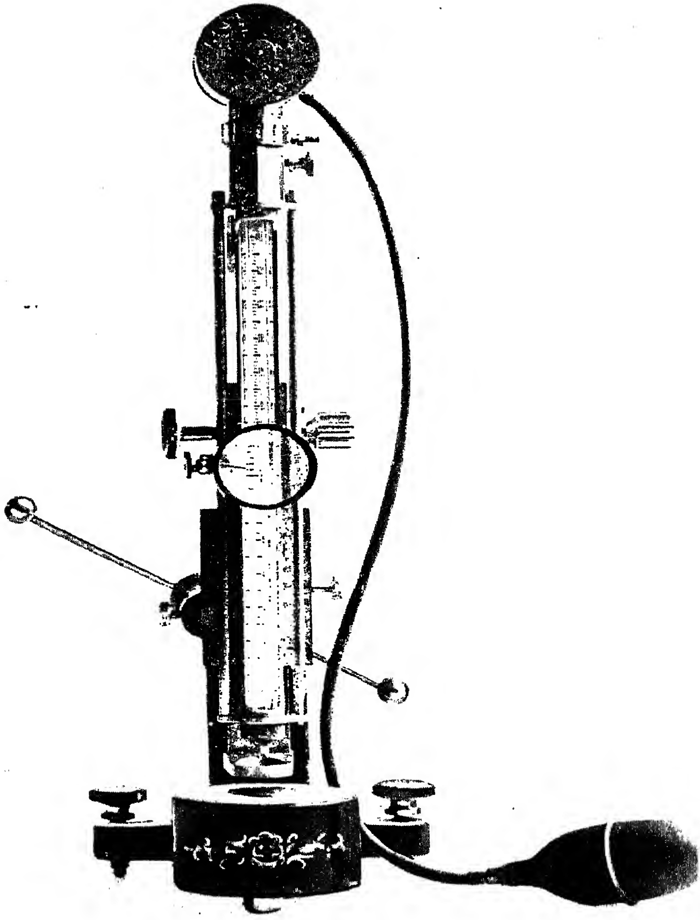
Avery's Brinell machine consists of a system of levers which carry the pressure applied by the straining screw. The load is indicated by an upward movement of balance weights, at which point the pressure is stopped. Any further pressure applied through accident or otherwise cannot increase the pressure on the specimen by reason of the action of the levers being arranged to take either 500 or 3,000 kilos and no more. For the 500-kilo test weights of this denomination only are hung from the steelyard.

The following table of Brinell numbers will be found useful :

TABLE OF HARDNESS NUMERALS
Steel Ball of 10 mm. diameter

Diameter of impression mm.	Hardness numeral Pressure kg.		Diameter of impression mm.	Hardness numeral Pressure kg.		Diameter of impression mm.	Hardness numeral Pressure kg.		Diameter of impression mm.	Hardness numeral Pressure kg.	
	3,000	500		3,000	500		3,000	500		3,000	500
2.—	*946*	158	3.25	*351*	59	4.50	*179*	29.7	5.75	*105*	17.5
2.05	*898*	150	3.30	*340*	57	4.55	*174*	29.1	5.80	*103*	17.2
2.10	*857*	143	3.35	*332*	55	4.60	*170*	28.4	5.85	*101*	16.9
2.15	*817*	136	3.40	*321*	54	4.65	*166*	27.8	5.90	*99*	16.6
2.20	*782*	130	3.45	*311*	52	4.70	*163*	27.2	5.95	*97*	16.2
2.25	*744*	124	3.50	*302*	50	4.75	*159*	26.5	6.—	*95*	15.9
2.30	*713*	119	3.55	*293*	49	4.80	*156*	25.9	6.05	*94*	15.6
2.35	*683*	114	3.60	*286*	48	4.85	*153*	25.4	6.10	*92*	15.3
2.40	*652*	109	3.65	*277*	46	4.90	*149*	24.9	6.15	*90*	15.1
2.45	*627*	105	3.70	*269*	45	4.95	*146*	24.4	6.20	*89*	14.8
2.50	*600*	100	3.75	*262*	44	5.—	*143*	23.8	6.25	*87*	14.5
2.55	*578*	96	3.80	*255*	43	5.05	*140*	23.3	6.30	*86*	14.3
2.60	*555*	93	3.85	*248*	41	5.10	*137*	22.8	6.35	*84*	14
2.65	*532*	89	3.90	*241*	40	5.15	*134*	22.3	6.40	*82*	13.8
2.70	*512*	86	3.95	*235*	39	5.20	*131*	21.8	6.45	*81*	13.5
2.75	*495*	83	4.—	*228*	38	5.25	*128*	21.5	6.50	*80*	13.3
2.80	*477*	80	4.05	*223*	37	5.30	*126*	21	6.55	*79*	13.1
2.85	*460*	77	4.10	*217*	36	5.35	*124*	20.6	6.60	*77*	12.8
2.90	*444*	74	4.15	*212*	35	5.40	*121*	20.1	6.65	*76*	12.6
2.95	*430*	73	4.20	*207*	34.5	5.45	*118*	19.7	6.70	*74*	12.4
3.—	*418*	70	4.25	*202*	33.6	5.50	*116*	19.3	6.75	*73*	12.2
3.05	*402*	67	4.30	*196*	32.6	5.55	*114*	19	6.80	*71.5*	11.9
3.10	*387*	65	4.35	*192*	32	5.60	*112*	18.6	6.85	*70*	11.7
3.15	*375*	63	4.40	*187*	31.2	5.65	*109*	18.2	6.90	*69*	11.5
3.20	*364*	61	4.45	*183*	30.4	5.70	*107*	17.8	6.95	*68*	11.3

PLATE XXXI.



SHORE SCHLEROSCOPE

The machine will test specimens up to a maximum height of 12 in. The distance from the standard to the pressure ball is 6 in., therefore allowing for a diameter of specimen of 12 in. Larger specimens may be tested out of centre. The pressure ball is so arranged that it can be easily removed for inspection.

A section of the Brinell indentation is shown on Fig. 252.

Chisel Indentation Test—This test is made by forcing a chisel of the form shown on Fig. 253 against the surface of the metal. It is convenient to grind the edge to a curve whose radius is 1 in. and to give the edge an angle of 60 deg. The indentation made is of the form shown on Fig. 254. Relative hardness is

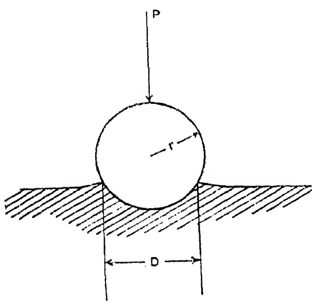
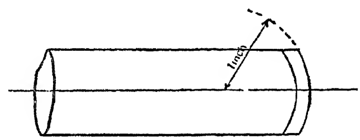
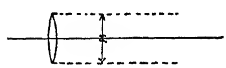


Fig. 252.—Brinell Indentation, Section through Specimen.



INDENTING CHISEL
Fig. 253.



INDENTATION
Fig. 254.

obtained by comparing the length H of the indentations. These lengths are easily measured. For comparative work this is a very useful method.

In the Martens Hardness Machine the specimen is indented by a steel ball 2 mm. in diameter. The load is applied by water pressure on a flexible diaphragm.

In this case the *depth* of the indentation is indicated by the height of a capillary mercury column. The relative movement of the ball and the surface of the specimen is utilized to displace the mercury. The depth of depression may be read to a small fraction of a millimetre.

In the "Scleroscope" a diamond is embedded in a piece of steel about as large as a revolver bullet and is allowed to fall so as to strike the sample. The height of fall is always the same, but the height of rebound is greater with the harder than with the softer specimens. The height of this rebound is used as a measure of the hardness.

The main defect of this device seems to be that the rebound

is affected by the nature of the bed upon which the sample rests when struck, and also by the size of the test pieces (Plate XXXI).

In the following table are complete hardness scales taken with each of the three methods ("Sclerometer," "Scleroscope," and "Brinell" test), collected by Mr. C. A. M. Smith * :

COMPARATIVE TABLE OF HARDNESS SCALES

Metal.	Sclerometer.	Scleroscope.	Brinell.
Lead	1·0	1·0	1·0
Tin	2·5	3·0	2·5
Zinc	6·0	7·0	7·5
Copper, soft	8·0	8·0	—
,, hard	—	12·0	12·0
Softest iron	15·0	—	14·5
Mild steel	21·0	22·0	16–24
Soft cast iron	21–24	24·0	24·0
Rail steel	24·0	27·0	26·35
Hard cast iron	36·0	40·0	35·0
Hard white iron	72·0	70·0	75·0
Hardened steel	—	95·0	93·0

These figures appear to show that the three systems result in scales not very dissimilar.

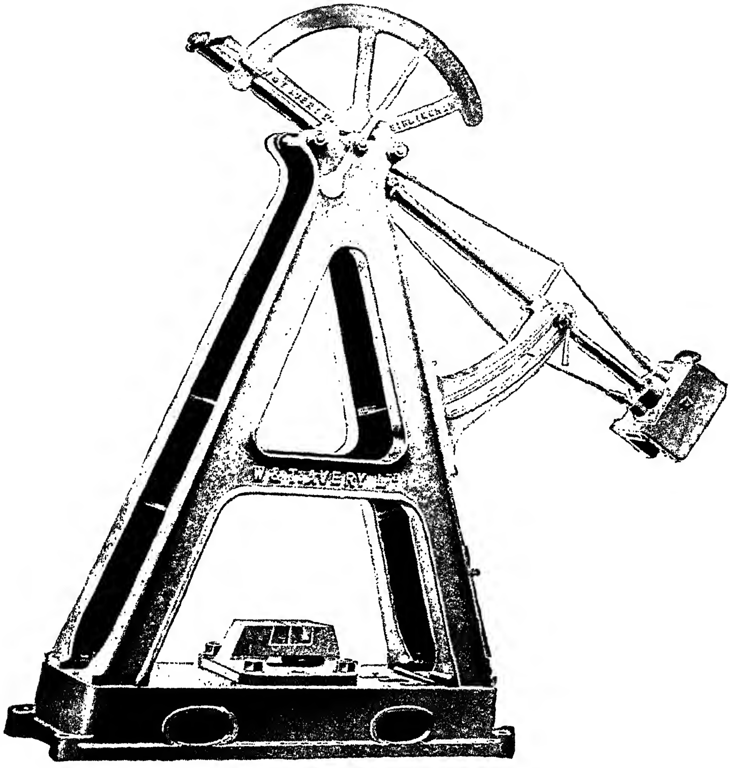
227. Drilling Tests for Hardness—This is a shop test carried out by comparing the weight of metal removed by a drill of prescribed size when being driven into the metal at a certain speed. It is often used in comparing cast irons.

† IZOD IMPACT TESTING MACHINE—The machine as made by Messrs. W. & T. Avery is shown in Plate XXXII (*a*). A heavy pendulum is allowed to swing through an angle of about 120 deg. and when at its lowest point to strike the specimen, which is gripped in a vice fixed to the baseplate of the machine (Plate XXXII (*b*)). It will be evident that after breaking the specimen, the height through which the pendulum rises will be less than that through which it fell, the difference representing the work required to break the specimen, neglecting losses at impact. This difference is registered on a graduated arc fixed near the top of the machine. The pendulum is fitted with a hardened knife-edge at its centre of percussion, which strikes the specimen.

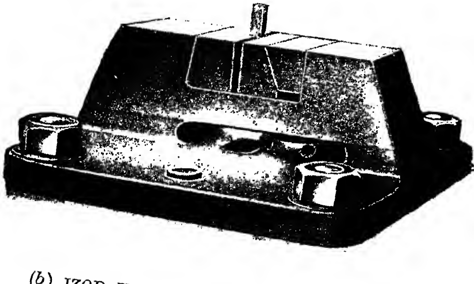
* From "Handbook on Testing," C. A. M. Smith.

† "Engineering," September 25, 1903. See also "Proc. Inst. Mech. Eng." (1908).

PLATE XXXII.



(*a*) IZOD IMPACT MACHINE



(*b*) IZOD IMPACT MACHINE BAR HOLDERS

Two standard specimens are shown in Fig. 255. Gauges are provided with the machine, to which the notches should be cut, and it will be evident that the fillets at the bottom of the notches should be as exact as possible. Gauges are also provided for fixing the specimens in the machine, so that the notch shall be on a level with the top of the vice and correctly facing the direction from which the pendulum will fall.

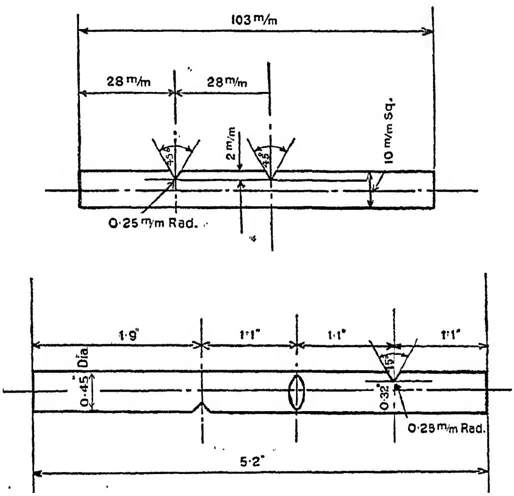


Fig. 255.—Izod Test Pieces.

The test is not very satisfactory in the case of ductile materials such as mild steel and wrought iron, as the specimens sometimes fail to break or the fractures are often very irregular. In such cases the results are liable to be somewhat inconsistent. The machine is hardly sensitive enough for testing cast iron, as in most cases the reading is inappreciable.

On pp. 522–3 are given some Izod, Tensile, and Brinell tests on two bars (A and B) of mild case-hardening steel containing about 0·15 per cent. C and 0·5 to 0·6 per cent. Mn. Before being turned the test pieces were baked in lime at 900° C. for three hours, and then allowed to cool in the furnace with everything closed. They were then heated to 820° C. and quenched in water, and reheated to 750° C. and quenched in water; and then machined to standard dimensions.

TENSILE TESTS

Mark on Specimen.	Izod Tests. Energy Abs. (ft. lb.)	Dia. (in.)	Yield Point. (tons sq. in.)	Max. (tons sq. in.)	Elong. per cent. on 2 in.	Brinell No. (3,000 kg. load).
A . .	78·0 88·8 80·2 82·2 78·9	0·564	28·8	38·7	31·2	309
B . .	72·3 104·1 97·0 69·4 97·8	0·564	26·0	35·3	33·8	340

The following table shows the degree of variation given by Tensile, Sankey, and Izod tests on test pieces cut from one bar of mild steel, which had been annealed at about 910° C.

		Probable Variation from Mean (per cent.) Peter's Formula.	Probable Variation from Mean (per cent.) Bessel's Formula.
Tensile (9 tests)	Diameter	0·058	0·071
	Area	0·126	0·141
	Yield point	6·22	5·63
	Maximum stress	0·488	0·477
	Elongation	2·15	1·98
Sankey (6 tests)	Number of bends to fracture	3·84	3·64
	Energy expended to fracture	3·92	3·76
	Yield point	5·47	5·03
	Maximum stress	1·23	1·18
	Izod impact test (18 tests)	14·5	14·2

The following figures are quoted by Messrs. W. & T. Avery and are interesting as showing results of ordinary tensile tests, Brinell tests, and Izod impact tests for four well-known steels. The correlation between these three tests has not yet been determined.

	Yield Point.	Ult. Stress.	Elongation.	R/A.	Average Izod Figure.	Brinell Figure.
	Tons per sq. in.	Tons per sq. in.	Per cent.	Per cent.		
Alloy steel	45·6	51·5	26·6	66·6	80	3·9 Different treatments
	49·0	57·6	25·2	62·5	59	3·8 Different treatments
Mild case-hardening steel	31·56	42·2	28·5	55·6	70	4·3 Different treatments
	30·88	42·72	27·0	52·0	43	4·35 Different treatments
Nickel chrome steel	47·36	53·32	22·0	62·8	27	3·7 Bad treatment
	55·32	61·72	20·0	58·5	55	3·5 Re-treated
	51·44	59·76	21·0	58·5	24	3·65 Bad treatment
	59·48	64·92	20·0	58·5	48	3·5 Re-treated
Nickel chrome steel	44·0	59·6	22·7	60·1	63·5	— Bar No. 1.
	43·1	58·1	22·7	56·7	19·2	— ,, ,, 2
	45·8	61·1	22·7	59·4	60·2	— ,, ,, 3

The above machine has a capacity of 120 ft. lb., but Messrs. Avery also make a smaller one of 73 ft. lb. capacity.

CHAPTER XXII

LIFE AND ENDURANCE OF MATERIALS—LOCAL FAILURES

THE feeling in the minds of most people when they see the completion of a new engineering structure or machine is that it is permanent and likely to remain so, apart from accidents. Such an idea is obviously wrong. No known material is likely to be everlasting, for one or more of several reasons.

The life of a new material may be imperilled by changes in its internal structure, or the member may become worn or damaged.

Timber suffers from both these causes. Its structure is cellular, and this may be destroyed by fungus growths which are the causes of ordinary and dry rot, by the attacks of minute insects, as in the "worming" effect in old furniture by which a number of small holes appear on the outside surface. Costly damage is often done to the timber used on marine works, especially where these are in tropical countries, owing to the ravages of certain marine insects. Many of the strength and elastic properties are influenced by the amount of moisture present in timber. All timber should be thoroughly well seasoned, natural seasoning being better than stove seasoning for the subsequent life of the wood, though slower. Recent experiments (by Mr. Carrington) show that the modulus of elasticity attains its highest value as the percentage of moisture gets low—say 3 per cent. or thereabouts. When perfect dryness is attained, the cellular structure becomes less perfect, and the modulus falls again. Wood does certainly seem to have a very long life if kept dry, as seen in the cases of pieces of very old domestic furniture and church fittings. The age of these runs to many centuries. In the thousands of years of ancient history, Egyptian relics of wood seem to have entirely disappeared. Some timbers survive best when kept wet at all times. Instances of such longevity were found when the piles supporting the Old London Bridge were withdrawn from the wet clay in which they had lain for centuries.

In the case of steel and some other metals, the chemical reactions involved in the processes of manufacture or the after-treatment may not be complete when the material is used, but

very long periods of time may be required to effect completion. However, internal changes take place very slowly, with ultimate results not necessarily injurious to the material. In the case of some copper alloys internal changes are more rapid and often disastrous.

Stone and brick offer great resistance so long as the atmosphere is warm and dry.

The worst effects on materials of this nature are caused by the absorption of moisture and its subsequent freezing. This results in cracks and splitting off of the surface, as in the case of bricks. The final result to the brickwork lining wet culverts may be very bad.

Another effect on some of the softer building stones is corrosion due to the minute quantities of acids to be found in the atmosphere. These slowly eat into the surface of the stone, and so cause a gradual crumbling away of the material. A more rigorous cause of surface disintegration is the purely mechanical result of the beating of rain combined with alternations of temperature. These wearing results of the weather on smooth surfaces come under the general head of "weathering," though this term more properly suggests the purely mechanical effects. The harder stones, like the granites, resist the mechanical effects of weather, but are unable to withstand the attacks of certain acids. Under normal conditions there are no internal changes taking place in building and engineering stones.

On the other hand, certain gases exist in very small quantities, even in fairly normal atmospheres, whose presence may be actually beneficial. One example of this is to be found in the hardening effect of CO_2 on stones and mortars containing an excess of CaO, and so producing a carbonate from an oxide, that is, a material like limestone instead of slaked lime.

From the above it may be assumed that the irons and steels are to be regarded as permanent substances during their normal engineering life. A few of the copper alloys are less permanent and stable. Timber is permanent for 100 years or more unless attacked by animal life or affected by moisture.

Masonry is permanent, but may be subject to surface effects. Mortars and stones containing calcium oxide are hardened by the carbon dioxide in the atmosphere.

Much harm may be caused in engineering members by "wear." This occurs in all cases where there is mechanical abrasion taking place. Mechanical wear in lubricated surfaces means loss of material and looseness in the fit. The only cure for this is better lubrication, and there are signs that attempts to produce naturally formed films will become more general. The abrasive wear of artificial paving flags is a subject to be taken into account by

their manufacturers. Great care has to be taken to secure a proper ratio of isolated hard pieces to the softer matrix material. If care is not taken the latter wears away too quickly, and leaves the former very prominent as isolated knobs.

In order to secure protection from decay of surfaces, protective coverings are often applied.

The black oxide or scale which forms on the surface of iron and steel after being heated to redness and exposed to the atmosphere is in itself a protective covering, but easily cracks and comes away from the surface when the metal is disturbed by blows or vibration. One result of further oxidation is the orange-coloured oxide Fe_2O_3, commonly known as iron rust, which shows itself, especially where the conditions are damp.

Iron surfaces may be protected from the rust-producing damp by metallic coatings, as in the zinc used in galvanizing iron and the tin-coated sheet iron known as tin plate. In neither case must the coating be broken or moisture will penetrate to the iron and corrosion set in. A third means of surface protection is by means of a coating of paint, which is really a coating of boiled linseed oil which oxidizes in the process of "drying" and forms a non-permanent protective covering. The colour in paint is used both for æsthetic reasons and to modify the appearance of the white lead which forms the body of the paint and helps in the drying. Where the position is an exposed one, the paint must be renewed at regular intervals of time. In the case of a large structure like the Forth Bridge there is a permanent staff of painters; when these have covered the bridge it is time to begin again. Of late years many patent paints, as aluminium paint, have been introduced.

228. Corrosion due to Electrical Causes—When two dissimilar metals are adjacent and exposed to the action of some acid an electrical current will be set up, with the result that the electrically negative of the two will be slowly wasted away. Steel boilers suffer acutely from such effects, especially where there are copper fireboxes.

A case of electrical corrosion which came under the writers' notice was that of the case-hardened pins of the conveyor chain used in a gasworks for removing the newly discharged and quenched coke. The flat links were evidently electrically positive to the harder surfaces of the pins, because it was found that the latter were badly eaten away where they touched the links. This effect was so striking that the pins bore the appearance of having been partly sheared under the pull of the chain (Fig. 256).

229. Local Failures—A designer must be careful that the smaller parts of large members are as strong as the structure as a

whole. The old saying about a chain and one of its links is still apt. Such failures may happen in large struts. A steel tubular strut may be well able to carry the general load on its middle

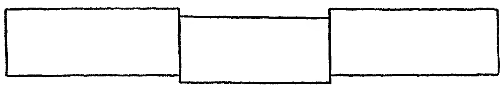


Fig. 256.

portion, but its ends may still suffer crinkling under the load. Reinforced-concrete members often suffer locally, also it is noticed that timber struts, such as pit props, give way by shearing close to the ends.

CHAPTER XXIII

OPTICAL METHOD OF SHOWING VARIATION OF STRESS

It has been pointed out in a former chapter that where the precise shape of a structural part is known, as well as the magnitude of the load which it carries, it is possible in many cases to compute the intensity of the stress at each point. Where the structural member is of a simple form, as a beam, shaft, or cylinder, the calculation presents no difficulty, and a complete graph in which the stress intensity is shown can be drawn without much trouble, but where the shape of the member is abnormal, or the load is of a complex nature, the matter can be simplified by a device which enables the variation of the stress to be seen at a glance. Such a plan has been developed by Professor Coker, F.R.S., of University College. In doing this he uses structural parts of transparent material and when these are loaded he exposes them to beams of polarized light. The effect of this is to produce a series of coloured bands across the loaded member, the variations in tint denoting different intensities of stress.

In order to find out how the tints corresponded to known stresses, Professor Coker loaded small tension bars of the same material and exposed them to polarized light, with the result that each bar assumed a uniform tint. The weight hanging on the bar was known as well as its dimensions, so that it was possible to calculate the stress. Such bars were made of parallel form and sufficiently long to cause the stress to be uniform.

In the table on page 529 Professor Coker gives the figures for the case of a beam subjected to a uniform bending moment. *

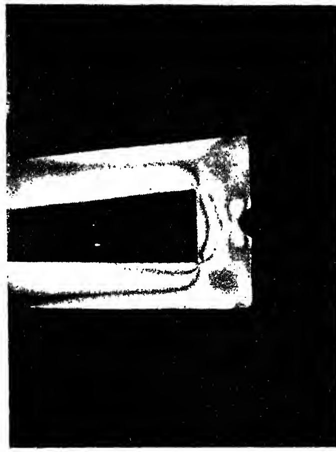
With a further increase of stress the scale of colours is repeated approximately for twice the intensity of stress. This is equally true for tension and compression.

On Plate XXXIII (*a*) is shown a black-and-white print of a transparent beam loaded so as to give compressive stress at the upper surface, tension at the lower surface, and zero on the horizontal central plane. In the original view showing the coloured

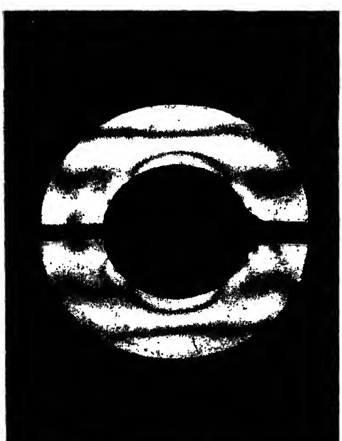
* "Inst. Mech. Eng. Min. Proc.," February 10, 1913. Lecture on Applications of Polarized Light (Coker).

PLATE XXXIII.

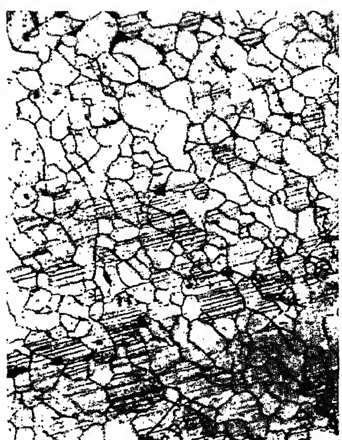
(*a*) UNIFORM BENDING MOMENT (COKER'S EXPERIMENT)



(*b*) SHAPED SECTION WITH THE LONG SIDES SQUEEZED TOGETHER (COKER'S EXPERIMENT)



(*c*) CIRCULAR LINK (COKER'S EXPERIMENT)



(*d*) MICROSCOPIC ANALYSIS OF STEEL : FERRITE

bands these are, from centre to upper surface, white, yellow, purple, blue, the blue indicating the maximum stress and the white zero. Below the neutral surface the ranges of colour are the same, but the stresses are of the opposite kind. From centre to upper surface the bands are white, darker, white, darker, and white again. The meaning of this arrangement of shading seems to suggest that there are at least two repetitions of "order" or "octave."

On Plate XXXIII (*b*) is shown a very striking example. This represents part of a U-shaped section with the free ends forcibly squeezed together. In this view the greatest stresses are shown by the darker bands, which, in the original, are dark blue.

Comparative Table of Tension and Compression Stresses corresponding to a Given Colour

Order.	Colour.	Stress.
I	Black	0
	Grey	3·5
	White	5·5
	Straw	8
	Orange	10
	Brick red	10·5
	Purple	11
II	Blue	13
	Yellow	18
	Red	21
	Purple	22

On Plate XXXIII (*c*) is shown a view of a black-and-white photographic print taken from a transparent ring under a vertical extending pull.

The variations of tension stress on the inside of the ring are shown remarkably well by the two crescent-shaped dark lines. In the original appearance, the highest and lowest points are strongly marked by blue areas.

For the purposes described glass is too brittle and too easily fractured and it has been found that the most suitable material is one of the nitro-celluloses such as xylonite. These carry the loads without fracture and are to some extent ductile.

Thus it should be possible when a difficulty as to stress variation occurs in connexion with the loading of some particular member to make a model of the member in the above material and load it as nearly as possible in the same way as the original design. If this model is now exposed to a beam of polarized light, the coloured bands produced will show how the stress varies from point to point. It is then reasonable to assume that

the stress variation in the original member from which the model has been copied is of the same nature as in the model.

230. Microscopic Examination of Metals (chiefly of Steel)—In what has been said previously, the general strength properties of finished engineering material have been examined. These include the finding of the P limit, the yield point, and the ultimate strength.

It now remains to discuss briefly the manner in which these metals can be examined microscopically with a view to ascertaining in what way they are built up and how their constituent materials are disposed.

It is not possible within the present limits to enter into a complete discussion of the microstructure of metals, but merely to describe the principal facts which have been discovered by observers, and to note the methods which have been employed.

231. General Constitution of Steel—When a sample of smoothly polished steel is observed through a powerful microscope

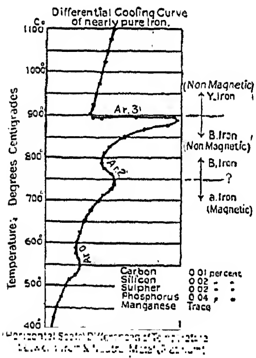


FIG. 257.

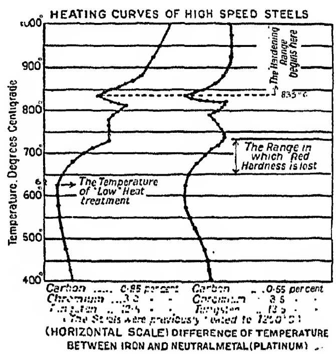


FIG. 258.

it is seen in the first place that the field of view is divided up into a series of polygonal areas which are sections through the metallic crystals of which the steel is made up. When molten steel is allowed to cool, its constitution undergoes a number of changes as the temperature falls. The principal points to be noted are the temperature when solidification begins and the temperature of recalescence, which occurs after solidification. When the second of the above points is arrived at there is a large quantity of heat given out. The changes which take place during the cooling of pure iron are shown on Fig. 257.

The heating curve for a sample of high-speed steel as obtained by Professor Carpenter is shown on Fig. 258. The chief constituent of steel is ferrite, which is practically pure iron (Plate

XXXIII (*d*)). In addition to this, the next constituent to be noted is cementite, which is a carbide of iron having the formula Fe_3C. In the actual steel the ferrite and cementite occur together in different proportions, and with their particles arranged according to a number of different formations. In Fig. 259 (p. 534) is a view showing the beginning of the formation of the eutectic alloy of ferrite and cementite which is known as pearlite.

With the magnification of, say, 300 to 1, the polished surface of steel presents the appearance of being divided up into a series of polygons with rounded corners and lying together somewhat like the counties on a map.

These areas represent plain sections of the crystal grains of which the steel is constituted. Sometimes between these grains may be thin layers of one of the substances just mentioned.

In order to be able to make it possible to appreciate this subdivision, it is necessary to polish the surface of the specimen and afterwards etch it. The size of specimen frequently used for steel is a square piece with about 1 in. length of side and $\frac{1}{4}$ in. thick. Both larger and smaller sizes are also used. When a specimen has been cut out of the parent mass for purposes of examination, the flat surface is ground . The object in grinding is to remove all irregularities and grooves or scratches from the surface, and this must be done in stages. First, the surface is filed with coarse and fine files until the smoothest surface which can be attained by means of a file has been reached. The surface is then further ground with a progressive series of emery cloth and emery paper. The final stage in the polishing is effected generally in some kind of grinding machine. The machine used by Stead consists of a disc over which is stretched khaki cloth. The cloth is held in position by means of a ring which fits round the disc and between the layers of the cloth is placed a quantity of diamantine powder or some other kind of polishing powder. Stead's machine runs about 1,000 revolutions per minute, and it is arranged that water can flow over the polishing surface of the block. The specimen is held between fingers and thumb against the face of the rotating block, and so is drawn slowly across it. The pressure should be gradually reduced. By following out this process a satisfactory polish can be obtained with practice. The same effect may be obtained by the more laborious process of hand-polishing. It should be noted that the harder the substance, the more easy it is to obtain a satisfactory polish. When viewed through the microscope, the polished surface will have somewhat of the appearance already described, but it will not be easy clearly to distinguish one crystal from another or one part of a crystal from another part of the same crystal. In order to emphasize these differences, it is usual to etch slightly

the surface of the metal. Different etching reagents are used by different observers, each being intended to produce a different effect. In order to reveal the presence of phosphorus in steel a 10 per cent. solution of copper ammonium chloride is used. For the rapid development of carbon steels a 5 per cent. solution of picric acid in alcohol plus 2 per cent. of nitric acid has been used. Other etching solutions which have been used with more or less success are ferric chloride for brasses and bronzes ; 3 per cent. solution of sulphuric acid in water to be used when taking auto-prints, a 1 per cent. solution of nitric acid in isoamyl-alcohol, and a 4 per cent. solution of nitric acid in isoamyl-alcohol. When examining brittle and non-brittle steels, especially when establishing a difference between these, a 4 per cent. solution of nitric acid in isoamyl-alcohol has been found useful, as also a 20 per cent. solution of hydrochloric acid in isoamyl-alcohol along with one-third of its volume of a saturated solution of nitranylene or of nitrophenol. Etching with iodine is also used.

The result of exposing the polished surfaces to the effects of these etching solutions is often to accentuate the contrasts of the tints of the constituent areas.

Another way of doing this is by what Osmond calls polishing in bas-relief. This consists in rubbing in such a way as to remove more material from the soft than from the hard portions of the exposed areas, so as to make the latter stand out in relief. The result is produced by rubbing on parchment roughly powdered with rouge. The bas-relief effect when exposed to inclined illumination is to produce shadow lines along the boundaries.

With high magnification much of the detail is accentuated. A similar effect is sometimes seen in artificial paving flags in which the constituent materials have not been well proportioned. The effect of the incessant tramp of feet is to wear away the softer cement mortar and to leave the granite lumps standing somewhat above the general level.

The reader should study carefully the work of the French scientist, Osmond.* A thorough knowledge of microscopic metallurgy can be acquired only by means of years of study and practice.

232. Summary of Constituents of Steel—FERRITE—This is the chief constituent of carburized iron and consists of what is practically pure iron. The ferrite as seen through the microscope appears to have a specular polish, when the specimen has been prepared in "bas-relief" (Plate XXXIII (*d*)).

CEMENTITE AND PEARLITE—Cementite is a carbide of iron and forms another of the constituents of the steel. From what

* "Microscopic Analysis of Metals," by Floris Osmond.

PLATE XXXIV.

(*a*) MARTENSITE

(*b*) HARDENED HIGH-SPEED STEEL

(*c*) SOFT IRON, BEST YORKSHIRE

(*d*) MARTENSITE AND TROOSTITE

has been discovered the carbide satisfies the formula Fe_3C. It occurs in one of two forms, first in rectilinear lamellæ, forming a network which divides the mass into grains, and in this way isolates them from each other; secondly, as long, parallel lines placed near together, forming white bands with narrow black borders, in which form it is known as pearlite. It is so called because under rays of oblique light it exhibits the rainbow colouring of mother of pearl. Sorby has shown that hard steels are mixtures of pearlite and independent cementite, while soft steels are mixtures of pearlite and ferrite, and in intermediate steels the whole mass is composed of ferrite alone. Thus, the larger the percentage of cementite, the harder the steel, the cementite itself being an extremely hard substance.

SORBITE—The appearance of this substance is as if the white bands and the black lines had become merged into a dark grey mass. In some cases it occurs independently of the pearlite or side by side with it. Osmond says: "Sorbite may be considered as pearlite which has not been able to separate into ferrite and cementite by reason of lack of time or from some other cause, and it seems to be true that it ought to contain a little more 'hardening carbon' than free pearlite."

MARTENSITE—This substance has somewhat the appearance of pearlite with bands which do not run parallel to one another but show a general tendency to cross one another in such a way as to form a number of equilateral triangles, that is to say, in martensite there are three systems of fibres instead of two. This constituent shows itself after steel has been quenched in cold water when the temperature of the steel has been a little higher than the maximum temperature of the critical interval (Plate XXXIV (*a*)).

TROOSTITE—This constituent is obtained by quenching carbon steel during the period of transformation. With bas-relief polish projections become visible, and between these is a border of some material of medium hardness. The hard projections are composed of martensite and the hollow of ferrite. The borders are composed of a material known as troostite. Tempering tends to transform martensite into troostite.

AUSTENITE—Austenite is a material very much softer than martensite, so soft that it can be scratched with a sewing needle.

The term hardenite is often used to denote martensite.

Hardened high-speed steel is shown on Plate XXXIV (*b*). A photograph of soft iron, best Yorkshire, is given on Plate XXXIV (*c*). In this case the black areas represent the slag.

The constitution of the iron carbon compounds is very well shown on Fig. 260.

Material.	Commercial Elastic Limit.			Breaking Strength.			Modulus of Elasticity.	
	Tension.	Com-pression.	Shear.	Tension.	Compression.	Shear.	Direct (E).	Shearing (G).
				34,840	145,600	31,360	—	—
Cast iron	—	—	—	25,760	107,500	—	—	—
				15,230	67,200	19,000	—	—
Soft	—	—	—	12,720	60,300	—	8,405,000	—
Medium	—	—	—	25,200	98,560	—	14,366,000	—
Hard	—	—	—	17,870	97,440	—	14,720,000	—
Malleable cast iron (mean)	21,000	—	20,000	46,500	48,000	—	10,000,000	9,000,000
Steel, Soft, C=0·198%	29,000	—	—	59,000	—	—	30,000,000	—
,, Medium C=0·275%	22,000	—	—	65,000	—	—	30,000,000	—
,, Hard, C=0·514%	44,000	—	—	104,500	—	—	30,000,000	—
Steel for tools and springs (mean)	—	—	—	125,000	—	—	—	—
Rail steel, C=0·4–0·45% (mean)	45,000	—	—	75,000	—	—	—	—
Tyre steel, C=0·25% (mean)	—	—	—	97,000	—	—	—	—
Structural, C=0·10%	—	—	—	51,650	—	—	—	—
,, C=0·15%	—	—	—	64,670	—	—	—	—
Boiler shell steel (mean)	38,000	—	—	60,000	—	—	—	—
Flues, fireboxes, rivets	35,000	—	—	55,000	—	—	—	—
Open-hearth bars (mean)	33,500	—	—	53,000	—	—	—	—
Steel plates, C=0·25% (Unwin)	42,000	38,000	—	65,000	—	50,000		
,, ,, C=0·5 % (Unwin)	47,000	49,000	—	78,000	—	56,000	31,000,000	13,000,000
,, ,, C=1·0 % (Unwin)	67,000	71,000	—	110,000	—	83,000		
Average mild steel	—	—	—	—	—	49,000	—	—
Steel castings for—								
Ship frame	—	—	—	66,300	—	—	—	—
Engine parts	—	—	—	77,000	—	—	—	—
Roller path	—	—	—	79,000	—	—	—	—

Material.	Commercial Elastic Limit.			Breaking Strength.			Modulus of Elasticity.	
	Tension.	Compression.	Shear.	Tension.	Compression.	Shear.	Direct (E).	Shearing (G).
				34,840	145,600	31,360	—	—
Cast iron	—	—	—	25,760	107,500	—	—	—
				15,230	67,200	19,000	—	—
Soft	—	—	—	12,720	60,300	—	8,405,000	—
Medium	—	—	—	25,200	98,560	—	14,366,000	—
Hard	—	—	—	17,870	97,440	—	14,720,000	—
Malleable cast iron (mean)	21,000	—	20,000	46,500	48,000	—	10,000,000	9,000,000
Steel, Soft, C=0·198%	29,000	—	—	59,000	—	—	30,000,000	—
" Medium C=0·275%	22,000	—	—	65,000	—	—	30,000,000	—
" Hard, C=0·514%	44,000	—	—	104,500	—	—	30,000,000	—
Steel for tools and springs (mean)	—	—	—	125,000	—	—	—	—
Rail steel, C=0·4–0·45% (mean)	45,000	—	—	75,000	—	—	—	—
Tyre steel, C=0·25% (mean)	—	—	—	97,000	—	—	—	—
Structural, C=0·10%	—	—	—	51,650	—	—	—	—
" C=0·15%	—	—	—	64,670	—	—	—	—
Boiler shell steel (mean)	38,000	—	—	60,000	—	—	—	—
Flues, fireboxes, rivets	35,000	—	—	55,000	—	—	—	—
Open-hearth bars (mean)	33,500	—	—	53,000	—	—	—	—
Steel plates, C=0·25% (Unwin)	42,000	38,000	—	65,000	—	50,000		
" " C=0·5 % (Unwin)	47,000	49,000	—	78,000	—	56,000	31,000,000	13,000,000
" " C=1·0 % (Unwin)	67,000	71,000	—	110,000	—	83,000		
Average mild steel	—	—	—	—	—	49,000	—	—
Steel castings for—								
Ship frame	—	—	—	66,300	—	—	—	—
Engine parts	—	—	—	77,000	—	—	—	—
Roller path	—	—	—	79,000	—	—	—	—

Table I.—Table of Strengths in Lb. per Sq. In.—*continued*

Material.	Commercial Elastic Limit.			Breaking Strength.			Modulus of Elasticity.	
	Tension.	Compression.	Shear.	Tension.	Compression.	Shear.	Direct (E).	Shearing (G).
Steel castings	29,000 to 42,000	—	—	51,000 to 76,000	—	—	—	—
Steel for reinforcement of concrete	31,360 to 48,160	—	—	49,000 to 70,000	—	—	—	—
Steel wire for reinforcement	62,000 to 78,000	—	—	71,000 to 85,000	—	—	—	—
Vanadium steel, C=1·1%, V=0·0%	67,000	—	—	—	—	—	—	—
C=1·1%, V=1·11%	120,000	—	—	—	—	—	—	—
Nickel steel, C=0·35%, Ni=3·5%	78,400	—	—	112,000	—	—	—	—
Chromium steel, C=0·35%, Cr=1·5%	78,400	—	—	112,000	—	—	—	—
Nickel chrom. steel, C=0·3%, Ni=3·5%, Cr=0·5%	100,800	—	—	123,200	—	—	—	—
Wrought iron—								
Hammered bars	24,750	—	—	42,100	—	—	—	—
Plates (with grain)	24,000	24,000	15,000	47,000 to 49,300	—	36,000	26,000,000	14,000,000
Copper, cast	—	—	—	19,000 to 23,000	—	—	—	—
,, rolled plates	5,600	—	3,000	31,000	—	—	15,000,000	5,600,000
,, trolley wire	—	—	—	50,600	—	—	17,000,000	—
,, ,, ,, annealed	—	—	—	31,580	—	—	16,000,000	—
Brass	12,000 to 29,000	—	—	29,000 to 57,000	—	—	15,000,000	—
Gun metal	13,000 to 26,000	—	—	17,000 to 49,000	—	—	14,000,000	—
Manganese bronze	18,000 to 31,000	—	—	40,000 to 71,000	—	—	—	—
Phosphor bronze	18,000 to 24,000	—	—	22,000 to 35,000	—	—	14,000,000	—
Delta metal	17,000 to 51,000	—	—	36,000 to 74,000	—	—	12,000,000 to 13,000,000	—
Aluminium castings	6,000 to 24,000	—	—	—	—	—	—	—
,, rolled	24,640	—	—	—	—	—	12,000,000	—
Tin	—	—	—	4,700	—	—	—	—
Lead	1,500	—	—	2,500	—	—	—	—

TABLE I.—TABLE OF STRENGTHS IN LB. PER SQ. IN.—*continued*

Material.	Commercial Elastic Limit.			Breaking Strength.			Modulus of Elasticity.	
	Tension.	Compression.	Shear.	Tension.	Compression.	Shear.	Direct (E).	Shearing (G).
...c	3,200	—	—	7,500	—	—	—	—
...ralumin	—	—	—	58,000	—	—	—	—
...ood. Pines	—	—	—	6,700 to 13,000	—	—	1,000,000 to 1,600,000	—
..., Leafwoods	—	—	—	9,000 to 22,000	—	—	1,450,000	—
...oncrete	—	—	—	1,089	—	—	1,000,000	—
,,	—	—	—	2,722	—	—	1,750,000	—
,,	—	—	—	4,356	—	—	4,000,000	—
...icks	1,000 to 3,700	—	—	—	1,750 to 11,600	—	—	—
...ortar	250 to 1,200	—	—	—	300 to 15,000	—	—	—
...ickwork	500 to 2,000	—	—	—	800 to 2,800	—	1,400,000 to 2,800,000	—
...one	—	—	—	—	2,000 to 14,000	—	—	—
...eather belting	—	—	—	4,000 to 8,000	—	—	—	—
...lt joint (hook type)	—	—	—	700 to 800	—	—	—	—
...anilla rope	—	—	—	800 to 1,000	—	—	—	—
...tton driving rope	—	—	—	5,500 to 6,500	—	—	—	—
...rcelain. *Compression* results are higher for unglazed than glazed; *tension* results lower for unglazed than glazed	—	—	—	200 to 3,000	4,600 to 67,000	—	—	—

* For vitreous materials the yield is the first crack.
Artificial paving flags.—The cross-breaking strength on a flag 24 in. width, 18 in. span, $2\frac{1}{2}$ in. thickness, is 2,000 to 5,000 lb.

TABLE II.—WEIGHTS OF MATERIALS

Material.	Lb. in 1 cu. ft.
Wrought iron	485
Steel	499
Cast iron	450
Copper	552
Gun metal	528
Brass	525
Tin	455
Zinc	437
Lead, sheet	711
Aluminium	166
Sandstone	150
Granite	165
Bricks, soft	110
„ hard	134
Portland Cement concrete	138
Lime mortar	105
Wood, Pines	35
„ Leaf	53

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